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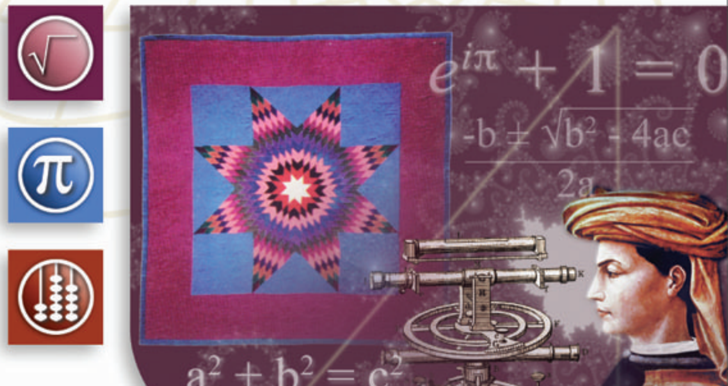

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VOLUME 1
Ab–Cy

Barry Max Brandenberger, Jr., Editor in Chief

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Preface

Mathematics has been defined as the "universal language of numbers," a language whose basic principles remain the same for all users. Many routine activities, from balancing a checkbook to measuring ingredients for a recipe, require a certain familiarity with mathematical concepts and applications. In Macmillan's *Mathematics* encyclopedia, users will learn more about the function of mathematics in everyday life, as well as its role as a tool for measurement, data analysis, and technological development. Interdisciplinary in scope, the encyclopedia provides students with a clear and comprehensive introduction to this vast topic through nearly three hundred entries commissioned especially for the set.

A Broader Look at Mathematics

For many people, mathematics and its basic disciplines—such as algebra, geometry, and trigonometry—are subjects of fear and dread. The goal of the *Mathematics* encyclopedia is to make these topics more accessible and interesting to students and the general user. Readers will learn that mathematics is much more than numbers alone—it is also an integral part of history and culture. Biographical entries highlight mathematicians who have made significant contributions to the field. Over thirty career-related articles offer examples of mathematics "on the job," whether it's a nutritionist calculating dietary needs or a photographer compensating for low-light conditions. Entries on applications explore the role of mathematics in our modern world, from everyday conveniences to global communication methods and a multitude of scientific and technological advances.

Organization of the Material

The authors who contributed entries to *Mathematics* bring a variety of expertise to the subject, and include members of academic institutions, math educators, and curriculum specialists. Contributors used their subject knowledge to write entries that are authoritative and up-to-date, but free of overly technological terms or scientific jargon. Many entries are illustrated, and numerous equations, tables, figures, and sidebars help illuminate the text. Unfamiliar terms and concepts are highlighted and defined in the margin, while cross-references direct users to articles of related interest. Most entries feature a selected bibliography, including Internet resources. Each volume includes a topical outline, glossary, and index, with a cumulative index to the entire set appearing in volume 4.

Acknowledgments and Thanks

I would like to acknowledge the support of my editorial advisors, Pam Garner and Dr. Ray Carry, for their valuable contributions to the scope and

overall design of this project. I would also like to thank Hélène Potter and Cindy Clendenon for their unending patience and guidance in making the initial ideas for this encyclopedia into a real product. Finally, I would like to thank my associate editor, Dr. Lucia McKay, for her remarkable mathematical and literary mind, which remained sharp and focused even after reviewing hundreds of articles. I cannot thank her enough.

Barry Max Brandenberger, Jr.

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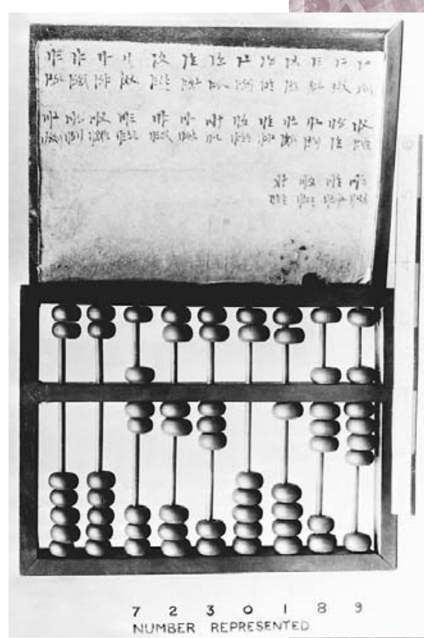
Abacus

The abacus is the most ancient calculating device known. It has endured over time and is still in use in some countries. An abacus consists of a wooden frame, rods, and beads. Each rod represents a different place value—ones, tens, hundreds, thousands, and so on. Each bead represents a number, usually 1 or 5, and can be moved along the rods. Addition and subtraction can easily be performed by moving beads along the wires of the abacus.

The word *abacus* is Latin. It is taken from the Greek word *abax*, which means "flat surface." The predecessors to the abacus—counting boards—were just that: flat surfaces. Often they were simply boards or tables on which pebbles or stones could be moved to show addition or subtraction. The earliest counting tables or boards may simply have been lines drawn in the sand. These evolved into actual tables with grooves in them to move the counters.

Since counting boards were often made from materials that deteriorated over time, few of them have been found. The oldest counting board that has been found is called the Salamis Tablet. It was found on the island of Salamis, a Greek island, in 1899. It was used by the Babylonians around 300 B.C.E. Drawings of people using counting boards have been found dating back to the same time period.

There is evidence that people were using abacuses in ancient Rome (753 B.C.E.–476 C.E.). A few hand abacuses from this time have been found. They are very small, fitting in the palm of your hand. They have slots with beads in them that can be moved back and forth in the slots similar to counters on a counting board. Since such a small number of these have been found, they probably were not widely used. However, they resemble the Chinese and Japanese abacuses, suggesting that the use of the abacus spread from Greece and Rome to China, and then to Japan and Russia.



This Chinese abacus is representing the number 7,230,189. Note the two beads on each rod of the upper deck and five beads on each rod of the lower deck. Beads are considered counted when moved towards the beam that separates the two decks.

The Suanpan

In China, the abacus is called a "suanpan." Little is known about its early use, but rules on how to use it appeared in the thirteenth century. The suanpan consists of two decks, an upper and a lower, separated by a divider. The upper deck has two beads in each column, and the lower deck has five beads in each column. Each of the two beads in the ones column in the top deck

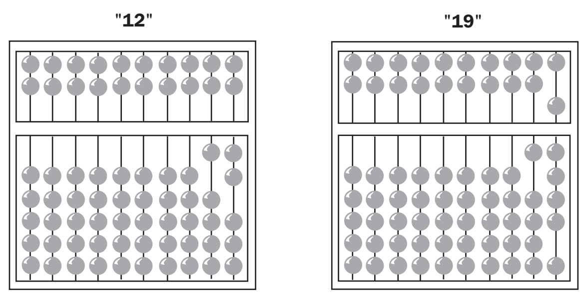
is worth 5, and each bead in the lower deck is worth 1. The column farthest to the right is the ones column. The next column to the left is the tens column, and so on. The abacus can then be read across just as if you were reading a number. Each column can be thought of in terms of **place value** and the total of the beads in each column as the **digit** for that place value.

place value in a number system, the power of the base assigned to each place; in base-10, the ones place, the tens place, the hundreds place, and so on

digit one of the symbols used in a number system to represent the multiplier of each place

The beads are moved toward the middle beam to show different numbers. For example, if three beads from the lower deck in the ones column have been moved toward the middle, the abacus shows the number 3. If one bead from the upper deck and three beads from the lower deck in the ones column have been moved to the middle, this equals 8, since the bead from the upper deck is worth 5.

To add numbers on the abacus, beads are moved toward the middle. To subtract numbers, beads are moved back to the edges of the frame. Look at the following simple calculation (12 + 7 = 19) using a suanpan.



The abacus on the left shows the number 12. There is one bead in the lower deck in the tens place, so the digit in the tens column is 1. There are two beads in the lower deck in the ones place, so the digit in the ones column is 2. This number is then read as 12. To add 7 to the abacus, simply move one bead in the ones column of the top deck (5) and two more beads in the ones column of the lower deck (2). Now the suanpan shows 9 in the ones column and 10 in the tens column equaling 19.

The Soroban

The Japanese abacus is called the soroban. Although not used widely until the seventeenth century, the soroban is still used today. Japanese students first learn the abacus in their teens, and sometimes attend special abacus schools. Contests have even taken place between users of the soroban and the modern calculator. Most often, the soroban wins. A skilled person is usually able to calculate faster with a soroban than someone with a calculator.

The soroban differs only slightly from the Chinese abacus. Instead of two rows of beads in the upper deck, there is only one row. In the lower deck, instead of five rows of beads, there are only four. The beads equal the same amount as in the Chinese abacus, but with one less bead, there is no carrying of numbers. For example, on the suanpan, the number 10 can be shown by moving the two beads in the upper deck of the ones column or only one bead in the upper deck of the tens column. On the soroban, 10 can only be shown in the tens column. The beads in the ones column only add up to 9 (one bead worth 5 and four beads each worth 1).

THE ABACUS VERSUS THE CALCULATOR

In late 1946 a Japanese postal official highly skilled in using the soroban (Japanese abacus) engaged in a contest with an American soldier in adding, subtracting, and multiplying numbers. The American used what was then the most modern electromechanical calculator. In four of five contests, the Japanese official with the soroban was faster, being beaten only in the multiplication problems.

The Schoty

The Russian abacus is called a schoty. It came into use in the 1600s. Little is known about how it came to be. The schoty is different from other abacuses in that it is not divided into decks. Also, the beads on a schoty move on horizontal rather than vertical wires. Each wire has ten beads and each bead is worth 1 in the ones column, 10 in the tens column, and so on. The schoty also has a wire for quarters of a ruble, the Russian currency. The two middle beads in each row are dark colored. The schoty shows 0 when all of the beads are moved to the right. Beads are moved from left to right to show numbers. The schoty is still used in modern Russia. SEE ALSO Calculators; Mathematical Devices, Early.

Kelly J. Martinson

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Absolute Value

Absolute value is an operation in mathematics, written as bars on either side of the expression. For example, the absolute value of -1 is written as $|-1|$.

Absolute value can be thought of in three ways. First, the absolute value of any number is defined as the positive of that number. For example, $|8| = 8$ and $|-8| = 8$. Second, one absolute value equation can yield two solutions. For example, if we solve the equation $|x| = 2$, not only does $x = 2$ but also $x = -2$ because $|2| = 2$ and $|-2| = 2$.

Third, absolute value is defined as the distance, without regard to direction, that any number is from 0 on the **real number** line. Consider a formula for the distance on the real number line as $|k - 0|$, in which k is any real number. Then, for example, the distance that 11 is from 0 would be 11 (because $|11 - 0| = 11$). Likewise, the absolute value of 11 is equal to 11. The distance for -11 will also equal 11 (because $|-11 - 0| = |-11| = 11$), and the absolute value of -11 is 11.

real number a number that has no imaginary part; a set composed of all the rational and irrational numbers

Thus, the absolute value of any real number is equal to the absolute value of its distance from 0 on the number line. Furthermore, if the absolute value is not used in the above formula $|k - 0|$, the result for any negative number will be a negative distance. Absolute value helps improve formulas in order to obtain realistic solutions. SEE ALSO Number Line; Numbers, Real.

Michael Ota

Absolute Zero

In mathematics, there is no smallest number. It is always possible to find a number smaller than any number given. Zero is not the smallest number because any negative number is smaller than zero. The number line extends to infinity in both the positive and negative directions. However, when

measuring things, it is often necessary to have a smallest number. If a car is stopped, it cannot go any slower. The temperature scale also has a lowest possible temperature, called "absolute zero." This is somewhat confusing, because temperatures measured on either the Fahrenheit or Celsius temperature scales are often negative. In some countries, temperatures below zero are quite common in the winter. So, before talking about absolute zero, some temperature scales should be explored.

Temperature Scales

In the United States, temperatures are usually reported on the Fahrenheit temperature scale. On this scale, water freezes at 32° F and water boils at 212° F. Temperatures below zero ("negative temperatures") are common, especially in the northern states. Thus 0° F is not the coldest possible temperature.

Scientific measurements of temperature in the United States and most other countries use the Celsius temperature scale. On this scale, water freezes at 0° C and water boils at 100° C. Nonetheless, 0° C is not the coldest possible temperature. In some parts of the United States, days or weeks may go by in winter when the temperature never rises *above* 0° C. Therefore, 0° C is also not the coldest possible temperature. However, there is a coldest possible temperature on both the Fahrenheit and Celsius scales, called absolute zero.✶

✶**Absolute zero on the Fahrenheit scale is −459 degrees.**

Determining Absolute Zero

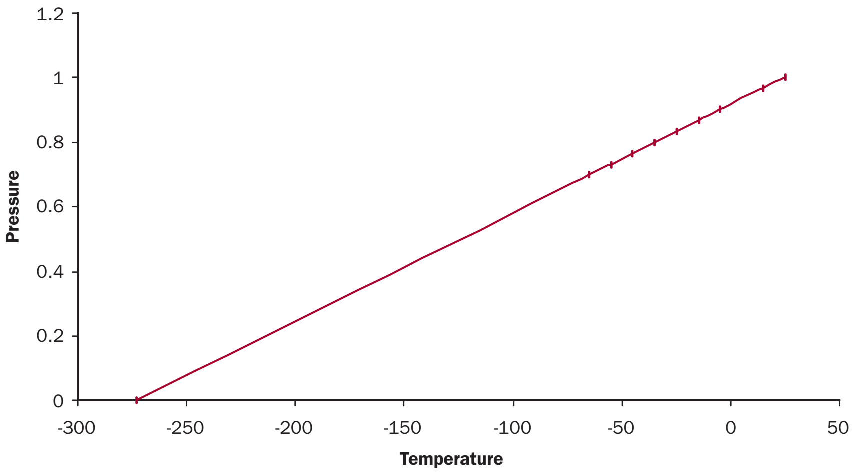
Suppose an experiment is done to determine if there is a lowest possible temperature. Helium gas is put in a cylinder with a pressure gauge. Helium is chosen because the atoms are very small and the attractive forces between helium atoms are also very small. A gas whose atoms exert no forces on each other and whose atoms have no volume is called an "ideal gas." Ideal gases do not exist, but helium behaves like an ideal gas if the temperature is relatively high (room temperature) and the pressure is low (atmospheric pressure).

The cylinder and gas have an initial temperature of 25° C. The cylinder is placed in a refrigerator and the temperature in the refrigerator is lowered to 15° C. The pressure of the gas in the cylinder also goes down (because the gas molecules are going slower). If the temperature is lowered to 5° C, the pressure will go down even more. This is done several more times and a graph is drawn of the data (see figure).

Notice that all of the data points fall along a straight line. Now it can be asked, at what temperature will the gas have no pressure? Of course a real gas will turn to a liquid or solid at some low temperature, but the temperature at which an ideal gas would have no pressure can be **extrapolated.** This temperature turns out to be approximately −273° C.

extrapolate to extend beyond the observations; to infer values of a variable outside the range of the observations

Although we picked helium gas for our thought experiment, the type of gas is not critical. At sufficiently high temperatures and low pressures, most gases behave like ideal gases. Consequently, at appropriately high temperatures and low pressures, all gases behave as if they would have no pressure at −273° C. This temperature is known as absolute zero.



Absolute Temperature Scales. Scientists have defined an absolute temperature scale that starts at absolute zero and uses a unit the same size as the Celsius degree. This absolute temperature scale is called the Kelvin scale. This temperature scale does not use a degree symbol. The unit of temperature is the kelvin. On this scale, water freezes at 273 K (read as 273 kelvin) and boils at 373 K. There is also an absolute temperature scale, called the "Rankine scale," that uses a unit the same size as the Fahrenheit degree. However, it is rarely used any more except in some engineering applications in the United States.

Achieving Absolute Zero. Is it possible to build a refrigerator that will cool an object to absolute zero? Surprisingly, the answer is no. This is not just a matter of building a better refrigerator. There are theoretical reasons why a temperature of absolute zero is impossible to achieve. The impossibility of achieving absolute zero is usually called the third law of thermodynamics. But although it prohibits reaching absolute zero, it does not prevent obtaining temperatures as close to absolute zero as we wish to get.

Low Temperatures in Nature

What are the lowest temperatures that occur in nature? The lowest natural temperature ever recorded on Earth was −89° C (recorded in Vostok, Antarctica on July 21, 1983). Other objects in the solar system can have much lower surface temperatures. Triton, a satellite of Neptune, was observed by *Voyager 2* to have a surface temperature of 37 K, making it the coldest known locale in the solar system. It is so cold that nitrogen freezes on the surface, making what looks like a pinkish frost. Triton has nitrogen **geysers** form when liquid nitrogen below the surface is vaporized by some internal heat source. The surface has lakes of frozen water. Pluto is only slightly warmer than Triton, with a temperature around 40 K to 60 K.

It might be thought that empty space is very cold. However, most space is not really empty. The space between the planets is not as cold as the surface of some of the planets. The few atoms and molecules in **interplanetary** space have a high **kinetic energy** (because of **planetary** magnetic fields and the **solar wind**), so they have a relatively high temperature. However,

MEASURING LOW TEMPERATURES

The temperature of liquid helium cannot be measured by using an ordinary thermometer. Among other problems, the mercury would freeze solid! So special methods must be used.

The most common method is to measure the resistance of certain types of wire. Platinum wire can be used at temperatures down to a few kelvin. In the liquid helium range, a phosphor-bronze wire with a small amount of lead added is used. At temperatures around 1.0 K, carbon resistance thermometers can be used. Below this temperature, various types of magnetic thermometers are used. These thermometers measure the magnetic properties of certain materials (such as magnetic susceptibility) that vary with temperature.

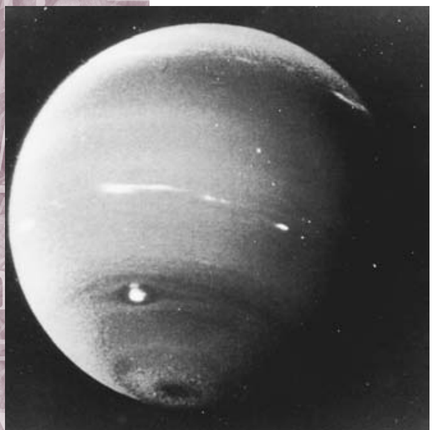
geyser a spring that occasionally spews streams of steam and hot water

interplanetary between planets; the space between the planets

kinetic energy the energy an object has as a consequence of its motion

planetary having to do with one of the planets

solar wind a stream of particles and radiation constantly pouring out of the Sun at high velocities; partially responsible for the formation of the tails of comets



The coldest recorded temperature in the solar system was found on a satellite (moon) of Neptune.

this hot gas will not warm an object in space, because there are so few atoms in the vacuum of interplanetary space.

The spaces between stars may also have quite high temperatures for the same reason as interplanetary space. **Interstellar** space can be filled with hot glowing gas, heated by nearby stars or strong **stellar** magnetic fields. The interstellar gas can sometimes have a temperature of millions of kelvins. However, the gas atoms are very far apart—farther apart than even in the best laboratory vacuum possible.

Intergalactic space is very cold. The radiation left over from the **Big Bang** at the beginning of the universe has a temperature of about 2.73 K. However, on Earth we have been able to produce much lower temperatures in laboratories.

Artificial Low Temperatures

Although a refrigerator cannot be built that will reach absolute zero, for nearly 100 years we have known how to build a refrigerator that will produce a temperature lower than the lowest naturally occurring temperature. Helium was first liquefied in 1908 by compressing and cooling the gas. Helium boils at 4.2 K. By rapidly pumping away the helium vapor, it is possible to lower the temperature to about 1.2 K. This happens because the helium atoms with the most energy are the ones that escape as vapor. As a result, pumping removes the higher energy atoms and leaves behind the atoms with lower kinetic energy.

The most common **isotope** of helium is helium-4. It has two neutrons and two protons in its nucleus. Helium-3 only has one neutron in its nucleus. By liquefying and pumping on a container of helium-3 it is possible to produce a temperature of around 0.3 K. This temperature is incredibly cold, but even lower temperatures have been recorded.

A mixture of helium-3 and helium-4 can produce even lower temperatures. At temperatures below 1.0 K, the mixture will separate into pure helium-3 and a saturated solution of helium-3 dissolved in helium-4. If the helium-3 atoms are pumped away from the mixture, more helium-3 atoms will dissolve from the pure helium-3 layer into the mixture. As might be expected, the helium-3 atoms with the most energy are the ones that will move into the mixture, leaving behind lower energy helium-3 atoms. This technique has produced temperatures as low as 0.002 K.

It is possible to produce even lower temperatures by using **magnetic traps** and other devices. At these extremely low temperatures, it becomes increasing difficult to state a precise definition of temperature. The **nucleus** of an atom, the conduction electrons of the atom, and the atom itself might all have different temperatures. The lowest temperature produced for liquid helium is 90 **microkelvins.**

The attempt to produce ever lower temperatures is not just competition for the sake of competition. The technology developed to produce and to measure low temperatures has broad applications in other fields. Also, the behavior of materials at extremely low temperatures tests the limits of our theoretical understanding of matter itself. For example, in 1996 Eric Cornell and a research team used magnetic trapping to produce a new state

interstellar between stars; the space between stars

stellar having to do with stars

intergalactic between galaxies; the space between the galaxies

Big Bang the singular event thought by most cosmologists to represent the beginning of our universe; at the moment of the big bang, all matter, energy, space, and time were concentrated into a single point

isotope one of several species of an atom that has the same number of protons and the same chemical properties, but different numbers of neutrons

magnetic trap a magnetic field configured in such a way that an ion or other charged particle can be held in place for an extended period of time

nucleus the dense, positive core of an atom that contains protons and neutrons

microkelvin one-millionth of a kelvin

of matter, called a "Bose-Einstein condensate." The existence of this state of matter confirmed our understanding of the **quantum** mechanical properties of matter. SEE ALSO Superconductivity; Temperature, Measurement of.

quantum a small packet of energy (matter and energy are equivalent)

Elliot Richmond

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Accountant

Accountants prepare, analyze, and verify financial records for individuals, businesses, and the government. Accountants may have their own businesses, work for accounting firms, or be employed by various types of businesses.

Some accountants specialize in tax matters and help people prepare income tax returns. Tax accountants also advise companies on the tax advantages and disadvantages of certain business decisions. These tasks involve a great deal of mathematics.

In preparing tax forms, accountants must add up income from the various sources. They use multiplication to compute tax savings that change with the number of exemptions, such as dependent children. They use subtraction to deduct expenses from income to calculate how much income must be taxed. Using all of these calculations, they determine how much tax their **clients** owe and develop and present various options to save the most money for their clients.

Accountants may review and assess budgets, performance, costs, and **assets** of different parts of a company. Computers are often used to prepare reports and graphs and to handle the detailed mathematics of business.

Many accountants prepare financial reports. An income statement summarizes a company's revenues (income) and expenses. A company's net earnings equal **revenues** minus expenses. Performance may be evaluated by comparing net earnings from one year to the next. Thus, there may be a percentage increase or decrease. The financial reports often contain graphs summarizing financial data.

Accountants use mathematical models to estimate how much a company's equipment **depreciates**. Accountants also use mathematics when budgeting, which entails controlling expenses based on the money available during a particular time period.

client an individual, business, or agency for whom services are provided by another individual, business, or industry; a patron or customer

assets real, tangible property held by a business corporation including collectible debts to the corporation

revenue the income produced by a source such as an investment or some other activity; the income produced by taxes and other sources and collected by a governmental unit

depreciate to lessen in value

People seek the advice of accountants to help them budget and prepare income tax forms.

Most accountant positions require at least a bachelor's degree in accounting. Large accounting firms often prefer people with a master's degree. SEE ALSO Financial Planner; Graphs.

Denise Prendergast

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Accuracy and Precision

Accuracy and precision both concern the quality of a measure. However, the terms have different meanings and should not be used as substitutes for one another.

Precision

Precision depends on the unit used to obtain a measure. The smaller the unit, the more precise the measure. Consider measures of time, such as 12 seconds and 12 days. A measurement of 12 seconds implies a time between 11.5 and 12.5 seconds. This measurement is precise to the nearest second, with a maximum potential error of 0.5 seconds. A time of 12 days is far less precise. Twelve days suggests a time between 11.5 and 12.5 days, yielding a potential error of 0.5 days, or 43,200 seconds! Because the potential error is greater, the measure is less precise. Thus, as the length of the unit increases, the measure becomes less precise.

The number of decimal places in a measurement also affects precision. A time of 12.1 seconds is more precise than a time of 12 seconds; it implies a measure precise to the nearest tenth of a second. The potential error in 12.1 seconds is 0.05 seconds, compared with a potential error of 0.5 seconds with a measure of 12 seconds.

Although students learn that adding zeros after a decimal point is acceptable, doing so can be misleading. The measures of 12 seconds and 12.0 seconds imply a difference in precision. The first figure is measured to the nearest second—a potential error of 0.5 seconds. The second figure is measured to the nearest tenth—a potential error of 0.05 seconds. Therefore, a measure of 12.0 seconds is more precise than a measure of 12 seconds.

Differing levels of precision can cause problems with arithmetic operations. Suppose one wishes to add 12.1 seconds and 14.47 seconds. The sum, 26.57 seconds, is misleading. The first time is between 12.05 seconds and 12.15 seconds, whereas the second is between 14.465 and 14.475 seconds. Consequently, the sum is between 26.515 seconds and 26.625 seconds. A report of 26.57 seconds suggests more precision than the actual result possesses.

The generally accepted practice is to report a sum or difference to the same precision as the least precise measure. Thus, the result in the preceding example should be reported to the nearest tenth of a second; that is,

rounding the sum to 26.6 seconds. Even so, the result may not be as precise as is thought. If the total is actually closer to 26.515 seconds, the sum to the nearest tenth is 26.5 seconds. Nevertheless, this practice usually provides acceptable results.

Multiplying and dividing measures can create a different problem. Suppose one wishes to calculate the area of a rectangle that measures 3.7 centimeters (cm) by 5.6 cm. Multiplication yields an area of 20.72 square centimeters. However, because the first measure is between 3.65 and 3.75 cm, and the second measure is between 5.55 and 5.65 cm, the area is somewhere between 20.2575 and 21.1875 square centimeters. Reporting the result to the nearest hundredth of a square centimeter is misleading. The accepted practice is to report the result using the fewest number of **significant digits** in the original measures. Since both 3.7 and 5.6 have two significant digits, the result is rounded to two significant digits and an area of 21 square centimeters is reported. Again, while the result may not even be this precise, this practice normally produces acceptable results.

Measurements in industrial settings such as a rubber manufacturing plant must be both accurate and precise. Here a technician is measuring tire pressure.

Accuracy

Rather than the **absolute** error to which precision refers, accuracy refers to the **relative** error in a measure. For example, if one makes a mistake by 5 centimeters in measuring two objects that are actually 100 and 1,000 cm, respectively, the second measure is more accurate than the first. The first has an error of 5 percent (5 cm out of 100 cm), whereas the second has an error of only 0.5 percent (5 cm out of 1,000 cm).

significant digits the digits reported in a measure that accurately reflect the precision of the measurement

absolute standing alone, without reference to arbitrary standards of measurement

relative defined in terms of or in relation to other quantities

How Are Precision and Accuracy Different? To illustrate the difference between precision and accuracy, suppose that a tape measure is used to measure the circumference of two circles—one small and the other large. Suppose a result of 15 cm for the first circle and 201 cm for the second circle are obtained. The two measures are equally precise; both are measures to the nearest centimeter. However, their accuracy may be quite different. Suppose the measurements are both about 0.3 cm too small. The relative errors for these measures are 0.3 cm out of 15.3 cm (about 1.96 percent) and 0.3 cm out of 201.3 cm (about 0.149 percent). The second measurement is more accurate because the error is smaller when compared with the actual measurement. Consequently, for any specific measuring tool, one can be equally precise with the measures. But accuracy is often greater with larger objects than with smaller ones.

Confusion can arise when using these terms. The tools one uses affect both the precision and accuracy of one's measurements. Measuring with a millimeter tape allows greater precision than measuring with an inch tape. Because the error using the millimeter tape should be less than the inch tape, accuracy also improves; the error compared with the actual length is likely to be smaller. Despite this possible confusion and the similarity of the ideas, it is important that the distinction between precision and accuracy be understood. SEE ALSO Estimation; Significant Figures or Digits.

Bob Horton

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Agnesi, Maria Gaëtana

Italian Mathematician and Philosopher
1718–1799

compendium a summary of a larger work or collection of works

algebra the branch of mathematics that deals with variables or unknowns representing the arithmetic numbers

precalculus the set of subjects and mathematical skills generally necessary to understand calculus

differential calculus the branch of mathematics primarily dealing with the solution of differential equations to find lengths, areas, and volumes of functions

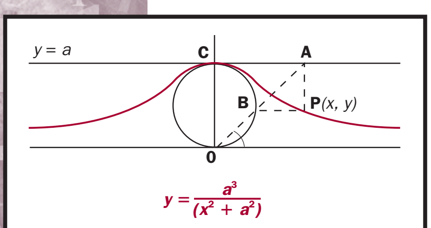
integral calculus the branch of mathematics dealing with the rate of change of functions with respect to their variables

Maria Gaëtana Agnesi was born in Milan, Italy. By the time she was 5 years old, she could speak both Italian and French. Her father was a professor of mathematics in Bologna, and Agnesi enjoyed a childhood of wealth and privilege. Her father provided her with tutors, and she participated in evening seminars, speaking to the guests in languages as varied as Latin, Greek, Hebrew, and Spanish.

In her teens, Agnesi mastered mathematics. She became a prolific writer and an advocate for the education of women. After her mother died, Agnesi managed the household of eight children, and educated her brothers. Her father remarried and after her stepmother died, Maria became housekeeper for twenty siblings.

Agnesi continued studying mathematics, mostly at night, and often to the point of exhaustion. In 1748 her mathematical **compendium**, *Instituzioni analitiche ad uso della gioventù italiana (Analytical Institutions)*, derived from teaching materials she wrote for her brothers, was published in two volumes. In this work, Agnesi had not only written clearly about **algebra**, **precalculus** mathematics, **differential calculus**, and **integral calculus**, but she had also added her conclusions and her own methods. *Analytical Institutions* remains the first surviving mathematical work written by a woman.

So successful was Agnesi's textbook that it became the standard text on the subject for the next 100 years. Her book was studied and admired not only in Agnesi's native Italy but also in France and Germany, and was translated into a number of other languages.



The bell-shaped curve shown in color is the curve traced by P as A moves along the line $y = a$. The point P is where the vertical line through A (which moves across the line $y = a$) crosses the horizontal line through B (which moves around the perimeter of the circle).

In his 1801 English translation of Agnesi's work, John Colson, the Cambridge (England) Lucasian Professor of Mathematics, made the mistake of confusing the Italian word for a versed sine curve, *aversiera*, with another Italian word for witch or wife of the devil, *avversiere*. Although 200 years have passed, the curve Colson misnamed the "Witch of Agnesi" still bears that name in many calculus texts as an illustration of an "even" function with a "maximum" value. For $a = 2$, for example, the maximum value of y will be 2. The curve illustrates many basic concepts in calculus.

Recognition

Agnesi was recognized during her lifetime with election to the Bologna Academy of Science. The Pope at the time was interested in mathematics and in 1750 made certain that Agnesi was invited to be an honorary lecturer in mathematics at the University of Bologna. However, Agnesi turned down the appointment and instead adopted a religious life, spending much of her time working among the elderly poor and sick women of Milan.

Although a hospice Agnesi founded has become famous throughout Italy, and Italian streets, a school, and scholarships have been named in her honor, Agnesi is perhaps best known today for her contributions to mathematics.✶ SEE ALSO Calculus.

Shirley B. Gray

✶On the centennial of Maria Gaëtana Agnesi's death in 1899, the Italian cities of Milan, Monza, and Masciago chose to remember her by naming streets after the noted mathematician and humanitarian.

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Agriculture

The shift from small, independent family farms to fewer, larger farms that rely heavily upon capital and technology began in 1793 with the invention of the **cotton gin**. The cotton gin enabled a single farm worker to clean 1,000 pounds of cotton in a day—a tremendous increase from the 50 pounds daily that an individual could clean without the gin. The continual adoption of new technologies in agriculture, along with the application of genetics, chemistry, physics, biology, and plant and animal nutrition, has created an agricultural industry in which mathematics play an increasingly important role.

cotton gin a machine that separates the seeds, hulls, and other undesired material from cotton

Crop Production Planning Models

Because most soils and climates are suitable for producing a variety of agricultural products, many farmers build a mathematical model to determine what crops to plant in which fields. Whereas some costs included in the model, such as insurance, taxes, **depreciation**, interest, land rents, and loan payments, are fixed, others vary widely, depending on the crop selected, the amount of labor and chemical to be applied, and other factors that can often be controlled by the farmer.

depreciate to lessen in value

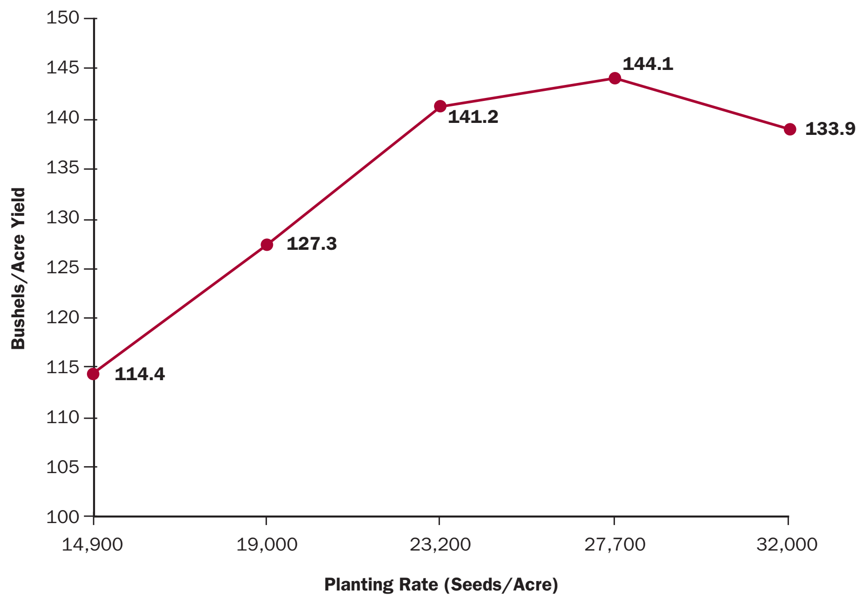
For example, corn seed is commonly sown at rates ranging from 19,000 to 32,000 seeds per acre. Because closely spaced plants compete for sunlight, water, and **nutrients**, excessive populations result in lower yield per plant. However, this decrease may be offset by an increase in total crop yield per acre. The line graph shows the effect of plant population on yield per acre of one particular corn hybrid.

nutrient a food substance or mineral required for the completion of the life cycle of an organism

Fertilizer selection requires calculations of how much of each nutrient is needed for the various crops. Nitrogen, for example, is available in solid, liquid, and gaseous man-made chemical fertilizers, as well as in organic fertilizers such as manure. Because each of these compounds contains a different percentage of nitrogen and because many fertilizers supply more than a single nutrient, the farmer's analysis of fertilizer costs and benefits often includes complicated algebraic equations.

revenue the income produced by a source such as an investment or some other activity; the income produced by taxes and other sources and collected by a governmental unit

After calculating projected expenses for the production of each crop, the farmer builds the **revenue** side of the model. After determining the area of



each field, the farmer then figures anticipated yield per acre, using weather expectations and historical data from the various fields. Acres, yield, and expected selling price are multiplied together to estimate revenue for each potential crop.

Careful consideration of these projected income and expense figures allows the farmer to determine which crops have the greatest likelihood of returning a profit and to plant accordingly. This exercise must be repeated each growing season for every field, as input costs and anticipated sales prices fluctuate.

Mathematics in the Field

commodities anything having economic value, such as agricultural products or valuable metals

Once crop selections have been made, farmers begin the processes of planting, nurturing, and harvesting farm **commodities**. These activities are likely to require the use of mathematics daily.

For example, to limit their effect on the environment, farmers use low application rates for fertilizers, herbicides, and insecticides. Typical application rates range from 0.5 to 16 ounces of active ingredient per acre. The chemicals are diluted with water for application. Mathematical formulas, such as the one below, are used to obtain the proper dosage through simultaneous adjustments of spray pressure, tractor speed, and chemical concentration.

$$\text{Gallons per acre} = \frac{5{,}940 \times \text{gallons per minute (per spray nozzle)}}{\text{miles per hour} \times \text{nozzle spacing (in inches)}}$$

calibrate act of systematically adjusting, checking, or standardizing the graduation of a measuring instrument

This is just one of many commonplace examples of the use of math in producing a crop. Farmers who irrigate must **calibrate** irrigation-pump engine speed to apply a given amount of water to a given area of cropland. Corn growers figure time remaining until harvest by calculating growing degree days, which is a measure of heat units needed by a corn plant to reach maturity. Formulas relating relative humidity and time to moisture content are used by farmers who must dry alfalfa or other forage crops before baling

Family farmers may still feed cows by hand, but the trend is toward larger corporate farms, which can increase profit margins by increasing numbers and by mechanizing tasks such as feeding livestock.

them as hay. The opportunities to use math in crop production are nearly endless.

Livestock Production

Decreasing **profit margins** and increasing demand for uniform, safe animal food products have boosted the significance of mathematics in livestock operations. Cattle, poultry, hog, and sheep producers keep individual productivity records on each animal, retaining for breeding purposes only those that efficiently convert feed into pounds of meat, milk, eggs, or wool.

profit margin the difference between the total cost of a good or service and the actual selling cost of that good or service, usually expressed as a percentage

For each animal, producers track rates of gain, feed consumption, carcass fat percentage, ratios of both more expensive and less desirable cuts of meat, and other data. Cost-effective **culling** and breeding decisions are made based on these numbers and formulas.

culling removing inferior plants or animals while keeping the best; also known as "thinning"

Additionally, livestock producers use algebraic formulas to create animal feed **rations** that are nutritionally balanced and cost effective. Feed components such as corn, alfalfa hay, oats, cottonseed hulls, and molasses all supply varying amounts of the micronutrients necessary for efficient livestock production. Complete rations are mixed using a combination of these and other ingredients. Careful attention to the balancing of rations not only controls costs but also increases productivity per animal.

rations the portion of feed that is given to a particular animal

Marketing

Many agricultural commodities can either be sold at harvest or stored for future sale. To estimate the amount of grain in storage and available for sale, geometric equations such as those in the table are used.

The decision of whether to sell or retain the harvest is essentially a mathematical exercise. Factors to consider include the cost of constructing on-farm

EQUATIONS FOR DETERMINING VOLUME OF STORED GRAIN

All measurements are in feet

Rectangular Bin	Bushels=0.8036 (length)(width)(average depth of grain)
Round Bin	Bushels=0.8036 (3.1416)(radius of base squared)(average depth of grain)
Piled grain (conical shaped pile)	Bushels=0.8036 (1.0472)(radius of base squared)(height of pile)

storage facilities or renting storage space from a local grain elevator, the potential increase in sales revenue if the product is sold at a higher price, the interest that could be earned if the crop were sold and money banked at harvest, and the risk involved if the price should decline or if part of the crop is lost as a result of spoilage.

Forward contracting, or locking in a sale price for future delivery, is done through **futures exchanges** such as the Chicago Board of Trade. Marketing through a futures exchange requires the farmer to use charts and graphs depicting market trends, trading volumes, **moving averages**, and other relevant data.

futures exchange a type of exchange where contracts are negotiated to deliver commodites at some fixed price at some time in the future

moving average a method of averaging recent trends in relation to long term averages, it uses recent data (for example, the last ten days) to calculate an average that changes but still smooths out daily variations

Mathematics and Agricultural Productivity

In countries where little attention is given to the mathematical and scientific aspects of farming, up to 70 percent of the population earns its livelihood through agriculture. As math and science are diligently applied to agriculture, productivity increases so that fewer farmers are needed. Thus, the number of farmers in the United States has declined from nearly 7 million in the 1930s to less than 2 million, or under 3 percent of the population.

The application of mathematics, science, and technology to agriculture has dramatically increased productivity per acre, per animal, and per farm. A small fraction of the population is now able to produce food enough for the entire world. SEE ALSO Algebra.

John R. Wood and Alice Romberg Wood

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Air Traffic Controller

Air traffic controllers play an integral role in flight safety by monitoring flight plans and alerting pilots of potential problems they may experience while in the air.

Air traffic control is a fast-paced job that requires a mathematics background. The required math knowledge includes geometry and calculus. Additionally, air traffic controllers must understand geography and weather conditions, engineering, computers, aeronautics, and aviation regulations. A good air traffic controller must also possess strong communication and decision-making skills, the ability to read and understand symbols, and an aptitude for abstract reasoning and visualization of objects in three-dimensional space.

Using radar, computers, and optical scanning devices, air traffic controllers measure the vertical space, or altitude, between an aircraft and the ground, the speed of the aircraft in **knots**, the direction of the aircraft through the use of compass headings, and its distance from other aircraft. The air traffic controller monitors a plane's route based on a filed flight plan and watches for any deviation that could cause a collision.

Air traffic controllers use instrument flight rules to monitor all planes as they take off, travel to their destinations, and land. Pilots navigate using a very high-frequency **omnidirectional** range beacon that denotes the pilot's magnetic course and measures the distance from the plane to ground stations monitored by air traffic controllers.

Air traffic controllers must be able to remember rapidly changing data, keep track of numerous aircraft at once, and work calmly under stressful situations. In the United States, air traffic controllers work for the Federal Aviation Administration and must pass the federal civil service examination. Around the world, all air traffic controllers speak English, use **Greenwich Mean Time**, and measure **meridians** and **longitudes** in relation to the **Prime Meridian**. SEE ALSO Flight, Measurement of.

Lorraine Savage

knot nautical mile per hour

omnidirectional a device that transmits or receives energy in all directions

Greenwich Mean Time the time at Greenwich, England; used as the basis for universal time throughout the world

meridian a great circle passing through Earth's poles and a particular location

longitude one of the imaginary great circles beginning at the poles and extending around Earth; the geographic position east or west of the prime meridian

Prime Meridian the meridian that passes through Greenwich, England

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Algebra

Algebra is a branch of mathematics that uses **variables** to solve equations. When solving an algebraic problem, at least one variable will be unknown. Using the numbers and expressions that are given, the unknown variable(s) can be determined.

variable a symbol, such as letters, that may assume any one of a set of values known as the domain

Early Algebra

The history of algebra began in ancient Egypt and Babylon. The Rhind Papyrus, which dates to 1650 B.C.E., provides insight into the types of problems being solved at that time.

The Babylonians are credited with solving the first **quadratic equation**. Clay tablets that date to between 1800 and 1600 B.C.E. have been found that show evidence of a procedure similar to the quadratic equation. The Babylonians were also the first people to solve indeterminate equations, in which more than one variable is unknown.

quadratic equation an equation in which the variable is raised to the second power in at least one term when the equation is written in its simplest form

The Greek mathematician Diophantus continued the tradition of the ancient Egyptians and Babylonians into the common era. Diophantus is considered the "father of algebra," and he eventually furthered the discipline with his book *Arithmetica*. In the book he gives many solutions to very difficult indeterminate equations. It is important to note that, when solving equations, Diophantus was satisfied with any positive number whether it was a whole number or not.

law a principle of science that is highly reliable, has great predictive power, and represents the mathematical summary of experimental results

identity a mathematical statement much stronger than equality, which asserts that two expressions are the same for all values of the variables

polynomial an expression with more than one term

binomial theorem a theorem giving the procedure by which a binomial expression may be raised to any power without using successive multiplications

complex plane the mathematical abstraction on which complex numbers can be graphed; the *x*-axis is the real component and the *y*-axis is the imaginary component

quaternion a form of complex number consisting of a real scalar and an imaginary vector component with three dimensions

By the ninth century, an Egyptian mathematician, Abu Kamil, had stated and proved the basic **laws** and **identities** of algebra. In addition, he had solved many problems that were very complicated for his time.

Medieval Algebra. During medieval times, Islamic mathematicians made great strides in algebra. They were able to discuss high powers of an unknown variable and could work out basic algebraic **polynomials**. All of this was done without using modern symbolism. In addition, Islamic mathematicians also demonstrated knowledge of the **binomial theorem**.

Modern Algebra

An important development in algebra was the introduction of symbols for the unknown in the sixteenth century. As a result of the introduction of symbols, Book III of *La géometrie* by René Descartes strongly resembles a modern algebra text.

Descartes's most significant contribution to algebra was his development of analytical algebra. Analytical algebra reduces the solution of geometric problems to a series of algebraic ones. In 1799, German mathematician Carl Friedrich Gauss was able to prove Descartes's theory that every polynomial equation has at least one root in the **complex plane**.

Following Gauss's discovery, the focus of algebra began to shift from polynomial equations to studying the structure of abstract mathematical systems. The study of the **quaternion** became extensive during this period.

The study of algebra went on to become more interdisciplinary as people realized that the fundamental principles could be applied to many different disciplines. Today, algebra continues to be a branch of mathematics that people apply to a wide range of topics.

Current Status of Algebra. Today, algebra is an important day-to-day tool; it is not something that is only used in a math course. Algebra can be applied to all types of real-world situations. For example, algebra can be used to figure out how many right answers a person must get on a test to achieve a certain grade. If it is known what percent each question is worth, and what grade is desired, then the unknown variable is how many right answers one must get to reach the desired grade.

Not only is algebra used by people all the time in routine activities, but many professions also use algebra just as often. When companies figure out budgets, algebra is used. When stores order products, they use algebra. These are just two examples, but there are countless others.

Just as algebra has progressed in the past, it will continue to do so in the future. As it is applied to more disciplines, it will continue to evolve to better suit peoples' needs. Although algebra may be something not everyone enjoys, it is one branch of mathematics that is impossible to ignore. SEE ALSO DESCARTES AND HIS COORDINATE SYSTEM; MATHEMATICS, VERY OLD.

Brook E. Hall

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Algebra Tiles

Algebra tiles are used to model **integers** and **variables** to aid in learning addition, subtraction, multiplication, and division of positive and negative values. Algebra tiles are especially useful in modeling multiplication and division because their dimensions are based on the concept of area. They can be bought or made, and they usually consist of three shapes: small squares for integers, rectangles for the variable x, and slightly longer rectangles for the variable y. The widths of the tiles for integers, for x, and for y are the same: one unit. Notice that the length of the x-tile is unknown: it is *not* equal to a certain number of unit tiles. The positive tiles are one color, and the negative tiles are another color, as shown in Figure 1(a).

integer a positive whole number, its negative counterpart, or zero

variable a symbol, such as letters, that may assume any one of a set of values known as the domain

One positive integer and one negative integer make a **zero pair**, as shown in Figure 1b. Because adding or subtracting zero does not change a number (e.g., $8 + 0 = 8$), zero pairs will be the basis of simplifying expressions.

zero pair one positive integer and one negative integer

Addition, Subtraction, Multiplication, and Division with Algebra Tiles

Addition is the simplest operation to do with algebra tiles. Just draw tiles for each term and then cross out zero pairs until no zero pairs remain. The total number of tiles that are left over is the answer, as shown in examples (a) through (c) of Figure 2.

Addition and subtraction are opposites. Whereas the strategy for addition is to remove zero pairs, the strategy for subtraction is to insert zero pairs. To subtract, start by drawing tiles for the first term, and then take away the amount to be subtracted, as shown in examples (d) through (g) of Figure 2.

If a closet measures 3 feet by 4 feet, then one might quickly sketch a rectangle with sides measuring 3 units and 4 units. Multiplication with algebra tiles uses the same concept. The length and width are the factors, and the area is the product. Recall the rules for multiplying positive and negative numbers: if the signs of the factors are the same, the product is positive; if the signs are different, the product is negative.

To model the product 3×4 using algebra tiles, start with an off-center "t" as a guideline, as shown in Figure 3(a). Place the factors three and four in the upper right and lower left portions of the "t." Draw the edges of a 3-by-4 rectangle in the lower right portion of the "t." Then extend the sides of each algebra tile across and down the rectangle. Count the number of resulting integer squares inside the rectangle to find the answer. In this example, the answer is 12. Examples (b) through (d) show other multiplication problems.

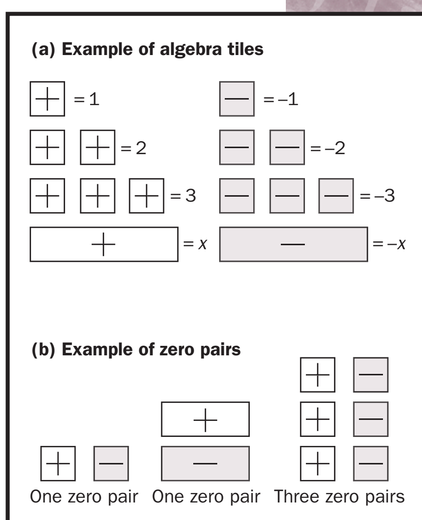


Figure 1. Examples of algebra tiles and zero pairs.

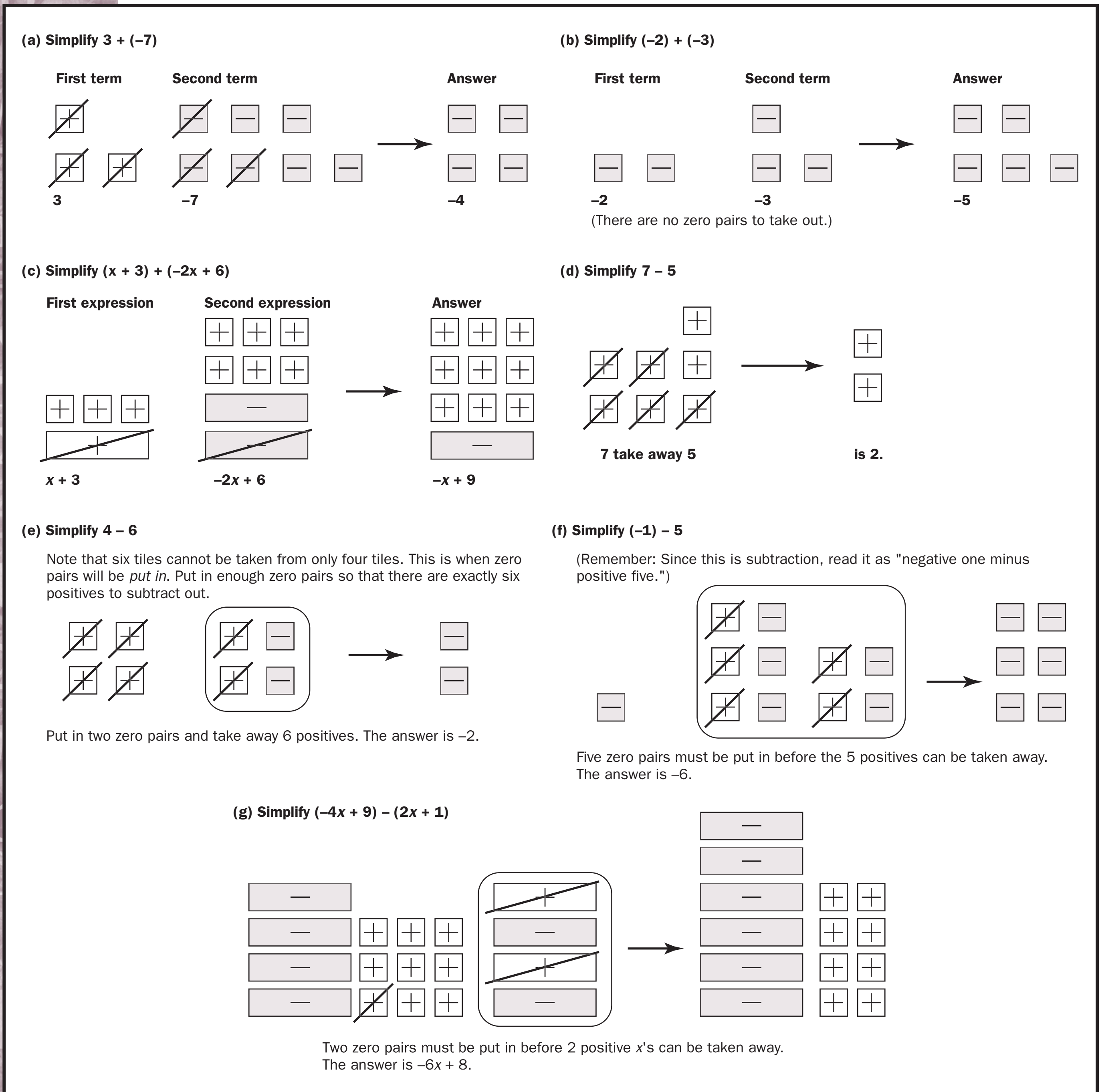


Figure 2. Examples of addition and subtraction using algebra tiles.

Division with algebra tiles employs the idea of division as sharing. Start by drawing the first term, and then divide its tiles into piles of equal amounts. The amount in *one* pile is the answer. See examples (e) and (f) in Figure 3.

Other Ways to Model Mathematics

Another conceptual way to represent integers and variables is to use cups and counters. Cups represent variables, and counters represent integers. Like algebra tiles, one color is used for positives, and another color is used for negatives. Cups are a great way to represent variables because, like a cup, variables

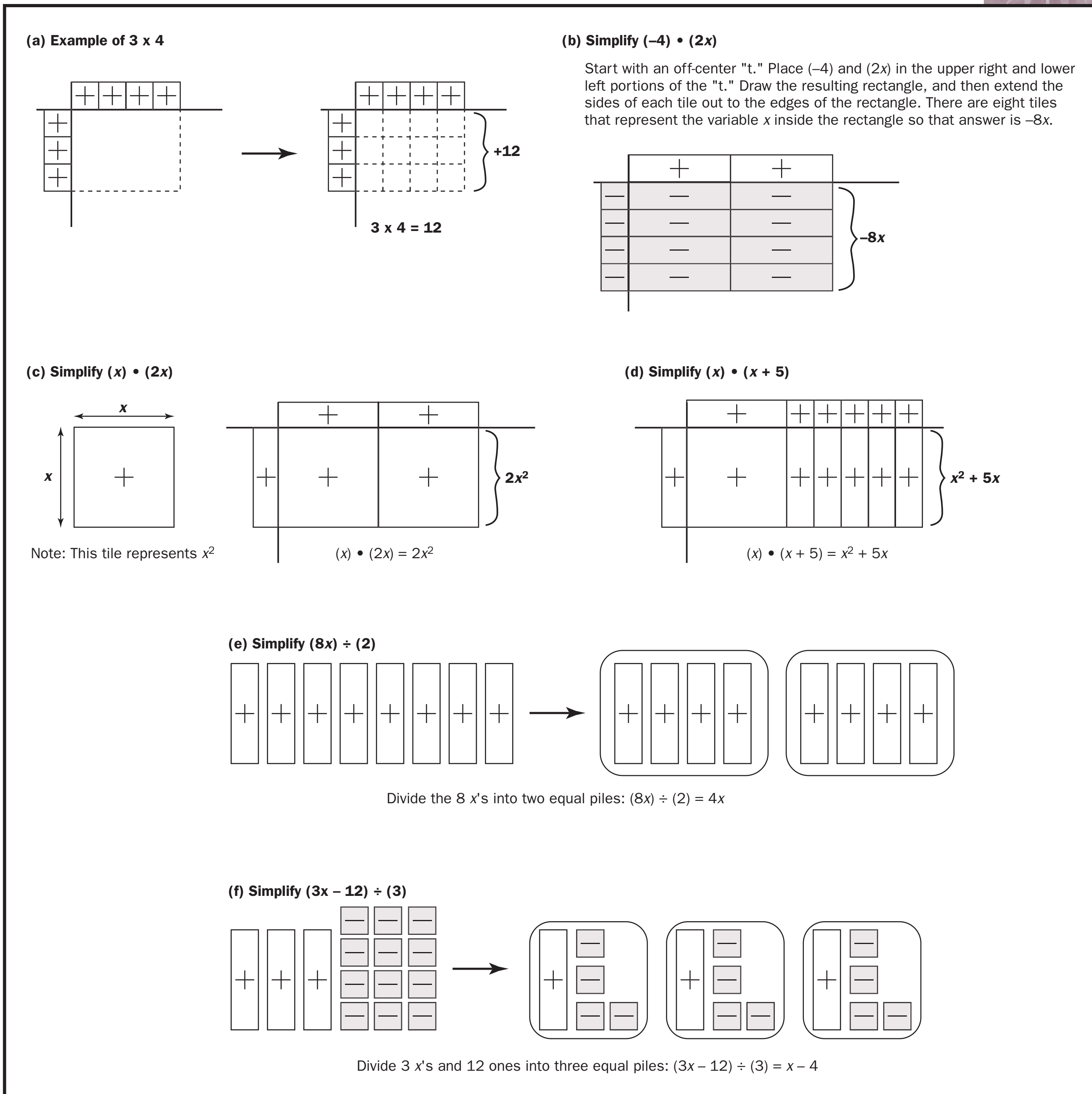
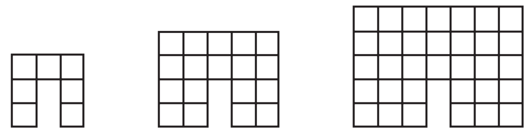


Figure 3. Examples of multiplication and division using algebra tiles.

can hold one, many, or no values. **Geoboards**, blocks, and even household items can be used to model proportions, slope, patterns, and functions.

The following example uses patterns of blocks to model building a tunnel. A student has built a tunnel using blocks. The height of the tunnel is two blocks, and the width of the tunnel is one block. See the drawings below. If the tunnel has one layer of thickness in all directions, then it will require seven blocks to build it; if it has two layers of thickness, then it will require eighteen blocks to build it, and so on. A table can be made to show how many cubes are required to make tunnels of different thicknesses.

geoboard a square board with pegs and holes for pegs used to create geometric figures



1 layer 2 layers 3 layers

Layers	Blocks
1	7 = (3 x 3) – 2
2	18 = (4 x 5) – 2
3	33 = (5 x 7) – 2
•	•
•	•
•	•

By studying the blocks and the table of relationships between layers and blocks, a pattern can be seen. For one layer, the tunnel requires enough blocks to make a 3×3 square less two blocks. For two layers, it requires enough blocks to make a 4×5 rectangle less two blocks. For three layers, it requires enough blocks to make a 5×7 rectangle less two blocks.

To predict the number of blocks required for x layers, a function can be written in which y is the number of blocks required for a particular number of layers, x. The equation can be written as $y = [(x + 2)(2x + 1)] - 2$. The number of blocks needed for four layers is $y = (4 + 2)(8 + 1) - 2 = 52$. This prediction follows the pattern in the table. For ninety-eight layers, 19,698 blocks are required. This example with blocks shows that any pattern made with models can be studied to try to write a function and make predictions. SEE ALSO Functions and Equations.

Michelle R. Michael

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Algorithms for Arithmetic

algorithm a rule or procedure used to solve a mathematical problem

An **algorithm** is a sequence of steps or instructions that outline how to solve a particular problem. One can think of an algorithm as a problem-solving formula or recipe. The term "algorithm" derives its name from al-Khwarizmi (c. 780–c. 850), an Arab mathematician who wrote an influential book on algebraic methods.

digit one of the symbols used in a number system to represent the multiplier of each place

power the number of times a number is to be multiplied by itself in an expression

base-10 a number system in which each place represents a power of 10 larger than the place to its right

integer a positive whole number, its negative counterpart, or zero

In the Hindu-Arabic number system, often referred to as the Arabic system, ten numerals or digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) are used in a sequence to form a number. Each **digit**, when the number is written in expanded form, represents a multiple of a **power** of ten; for example, $247 = 2 \times 10^2 + 4 \times 10^1 + 2 \times 10^0$. Therefore, this number system is called a **base-10**, or decimal, number system.

An algorithm that uses ten to an **integer** power n, 10^n, to perform calculations is a base-10 algorithm. Many of the rules that are used in fundamental arithmetic are actually algorithms for the base-10 system.

The base-10 number system is used every day and may seem ordinary, but it is a very powerful and elegant system with which to express numbers. Although most people may not be aware of it, when adding and subtracting numbers, for example, one is essentially using shortcut rules that are based on base-10 algorithms.

Addition Algorithms

Consider 448 + 246.

$$\begin{aligned} 448 &= 400 + 40 + 8 \\ \underline{246} &\underline{= 200 + 40 + 6\quad} \\ &= 600 + 80 + 14 \\ &= 600 + 80 + (10 + 4) \\ &= 694 \end{aligned}$$

The base-10 algorithm for the same problem is shown explicitly below.

$$\begin{aligned} 448 &= 4 \times 10^2 + 4 \times 10^1 + 8 \times 10^0 \\ \underline{246} &\underline{= 2 \times 10^2 + 4 \times 10^1 + 6 \times 10^0} \\ &= 6 \times 10^2 + 8 \times 10^1 + 14 \times 10^0 \end{aligned}$$

But $14 \times 10^0 = (10 + 4) \times 10^0 = 10 + 4 \times 10^0$. The 10 here is responsible for the 1 "carry-over" to 8, the first digit on the left. Therefore,

$$\begin{aligned} &= 6 \times 10^2 + (8 + 1) \times 10^1 + 4 \times 10^0 \\ &= 6 \times 10^2 + 9 \times 10^1 + 4 \times 10^0 \\ &= 600 + 90 + 4 \\ &= 694. \end{aligned}$$

Subtraction Algorithms

A base-10 algorithm for solving the subtraction problem $764 - 347$ follows.

$$764 - 347 = (700 + 60 + 4) - (300 + 40 + 7)$$

Here the idea is to subtract corresponding numbers: 7 from 4, 40 from 60, and 300 from 700. It is known that 7 is greater than 4, and 7 from 4 is -3, but suppose nothing is known about negative numbers. In the first parentheses, write 60 as 50 + 10.

$$764 - 347 = (700 + 50 + 10 + 4) - (300 + 40 + 7)$$

In the first parenthesis, add 10 to 4, so 14 is greater than 7.

$$764 - 347 = (700 + 50 + 14) - (300 + 40 + 7)$$

So now the resulting problem is to subtract 7 from 14, 40 from 50, and 300 from 700.

$$\begin{aligned} 764 - 347 &= (700 - 300) + (50 - 40) + (14 - 7) \\ &= 400 + 10 + 7 \\ &= 417 \end{aligned}$$

Another algorithm for subtraction is called "subtraction by taking complements." Consider $764 - 347$ again. Adding the same number, c, to both the terms will not affect the answer, because $c + (-c) = 0$. So effectively the value does not change.

$$764 - 347 = (764 + c) - (347 + c)$$

Choose c so that $347 + c$ becomes a multiple of 10, say 1,000. Therefore, $c = 653$.

$$\begin{aligned} 764 - 347 &= (764 + 653) - (347 + 653) \\ &= 1{,}417 - 1{,}000 \\ &= 417 \end{aligned}$$

Multiplication Algorithms

The following presents a multiplication algorithm in a vertical and horizontal format. Consider 24×12. To understand this, write 24 as $20 + 4$, and 12 as $10 + 2$.

$$\begin{array}{rl} \text{Vertical format: } & 24 = (20 + 4) \\ & \underline{\times\ 12 = (10 + 2)} \\ & 8 = 2 \times 4 \\ & 40 = 2 \times 20 \\ & 40 = 10 \times 4 \\ & \underline{200} = 10 \times 20 \\ & 288 \end{array}$$

Horizontal format: $12 \times 24 = (10 + 2) \times (20 + 4)$

distributive property property such that the result of an operation on the various parts collected into a whole is the same as the operation performed separately on the parts before collection into the whole; for example, multiplication is distributive over the arithmetic sum of different numbers

Using the **distributive property** of multiplication over addition, one can write

$$\begin{aligned} &= (10 \times 20) + (10 \times 4) + (2 \times 20) + (2 \times 4) \\ &= 200 + 40 + 40 + 8 \\ &= 288. \end{aligned}$$

A Division Algorithm

divisor the number by which a dividend is divided; the denominator of a fraction

Consider the division of a decimal number by another decimal number; for example, 28.75026 divided by 21.685. By using a division algorithm, this problem can be solved in three steps. The first step is to multiply the **divisor** by a power of ten that makes the divisor a whole number. Multiplying by 10 moves the decimal point one place to the right. Therefore, the divisor should be multiplied by 1,000, or 10^3, because there are three digits to the right of the decimal point in the divisor.

dividend the number to be divided; the numerator in a fraction

The second step is to multiply the numerator by 10^3. This is a necessary step to keep the division answer unchanged because multiplying both the **dividend** and the divisor by the same number keeps the fraction unchanged. Therefore, the decimal point in the dividend is moved three places to the right.

The resulting problem is to divide 28,750.26 by 21,685. The third step is to proceed with long division to get the answer.

Consider the following problem. There are 224 bananas. Each crate is packed with 16 bananas. How many crates are needed to pack all the bananas? Find the number that results from dividing 224 by 16. In other words, how many groups of 16 bananas are there in 224 total bananas?

In the first step of the division algorithm for 224/16, find the number of bananas that would be packed in the first ten crates and subtract it from the total bananas.

$$\begin{array}{r} 224 \\ \underline{-160} \\ 64 \end{array} = 10 \text{ groups of } 16$$

The next step is to figure out how many groups of 16 consist of 64.

$$\begin{array}{r} 64 \\ \underline{-\ 64} \\ 0 \end{array} \begin{array}{l} \\ = 4 \text{ groups of } 16 \\ = (10 + 4) \text{ groups of } 16 \end{array}$$

Therefore, 224/16 = 14. In fact, all arithmetic calculations are based on base-10 algorithms. SEE ALSO Bases; Decimals; Powers and Exponents.

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Alternative Fuel and Energy

The entire world is aware of the potential and real effects of environmental pollution. The continued alteration of the natural world is a topic of great concern. One of the kinds of pollution considered to be responsible for global climate change is air pollution. Natural influences are constantly putting a wide variety of chemicals into the atmosphere. However, in the last century humans have increased the output of atmospheric chemicals as a result of emissions from industrial plants and automobiles.

Governments around the world have passed laws forcing industries to reduce toxic emissions. Unfortunately, one of the leading sources of pollution is the exhaust from gasoline–powered vehicles, which emit carbon dioxide among other polluting agents, as a by-product of internal combustion engines. This gas has been found to contribute to global change by trapping the Sun's heat at Earth's surface. As the heat remains on the surface, temperatures begin to rise in a process similar to what occurs in a greenhouse. Though tremendous advances have been made in creating alternative fuels that are much cleaner and more efficient than those currently in use, traditional fossil fuels are still the dominant fuels used in transportation.

Some of the more promising alternative fuels are compressed natural gas (CNG), M85, a combination of 85 percent **methanol** and 15 percent gasoline, liquefied natural gas (LNG), electricity, biodiesel (made from animal and plant oils), and hydrogen. Most of these alternative fuels still have

methanol an alcohol consisting of a single carbon bonded to three hydrogen atoms and an O–H group

Solar panels are a nonpolluting substitute for fossil fuel-based electricity.

some elements of fossil fuels in them, yet the reason they are preferred to ordinary gasoline is that they burn cleaner and more efficiently. When used for transportation for example, natural gas produces far less carbon monoxide, carbon dioxide, and particulate matter than gasoline. In a fuel cell vehicle, which is powered by hydrogen, the by-product is water vapor.

Since these fuels are available, one would expect them to be in more widespread use. But they are not. This is the result of the mathematics of economy. Because alternative fuels are a fairly recent development, the **infrastructure** to support them is not in place yet, and these fuels end up being more expensive than traditional fuels. Usually, though, when figuring cost comparisons of alternative fuels to traditional fuels, externality costs are not included. These **externalities** actually increase the costs associated with fossil fuels. For example, future environmental costs is a variable often not considered.

infrastructure the foundation or permanent installations necessary for a structure or system to operate

externality a factor that is not part of a system but still affects it

Eventually, cars will not run solely on gasoline. The supplies of petroleum in the world are limited. As petroleum supplies begin to diminish, it will also become more difficult to extract what remains. In the meantime, consumers wonder why the cleaner cars and fuels are not available now. Once again, the explanation is cost. Alternative fuel vehicles are more expensive because the infrastructure to produce them is not yet in place. Fueling stations for alternative fuels are not as readily available as for fossil fuels. Gas stations can be found all over the United States. It is more difficult to find a filling station that caters to alternative fuels. This added effort deters consumers from purchasing an alternative fuel vehicle. In addition, the general public is more likely to buy a car that runs on gasoline since that is what they are used to, and because they are not as educated on the subject of alternative fuels as they could be.

If gasoline prices continue to rise as they have in recent years, it will be financially practical for the United States and other countries to commit to bringing down the costs associated with alternative fuels. Whether this is done through government incentives or subsidies, without lower prices it will be difficult to change the **paradigm** of consumers toward one that involves a greater amount of environmental accountability. Eventually countries will be forced to take a more **proactive** approach to alternative fuels because fossil fuels are a non-renewable resource.

paradigm an example, pattern, or way of thinking

proactive taking action based on prediction of future situations

There are many factors affecting air pollution. The chemical problems can be addressed, but it is the mathematics of economy that prevents the world from using alternative fuels in industry and motor vehicles. When it comes to decreasing air pollution, it is definitely a mathematical problem.

Brook E. Hall

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Analog and Digital

Transmitting information from one place to another has always been of great interest to humankind. Before the discovery of electricity and the invention of electronic devices, the principal means for transmitting information were limited to sound, writing and printing, and light. Since the discovery of electricity, transmitting information with electrical signals through wires and with electromagnetic waves through the air have come to dominate information technology.

Most electronic transmission of information makes use of a very limited symbol set—usually **binary numbers** consisting of groups of binary digits (0s and 1s). An electrical current or a magnetic field can very easily be interpreted as being either on or off, or in one direction or the other, and so is well suited to representing binary digits.

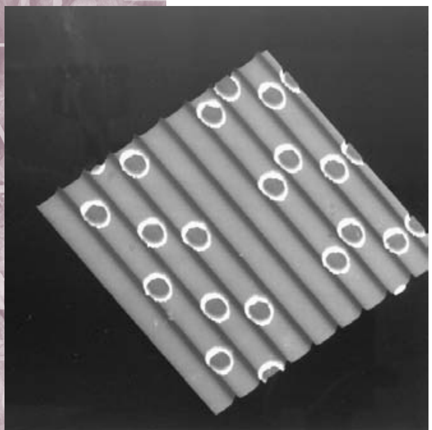
binary number a base-2 number; a number that uses only the binary digits 1 and 0

Binary numbers are the building blocks of **digital** technology. To put digital signals into context, we must first understand how information messages can be transmitted.

digital describes information technology that uses discrete values of a physical quantity to transmit information

Transmitting Messages

Every message transmitted contains information in a coded form. What is actually transmitted is a signal that conveys the information by some physical means. A voice message is sent using sound waves, which are pressure



Data marks burned on this optical disc illustrate the discrete nature of digital information.

variations in a fluid (such as air) or a solid. A written or printed message involves ink marks on paper or dark and light spots on a computer monitor or television screen. A picture message uses paint or chalk on a canvas. An electronic message involves movement of electrons.

No matter how the message is transmitted, understanding it requires interpretation and reconstruction of what is received to reveal what the sender intended to convey; namely, the information. Until the message is interpreted by the receiver, the information remains hidden. It is coded in and represented by the physical changes—variations and arrangements of pressure, color, inkspots, electrical voltage, electromagnetic waves, light patterns, and so forth—that are involved in the actual transmission of the message.

Analog and Digital Messages. Whether a message is classified as analog or digital depends on the type of variation in the physical changes. If the variation is continuous, then the message is said to be analog. If the information in the message is represented by variations that go in distinct steps, then the message is digital. For example, a digital clock uses digits to show each distinct minute. An analog clock shows the time by an analogy (comparison) between the time and the position of the clock hands.

An ordinary voice message, conveyed directly from a speaker to a hearer by pressure changes in the air, is an analog message because the pressure changes are continuous: the pressure varies smoothly up and down to create sound. An ordinary photograph is an analog message—colors vary continuously from one part of the picture to another.

A message conveyed using an alphabet is digital. Each unit in the message is a particular letter. If the signal is clear, then each symbol in the message is immediately recognizable as one letter or another.

Advantages of Digital Information

The power of digital signals comes from the fact that the receiver knows that each symbol is supposed to be a member of the symbol set being used. For example, when one is reading a message written in particularly bad handwriting, although it may be difficult to decide what some of the letters are, one knows that there are only twenty-six possibilities (in the Roman alphabet) for each letter of the message. This often makes it possible to read successfully a message written in quite a scrawl.

base-10 a number system in which each place represents a power of 10 larger than the place to its right

base-2 a binary number system in which each place represents a power of 2 larger than the place to its right

The smaller the number of possible symbols that can be transmitted, the easier it is to read a message in which the signal has been degraded. For example, if one receives a message that is known to be in the **base-10** number system, then each symbol must be one of the ten decimal digits. Even very hastily written numbers can be read. However, using even this limited set, it is sometimes hard to distinguish between 5s and 6s or between 7s and 9s. In contrast, if the set of allowable symbols is limited to 0 and 1 (the binary or **base-2** digits), then the chance of successfully interpreting even a poor message increases.

bit a single binary digit, 1 or 0

Combining binary digits (or **bits**) in large groups can be used to represent information of any desired complexity. A string of eight bits is known

Each piano key generates a distinct, digital "message" of sound. But the music of a complex piano tune is analog because the listener hears a blending of notes, continuously changing melody, and layered harmonies.

as a **byte**. With eight bits in a byte, there are 256 possible different arrangements for a string of eight 0s and 1s, ranging from 0 to 255. Hence, one byte can represent any one of 256 colors at a particular point on a cathode-ray tube, or it can represent any one of 256 characters in an "alphabet." Such an alphabet can represent all the lower-case letters, all upper-case letters, all decimal digits, many punctuation characters, and many special symbols such as a smiley face, a frowny face, arrows, and icons.

byte a group of eight binary digits; represents a single character of text

fidelity in information theory a measure of how close the information received is to the information sent

By digitally representing the rich alphabet of a written language, using groups of bits and bytes, messages can be transmitted that can be interpreted much more reliably than if the same messages were transmitted in analog form. One can easily see the difference by comparing a fax message with an e-mail message. The image of the printed page is conveyed in a fax by continuous variations of electrical current that are reassembled at the receiver to make another image of the page. Noise on the line degrades the transmitted signal and makes the received image fuzzy. In contrast, the information content of an e-mail message is conveyed through electrical signals that represent binary digits. The receiver has only to decide whether each bit received is a 0 or a 1, which is a relatively easy choice.

Cellular telephones, digital versatile disc (DVD) recordings, compact disc recordings, and satellite TV all use digital signals to increase the **fidelity** of the information received. Complex pictures and sounds are divided into very many small pieces by sampling; that is, by looking at the sound or light individually at each of very many points or times in the picture or sound. Information about each of these points is converted to a series of binary digits, which is transmitted. The receiver then converts this digital signal back into the individual tiny pieces of the picture or sound and assembles them again to make an accurate representation of the original. By doing a large amount of processing, a message can be transmitted very accurately using digital signal methods. SEE ALSO Bases; Communication Methods; Compact Disc, DVD, and MP3 Technology; Computers and the Binary System.

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Angles, Measurement of

The problem of measuring angles has been considered by mathematicians since ancient times. The ability to measure angles with some precision is important in fields such as astronomy, physics, engineering, and architecture, as well as in various practical supporting fields such as carpentry. It is difficult to imagine the ancient Egyptians trying to construct the pyramids without a precise system for measuring angles, particularly **right angles**.

right angle the angle formed by perpendicular lines; it measures 90 degrees

set a collection of objects defined by a rule such that it is possible to determine exactly which objects are members of the set

The problem of measuring angles is compounded by the need for a satisfactory definition of the object to be measured. It was not until the concept of a **set** was developed in the late nineteenth century that the modern definitions of geometric objects such as sets of points were possible, and it was well into the twentieth century before such definitions began to appear in geometry textbooks. As late as the 1920s an angle was typically defined by one of three methods:

1. The difference in inclination (slant) to each other of two lines that intersect (meet or cross);
2. The amount of turning (rotation) necessary to bring one side to the position of the other; or
3. The portion of the **plane** included between its sides.

plane generally considered an undefinable term, a plane is a flat surface extending in all directions without end, and that has no thickness

By the 1940s, there was some movement toward the modern approach in definitions such as "the geometric figure formed by two **rays** drawn from the same point," but it was not until the 1960s that the modern definition of an angle as "the **union** of two rays having a common endpoint" was commonly used in textbooks.

ray half line; a line segment that originates at a point and extends without bound

union a set containing all of the members of two other sets

While the modern definition rightly focuses attention on the points forming the object itself, the older definitions seem to hint more at the measurement of that object. In particular, the first two definitions above suggest a rotational aspect of angle measurement. On the other hand, the third definition suggests a more static (without rotation) aspect of angle measurement. There are numerous applications of both aspects. For example, civil engineers measuring the grade (inclination) of a newly cut road pan are interested in the amount to turn the **transit** above or below a horizontal line, but an opthamologist measuring a patient's field of vision is interested in the portion of a plane the patient can see without moving her or his eyes.

transit a surveyor's instrument with a rotating telescope that is used to measure angles and elevations

Early Measurements of Angles

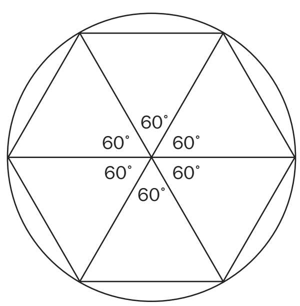
Ancient civilizations were first concerned with measuring right angles, because of the importance of right angles in the construction of buildings and in surveying. Early instruments for measuring right angles included tools resembling the modern carpenter's square, the baculum (sometimes called

the geometric cross or cross-staff), and a rope stretched to form a 3-4-5 right triangle.

The baculum consisted of a flat rod approximately 4 feet in length with a cross-piece that could be slid along the rod while remaining always **perpendicular** to it. The idea of stretching a rope into the form of a known figure, such as the 3-4-5 right triangle was immortalized by the Egyptian "rope-stretchers."

The concept of measuring angles other than right angles requires some consistent unit of measure. Sometimes that unit angle appears to have been the right angle, as in the ancient Greek's use of the right angle in stating relationships among angles. For example, in the *Elements*, Euclid wrote that the three interior angles of a triangle are equal to two right angles. However, the ancient Babylonians chose a different unit angle which led to the development of the **degree**, the unit of angle measure which we know today.

The Angle-Circle Connection. Most historians think that the ancient Babylonians believed that the "circle" of the year consisted of 360 days. This is not a bad approximation given the crudeness of the ancient astronomical tools. Historians of mathematics also generally believe that the ancient Babylonians knew that the side of a **regular hexagon** inscribed in a circle is equal in length to the **radius** of the circle. This may have suggested to them a division of the full circle (360 "days") into six equal parts, each part consisting of 60 "days" as shown below. Once they divided the angle of an **equilateral triangle** into the 60 equal parts now called degrees, they further subdivided a degree into 60 equal parts, called minutes, and a minute into 60 equal parts called seconds.



Many scholars believe this explains the use of 60 as the base of the Babylonian system of angle measurement. However, others believe that the use of 60 as the base had more to do with the fact that 60 has many integral **divisors**. Work with fractional parts of the whole (60) is greatly simplified, because 2, 3, 4, 5, 6, 10, 12, 15, 20, and 30 are all divisors of 60.

Regardless of which account is accurate, the Babylonians' use of the degree as a unit of angle measure has survived to the present era, and one degree represents 1/360 of a circle. The genius of basing the measurement of angles on the circle is that it does not matter which circle is used: the angle cuts off the same fractional part of the circle, as shown below.

THE ROPE-STRETCHER LEGEND

By placing twelve equally spaced knots in a loop of rope, the ancient Egyptian rope-stretchers were able to form right angles easily and accurately. Ancient Greek historians give accounts of the annual flooding of the Nile River and the subsequent need for precise surveying techniques. These historians greatly admired the accomplishments of the rope-stretchers, probably in part due to the Greeks' admiration for the Egyptians' remarkable precision in building the pyramids. But some modern historians believe that the rope-stretchers' accomplishments were somewhat exaggerated.

perpendicular forming a right angle with a line or plane

degree 1/360 of a circle or complete rotation

regular hexagon a hexagon whose sides are all equal and whose angles are all equal

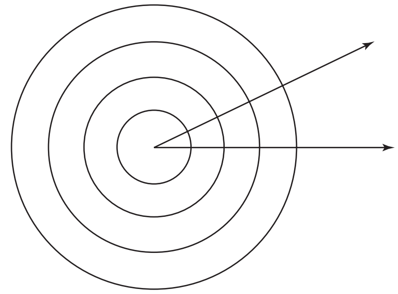
radius the line segment originating at the center of a circle or sphere and terminating on the circle or sphere; also the measure of that line segment

equilateral triangle a triangle whose sides and angles are equal

divisor the number by which a dividend is divided; the denominator of a fraction

EARLY INSTRUMENTS

Early instruments used for the measurement of angles of any size include the astrolabe, planisphere, and quadrant. These instruments are all variations of a protractor and are similar to the modern sextant. A sextant is an instrument used to measure the angle of inclination of celestial bodies in order to determine the latitude of the observer. Given the great interest in astronomy of ancient civilizations, it is not surprising that their instruments for the measurement of angles were such that they could be easily used to take astronomical readings.



It is also noteworthy that the Babylonian choice of 360 for the number of equal subdivisions of a circle was arbitrary. They might have subdivided the circle into any number of equal parts. In fact, the gradient—a seldom-used unit of angle measurement that can infrequently be encountered in engineering—is based on subdividing a right angle into 100 equal parts. Then, since a circle consists of four right angles, there are 400 gradients in a circle.

Making Indirect Measurements

Once the ancients had developed the capacity to measure angles with precision, they had one of the tools necessary for indirect measurement. Indirect measurement simply refers to methods for obtaining measurements of properties of figures, particularly distances, which are inaccessible to direct measurement. The work of engineers, surveyors, cartographers, and astronomers would be rendered virtually impossible without indirect measurement schemes. In fact, the branch of mathematics known as trigonometry arose as a response by ancient Greek mathematician-astronomers to the problem of obtaining inaccessible measurements: the distances and angles between certain celestial bodies.

In the period from about 250 B.C.E. to 200 C.E., Greek mathematicians undertook the first systematic study of relationships between ratios of distances and angle measures and between the lengths of **chords** of a circle and the measures of the angles in the circle, or **arcs** of the circle, which subtended (or cut off) those chords. As a result of this work, all that was needed to supply reasonable estimates of otherwise inaccessible celestial measurements was a fairly precise measure of the radius of Earth.

chord a straight line connecting the end points of an arc of a circle

arc a continuous portion of a circle; the portion of a circle between two line segments originating at the center of the circle

circumference the distance around a circle

The ancient Greek mathematician Eratosthenes of Cyrene had provided the means for such estimates by determining the **circumference** of Earth with astonishing accuracy in about 250 B.C.E. To do so, Eratosthenes had to use a form of indirect measurement. He observed that at high noon on the day of the summer solstice, the Sun shone directly down a deep well in the city of Syene in Egypt. He had also observed that at the same time (high noon on the day of the summer solstice) in the city of Alexandria, 500 miles due north of Syene, the Sun cast shadows showing that the angular distance between Syene and Alexandria was 1/50th of a circle. Therefore, Eratosthenes reasoned that the circumference of Earth should be 50 times the distance from Syene to Alexandria, or about 25,000 miles. The remarkable accuracy of this estimate is one of the crowning achievements of ancient astronomy. SEE ALSO Angles of Elevation and Depression; Astronomy, Measurements in; Circles, Measurement of; Distance, Measuring; Mathematics, Very Old; Navigation; Trigonometry.

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Angles of Elevation and Depression

Angles of elevation and depression are angles that are formed with the horizontal. If the line of sight is upward from the horizontal, the angle is an angle of elevation; if the line of sight is downward from the horizontal, the angle is an angle of depression. These types of angles and some **trigonometry** can be used to indirectly calculate heights of objects or distances between points. Alternatively, if the heights or distances are known, the angles can be determined.

trigonometry the branch of mathematics that studies triangles and trigonometric functions

Angles of Elevation

Finding a Flagpole's Height. Suppose that the height of a flagpole must be determined. The flagpole forms a right angle with level ground. Suppose the flagpole casts a shadow of 20 feet. See part (a) of the figure on the following page. The 20-foot shadow and the flagpole itself form two legs of a right triangle. The triangle's **hypotenuse** is formed by the line of sight extending from the end of the shadow (Point E) upwards toward the Sun. The angle of elevation of the Sun has one ray extending along the horizontal toward the bottom of the pole, and the other extending toward the Sun. Furthermore, suppose the angle of elevation measures 50°. Because three parts of this right triangle are known (the **right angle**, the shadow side, and the angle formed by the shadow and the hypotenuse), there is enough information to find the flagpole's height.

hypotenuse the long side of a right triangle; the side opposite the right angle

right angle the angle formed by perpendicular lines; it measures 90 degrees

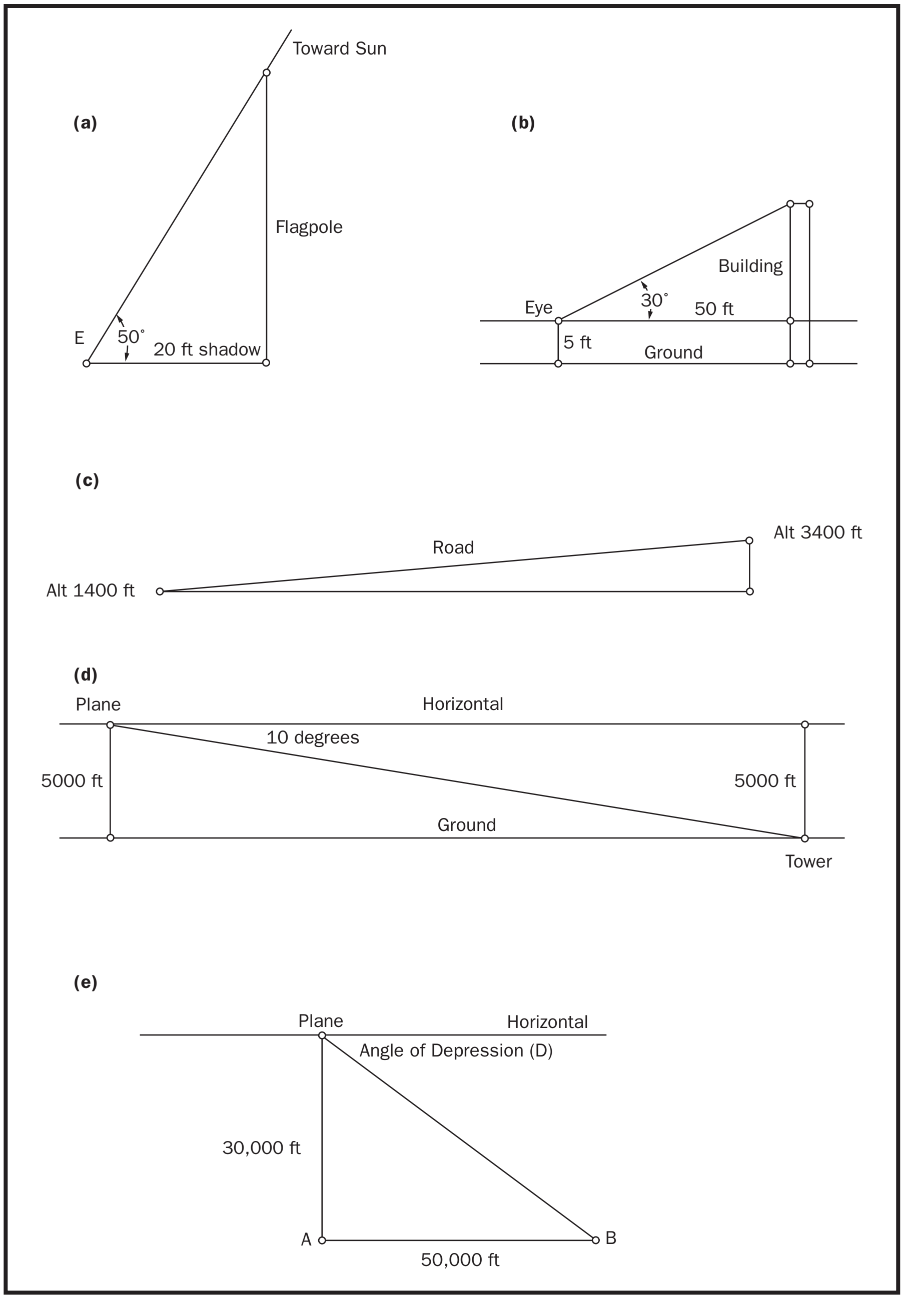
The relationship between the angle of elevation, the height of the flagpole, and the length of the shadow is represented by the equation $\tan 50° = \frac{h}{20}$, where h represents the height of the flagpole. Solving for h results in $h = 20\tan 50° \approx 23.8$ feet.

Determining a Building's Height from Eye Level. Care is often needed when working with angles of elevation to ensure that the correct measure is being determined. Suppose one wants to determine the height of a building. Perhaps there is no shadow, or perhaps the angle of elevation from a point on the ground cannot be measured. With a type of **protractor** that uses gravity, however, the angle of elevation from one's eye to the top of the building can be measured.

protractor a device used for measuring angles, usually consisting of a half circle marked in degrees

Suppose the angle measured using this type of protractor is 30°, and the horizontal distance to the building is 50 feet. As before, a right triangle is formed. The relationship determined by the given information is $\tan 30° = x/50$. Solving for x results in $x = 50\tan 30° \approx 29$ feet.

Examples of making indirect measurements using distances and angles of elevation and depression.



However, when determining the height of the building, the height of one's eye must be added because the angle was not measured from the ground. See part (b) of the figure. If one's eye is 5 feet above the ground, the building must be about 34 feet tall.

Calculating a Road's Angle of Elevation. Using the inverse trigonometric functions, one can also determine the angle of elevation if only the sides of a triangle are known. Suppose a road increases in altitude from 1,400 feet to 3,400 feet over the span of 10 miles (52,800 feet). See part (c) of the figure. The average angle of elevation of the road, called A, can be found by solving the equation $\tan A = \frac{2{,}000}{52{,}800}$. To solve for angle A, the inverse tan-

gent of both sides must be taken. This can be denoted as $\tan^{-1}A$. The result is $A = \tan^{-1} \frac{2{,}000}{52{,}800} \approx 2.17°$.

Angles of Depression

Determining Distance from an Airplane. Like an angle of elevation, an angle of depression is formed by a ray and the horizontal, but the ray is lower than the horizontal. Suppose that a pilot is flying at an altitude of 5,000 feet and spots a control tower ahead. If the pilot lowers her eyes from the horizontal at an angle of 10°, she can calculate her distance to the tower with right-triangle trigonometry.

See part (d) of the figure. If d represents the direct distance to the tower (measured along the hypotenuse), the pilot can find the distance by solving the equation $\sin 10° = \frac{5{,}000}{d}$, which yields a solution of approximately 28,800 feet, or close to $5\frac{1}{2}$ miles.

Similarly, the pilot can also determine the horizontal difference (the distance along the ground) between her current position and the tower. If g represents the horizontal difference, she solves the equation $\tan 10° = \frac{5{,}000}{g}$, which yields a solution of approximately 28,400 feet, or a little less than $5\frac{1}{2}$ miles. With such a small angle, the difference between the horizontal leg and the hypotenuse is small.

Finding an Angle of Depression. As with angles of elevation, if two of the sides of the right triangle are known, one can solve for an angle of depression. For example, suppose a plane is flying at an altitude of 30,000 feet directly above point A on the ground. Furthermore, suppose that point A is 50,000 feet from another point on the ground, point B. See part (e) of the figure. Note that the angle of depression from the plane (D) is **congruent** to the angle of elevation from point B.

The equation $\tan D = \frac{30{,}000}{50{,}000}$ can be used to determine the angle of depression, D, from the plane to point B. To do so, the **inverse tangent** of both sides is taken, resulting in $D = \tan^{-1} \frac{30{,}000}{50{,}000} \approx 31°$. SEE ALSO Angles, Measurement of; Mount Everest, Measurement of; Trigonometry.

Bob Horton

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Apollonius of Perga

Greek Geometer

262 B.C.E.–190 B.C.E.

Apollonius, known as "The Great Geometer" to his admirers was born in 262 B.C.E. in the Greek colonial town of Perga in Anatolia, a part of modern Turkey. Apparently Apollonius's intellectual ability was recognized early, and as a young man he attended the university in Alexandria, Egypt, where many of the great scholars of that time were gathered.

A HELPFUL (HORIZONTAL) HINT

A common mistake when working with angles of elevation and angles of depression is mislabeling. To avoid mislabeling, remember that angles of elevation and angles of depression are always formed with the horizontal, and never with the vertical.

congruent exactly the same everywhere; having exactly the same size and shape

inverse tangent the value of the argument of the tangent function that produces a given value of the function; the angle that produces a particular value of the tangent

Apollonius of Perga may not have known his contemporary Archimedes, but he did improve upon Archimedes' calculation of the value of π.

plane generally considered an undefinable term, a plane is a flat surface extending in all directions without end, and that has no thickness

ellipse one of the conic sections, it is defined as the locus of all points such that the sum of the distances from two points called the foci is constant

hyperbola a conic section; the locus of all points such that the absolute value of the difference in distance from two points called foci is a constant

parabola a conic section; the locus of all points such that the distance from a fixed point called the focus is equal to the perpendicular distance from a line

solid geometry the geometry of solid figures, spheres, and polyhedrons; the geometry of points, lines, surfaces, and solids in three-dimensional space

plane geometry the study of geometric figures, points, lines, and angles and their relationships when confined to a single plane

eccentric having a center of motion different from the geometric center of a circle

epicycle the curved path followed by planets in Ptolemey's model of the solar system; planets moved along a circle called the epicycle, whose center moved along a circular orbit around the sun

retrograde apparent motion of a planet from east to west, the reverse of normal motion; for the outer planets, due to the more rapid motion of Earth as it overtakes an outer planet

Apollonius's teachers had studied with Euclid (c. 330–c. 260 B.C.E.), who is regarded as the most outstanding mathematician of ancient times. Apollonius quickly gained a reputation for his thorough and creative approach to mathematics and was made a professor at the university.

Apollonius wrote a number of books on mathematics, especially geometry. He gathered, correlated, and summarized the mathematics of his predecessors. More importantly, he extended their work and made many creative and original contributions to the development of mathematics.

His best known work is contained in the eight volumes of *Conics*. Volumes I–IV survive in the original Greek, and volumes I–VII, like many other Greek intellectual works, survive in medieval Arabic translation. Volume VIII is lost but is known from references made to it by later Greek scholars.

Conics addressed the four types of curves that result when a solid cone is cut into sections by a **plane**: the circle, the **ellipse**, the **hyperbola**, and the **parabola**. Apollonius discovered and named the latter two curves. In *Conics*, he gives a thorough treatment of the theory of these curves and related matters, developing a total of 387 propositions.

Apollonius also demonstrated applications of the geometry of conic sections and curves to various mathematical problems. His work led to the separation of geometry into the two divisions of **solid geometry** and **plane geometry**.

In addition to *Conics*, Apollonius wrote at least eleven books, some in more than one volume. Of these, only one—*Cutting Off a Ratio* (also known as *On Proportional Section*)—survives in an Arabic translation. The others are known only by mention or discussion by other Greek mathematicians and authors.

In addition to his work in pure mathematics, Apollonius analyzed the properties of light and its reflection by curved mirrors, and he invented a sundial based on a conic section. Of particular importance was his application of geometry to astronomy. His use of elliptical orbits with **eccentric** and **epicylic** motion to explain the complex movement of planets, including their **retrograde** motion, was accepted until the time of Copernicus (1473–1543).

The work of Apollonius had an extensive effect on the subsequent development of mathematics and is still relevant today. Later mathematicians influenced by Apollonius's work include René Descartes (1596–1650) in the development of Cartesian mathematical science; Johannes Kepler (1571–1630) in his proposal of elliptical orbits for the planets; and Isaac Newton (1642–1727), who used conic sections in understanding the force of gravity. SEE ALSO CONIC SECTIONS; DESCARTES AND HIS COORDINATE SYSTEM; EUCLID AND HIS CONTRIBUTIONS; NEWTON, SIR ISAAC.

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Archaeologist

Archaeology is the science that deals with past human life and activities as shown by the **artifacts** left by ancient people. An archaeological site is carefully surveyed before, during, and at the end of a dig. Surveying involves measuring distances, directions, and angles using equipment such as the theodolite. The sites are then mapped into grids for excavation.

artifact something made by a human and left in an archaeological context

Archeologists use mathematics in several areas of their work. First, knowledge of simple geometry is necessary for measuring variables such as the height and length of artifacts, bones, and relics. An understanding of **scale drawing** is also a necessity. While excavating a site, the archaeologist draws a top plan to record what each area looks like throughout the dig. This top plan is drawn to scale as much as possible. Additionally, archaeologists need to understand how to reduce a square proportionately.

scale drawing a drawing in which all of the dimensions are reduced by some constant factor so that the proportions are preserved

A knowledge of simple statistics is useful for checking statements that involve comparisons, such as claims that the dimensions of artifacts change over time. An awareness of **probability** and **correlation** also leads to a better understanding of sampling. Scientific analyses carried out for purposes such as the identification of rocks or soil result in lists of figures that can only be clarified by statistical methods. SEE ALSO Distance; Measuring; Ratio, Rate, and Proportion; Statistical Analysis.

probability the likelihood an event will occur when compared to other possible outcomes

correlation the process of establishing a mutual or reciprocal relation between two things or sets of things

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Archimedes

Greek Mathematician and Inventor
287 B.C.E.–212 B.C.E.

Archimedes was the greatest mathematician of the ancient world and one of the greatest mathematicians of all time. He was born in the Greek city of Syracuse on the island of Sicily. As a young man, Archimedes studied with successors of Euclid at Alexandria, Egypt. He returned to Syracuse after his studies and spent the rest of his life there.

Archimedes is famous for his practical applications of mathematics. During his time in Egypt, Archimedes invented a device now known as the Archimedean screw. This device is still employed in many parts of the world to pump water for irrigation. A short time later, he invented the double pulley, which was used by merchants to haul boats out of water. Archimedes also expanded the principles of the common lever.

Although more commonly used to lift water, this Archimedes screw is moving plastics through a recycling plant.

EUREKA!

According to tradition, Archimedes discovered this first law of hydrostatics while trying to determine if a crown fabricated for King Hiero of Syracuse was solid gold or an alloy of gold and silver. While lowering himself into his bath, Archimedes observed that the water level rose. It struck him that the amount of water displaced equaled the immersed portion of his body. He was so overcome by this discovery that he immediately ran from his bath through the streets toward the royal palace shouting "Eureka!" (I have found it!).

In his lifetime, Archimedes was best known for his war machines. Some commentaries describe his huge catapults that could hurl enormous rocks great distances. Another of Archimedes' war machines could snatch ships out of the water and crush them.

Archimedes also studied the center of gravity, or the balancing point of geometric shapes; the specific gravity of **geometric solids**; and what traditionally became known as Archimedes' Principle, used to determine the weight of a body immersed in a liquid.

geometric solid one of the solids whose faces are regular polygons

Some of Archimedes' other discoveries include determining that the value of π is between $3\frac{10}{71}$ and $3\frac{1}{7}$, showing that the surface of a sphere is four times the area of its great circle, and finding that the volume of a sphere is two-thirds the volume of its circumscribed (bounding) cylinder. In making these discoveries, Archimedes used integration, an early form of calculus. Some historians claim that if Archimedes had had access to modern mathematics notation, he could have invented the calculus nearly 2,000 years earlier than Sir Isaac Newton. SEE ALSO Calculus.

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Architect

The path to becoming an architect is not an easy one. It begins in high school with good grades, particularly in such subjects as algebra, calculus, trigonometry, and physics. It then requires at least a bachelor's degree, and possibly a master's degree, from an accredited school of architecture. After a three- to five-year internship, aspiring architects must pass a licensing examination in the state they wish to practice. Along the way, an architect will become comfortable with numbers, measurements, and the principles of engineering. In addition, they develop solid computer skills, including the ability to use computer-aided design and drafting programs.

The architect's job begins with a sketch of the building the client wants, though a landscape architect will design natural environments rather than structures. After imagining the building, the architect—usually working as part of a team—must turn the concept into detailed drawings with realistic features so that the builder can construct it in the way it was envisioned, while staying within the customer's timeframe and budget.

It is not enough, though, for an architect to be an artist and mathematician; an architect also has to have good communications skills. Under conditions that are often stressful, an architect will have to communicate with builders throughout the design and construction process and make sure that they have the needed information. The architect also must communi-

cate with clients to ensure they are satisfied with the results of the project. SEE ALSO Computer-Aided Design; Landscape Architect; Morgan, Julia.

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Architecture

A basic human impulse is to look for patterns in our surroundings. An example is the impulse to find patterns in time. People organize their daily activities around natural rhythms, such as the rising and setting of the Sun. Similarly, people are also driven to find patterns in space. One of the tools used to do this is mathematics. Mathematics attempts to describe relationships and patterns that emerge within ordered, logical structures. Whereas many patterns occur in the natural world, many occur in objects that humans fabricate, including homes, office buildings, concert halls, and places of worship. Behind these buildings are the architects who translate these mathematical concepts into concrete form. Today, humans continue to marvel at the mathematical constructions of ancient architects—the pyramids, **ziggurats**, temples, stadiums, and even such technological feats as ancient irrigation projects.

ziggurat a tower built in ancient Babylonia with a pyramidal shape and stepped sides

Historical Architects

Architecture and mathematics have historically been disciplines that were indistinguishable. Architects were mathematicians, and mathematicians were often architects. In the sixth century, Byzantine emperor Justinian wanted the **Hagia Sophia** to be unlike any other building built before, so he assigned the job to two professors of mathematics named Isidoros and Anthemios. In the Islamic world, architects created a wealth of complex patterns, particularly in their elaborate tiling patterns.

Hagia Sophia Instanbul's most famous landmark, built by the emperor Justinian I in 537 C.E. and converted to a mosque in 1453 C.E.

It has been said that people used little mathematics during the medieval period, but the magnificent cathedrals of the Middle Ages sharply contradict that belief. Medieval stonemasons had a firm command of geometry, and they constructed their monuments according to mathematical principles. They did not write their mathematics down in books; instead, they wrote it down in the structures and spaces they created. These and other traditional architects developed architecture into a discipline that is, at the core, mathematical.

Architecture and Mathematical Principles

What are some of the mathematical principles that architects use? One is the use of scale. In designing a building, an architect starts with a schematic that represents the building, much as a map represents geography. When making the drawing, the architect uses a tool called an "architect's scale" calibrated in multiples of $\frac{1}{16}$ and $\frac{1}{32}$ of an inch. Using a scale enables the

Architectural drawings must be rendered with precision and to exact scale. Shown here is a drawing of a 17-story casino resort hotel.

architect to draw a diagram of a room or building in a way that represents its actual proportions. For example, if the architect is trying to draw a section of wall that measures 4 feet 3 inches, the markings on an architect's scale enables them to draw a line that consistently represents that distance on a smaller scale. The scale also ensures that a line representing a section of wall that will measure, as an example, 8 feet 6 inches is exactly twice as long as the 4-foot 3-inch line. Being able to measure scale enables an architect to reproduce complementary proportions throughout a structure.

geometer a person who uses the principles of geometry to aid in making measurements

quadratic equation an equation in which the variable is raised to the second power in at least one term when the equation is written in its simplest form

Closely related to scale is what architects and **geometers** refer to as the Golden Mean, or sometimes the Divine Proportion. The Golden Mean is expressed as the number 1.618 . . . and is arrived at by solving a **quadratic equation**. This equation defines in numbers the most aesthetically pleasing relationship of a rectangle. The use of the Golden Mean in designing buildings and rooms to maintain a pleasing sense of proportion—to avoid the appearance of "squatness," for example—has been a constant throughout the history of architecture.

azimuth the angle, measured along the horizon, between north and the position of an object or direction of movement

Sometimes architects rely on mathematics for practical rather than aesthetic purposes. For example, in designing an overhang for a building, architects can calculate sun altitudes and **azimuths**. This enables them to determine the best angle and size of the overhang to provide maximum shade in the summer and solar heat in the winter. To provide adequate natural lighting in a room, they generally calculate the area of windows such that it is at least 8 percent of the room's floor area. Architects will use human dimensions to determine, for example, the best height for a counter or how

much space is needed to walk and work between a kitchen counter and an island in the middle of the kitchen.

Architects also use geometry to determine the slope, or pitch, of a roof. This slope is usually expressed as a rise-over-run fraction; for example, a $\frac{6}{12}$ pitch means that for each horizontal foot, the roof line rises 6 inches; an $\frac{8}{12}$ roof would have a steeper pitch, a $\frac{4}{12}$ roof a gentler pitch. These measurements are critical if, for example, a dormer window is going to protrude from the roof. The architect has to ensure that the dormer is set back far enough from the front—but not too far—for the window and its surrounding structure to fit and be at the correct height. For architects, such precision is very important. SEE ALSO Architect; Golden Section; Scale Drawings and Models.

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Artists

Are artists and mathematicians two completely different kinds of people? Brain research suggests that artists may create their works by using the right side of their brains, whereas mathematicians may reason and calculate by using the left side of their brains. However, it is not necessarily true that art and math cannot coexist. Perhaps the most famous example of someone who excelled in art as well as math and science is Leonardo da Vinci. Indeed, many artists today use mathematical formulas and calculations in their work.

Anyone who mixes solutions as part of his or her work uses formulas and **proportions**. Photographers, for example, who develop their own film must mix dry chemicals and liquids to create the baths in which they immerse film and paper. Someone who works with ceramics mixes glazes and calculates formulas involving time and temperature for firing the glazed pottery. Most painters and graphic artists do not grind and mix their own pigments to make paint, but they do use the mathematics of proportion as they work with perspective. They also use math as they determine the dimensions for paper, canvas, and frames.

Moreover, all artists who sell their work to earn their living use mathematics to track costs, expenses, and income. Artists who work with expensive materials, such as gold, diamonds, and other precious metals and stones, may base their prices partly upon the current market price for these **commodities**. Sculptors, who may create a miniature or mold in wax and then cast it in bronze, may also consider their expenses for these procedures, as well as the cost of raw materials, as they determine the price of their artwork. Other artists adjust their prices based on the amount of time involved. As artists become known, their prices may move farther away from the cost

proportion the mathematical relation between one part and another part, or between a part and the whole; the equality of two ratios

commodities anything having economic value, such as agricultural products or valuable metals

Artists use mathematics throughout their creative process. This woman may use proportions to determine the perspective of her painting, as well as the size of her canvas or frame.

basis and be more related to their aesthetic and market value. Nonetheless, every artist who uses a gallery or an agent will become familiar with the mathematics of percent as they pay commissions on every work that is sold. SEE ALSO Ceramicist; Computer Graphic Artist; Percent; Photographer; Ratio, Rate, and Proportion.

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Astronaut

An astronaut is a person trained to perform a specific task on a craft designed and intended to travel beyond Earth's atmosphere. The term "astronaut" derives from the Greek words meaning "space sailor." The National Aeronautics and Space Administration (NASA) uses the term to designate all persons who have been launched as crew members aboard NASA spacecraft bound for Earth orbit and beyond. The term is also used by NASA to designate everyone selected to join the corps of NASA astronauts, whether they have been in space or not.

The Past

Between April 9, 1959, when the first seven U.S. astronauts were presented at a press conference in Washington, DC, and 1978, a total of 73 men were selected as astronauts by the National Aeronautics and Space Administration. Of these individuals, 43 flew in the Mercury, Gemini, Apollo, and Skylab programs. In 1978, 35 new astronaut candidates were selected for the Space Shuttle program, including six women. Since then, candidates have been recruited continuously, with selection occurring about every two years.

NASA selected the first group of astronauts in 1959. The original "Mercury seven" astronauts were all military test pilots with at least 1,500 hours of flight time. From an initial group of 508 men, 32 pilots were selected for further testing. These men were put through strenuous physiological and psychiatric examinations, reducing the number of potential candidates to 18. Finally, seven men were selected as America's first astronauts.

Each of the original astronauts flew in Project Mercury except Donald Slayton, who was grounded for medical reasons. Sixteen years later, Slayton finally got his chance to go into space when he was selected as an American crew member of the Apollo-Soyuz Test Project, the world's first international manned space flight.

Nine more astronauts were selected in September 1962. Two were civilian test pilots and seven were military test pilots. A third group of 14 military and civilian test pilots was selected in October 1963.

By 1964, the emphasis on test pilots had shifted, and NASA began looking more closely at academic qualifications. In October of 1964, over 400 applications were reviewed from individuals who met minimum physical requirements and also had a doctorate or equivalent experience in the natural

THE MERCURY SEVEN

The original seven astronauts were Air Force Captains L. Gordon Cooper, Jr., Virgil "Gus" Grissom, and Donald K. "Deke" Slayton; Marine Lieutenant Colonel John H. Glenn, Jr.; Navy Lieutenant M. Scott Carpenter; and Navy Lieutenant Commanders Walter M. Schirra, Jr. and Alan B. Shepard, Jr. These men were all test pilots with extensive experience flying high-performance aircraft at speeds greater than the speed of sound. They also met the strict height limitations NASA had imposed because of the restricted space in the capsules.

sciences, medicine, or engineering. From a pool of more than 400 applicants, six were selected in June 1965 as the first scientist-astronauts. Only one—Harrison H. Schmitt, a geologist—flew to the Moon. Three other scientist-astronauts orbited in Skylab, the space laboratory.

During the remainder of the 1960s and early 1970s, several more groups of pilot astronauts and scientist-astronauts were recruited. A fifth group of 19 more test pilots was named in April 1966, followed by a sixth group of 11 scientists in August 1967. A seventh group of seven astronauts was transferred to NASA in August 1969 from the U.S. Air Force after its Manned Orbiting Laboratory program was halted.

The concept of scientist-astronauts was expanded for the space shuttle program. A new category of astronaut was added, the **mission specialist**. The first group of astronaut candidates for the new space shuttle program was selected in January 1978. In July of 1978, the 35 candidates began a rigorous training and evaluation period at NASA's Johnson Space Center in Houston, Texas to qualify for subsequent assignment for future space shuttle flight crews. This group of 20 mission specialist astronauts and 15 pilot astronauts completed training and went from astronaut candidate status to astronaut (active status) in August 1979. Six of the 35 were women and four were minorities. Ten more groups of pilots and mission specialists have been added to the astronaut corps since 1979.

mission specialist an individual trained by NASA to perform a specific task or set of tasks onboard a spacecraft, whose duties do not include piloting the spacecraft

Since 1978, several additional groups have been selected with a mix of pilots and mission specialists. From 1959 through 2001, well over 300 people served as U.S. astronauts. There are currently over 100 astronauts actively serving in various capacities and an additional 16 astronaut candidates in training in the class of 2000 (Group 18). Over 100 astronauts have retired, resigned or been reassigned. Several former astronauts have entered politics. One former astronaut, John Glenn, served as a United States senator. Twenty-five astronauts are deceased.

Women in Space. To date, only 37 women have flown in space. This includes 29 Americans, three Russians, and five women from other nations. Some space experts considered using women astronauts from the very beginning. Women are generally smaller in stature and lighter in weight, so they would have had less difficulty fitting into the cramped Mercury capsule, which would have required less propellant mass to make orbit. The idea was considered seriously enough that initial testing was started. Ever political, the U.S. military knew that the Soviets had included women in their astronaut corps, and if the United States wanted to be the first to put a woman into space, it had to act quickly.

The result was the Mercury 13 program. Thirteen women pilots were originally selected: Myrtle "K." Cagle, Jerrie Cobb, Jan Dietrich, Marion Dietrich, Wally Funk, Jane Hart, Jean Hixson, Gene Nora Jessen, Irene Leverton, Sarah Ratley, Bernice "Bea" Steadman, Jerri Truhill, and Rhea Woltman. Cobb, Funk, and Woltman took a second series of fitness tests. Cobb and Funk went through a third stage of testing. Cobb rated in the top two percent of both men and women tested. However, NASA subsequently decided that astronauts had to be jet pilots and women, at that time, were not allowed to pilot jets, military or civilian. So the Mercury 13 program was disbanded and **cosmonaut** Valentina Tereshkova became the first

cosmonaut the term used by the Soviet Union and now used by the Russian Federation to refer to persons trained to go into space; synonomous with astronaut

The crew of space shuttle mission STS-98 illustrates the shift to increased crew specialization, with commander Kenneth Cockrell (seated, far right), pilot Mark Polansky (seated, far left), and mission specialists Robert Curbeam, Jr. (standing, far left), Marsha Ivins, and Thomas Jones (standing, far right).

woman in space in 1963. Two decades passed before Sally Ride would become the first American woman in space.

The Present

The goals of NASA for the twenty-first century include a continuing human presence in space. The International Space Station is designed and intended for long-term human occupation and may serve as a test platform for further exploration of space by humans, including trips to the Moon and Mars.

As of 2001, the crew of a space shuttle includes a commander, a pilot, and one or more mission specialists. The commander, pilot, and at least one mission specialist are selected from among NASA astronauts. The other crew position is usually a "payload specialist." Payload specialists may be cosmonauts or astronauts designated by other nations, individuals selected by a company flying a commercial payload aboard the spacecraft, or a person selected through some other formal selection process. They are scientists or engineers selected by their employer or country for their expertise in conducting a specific experiment or commercial venture on a space shuttle mission. Payload specialists are trained by NASA and are listed as members of the shuttle crew.

Pilot astronauts are trained to serve as both space shuttle commanders and pilots. The space flight commander has overall responsibility for the vehicle, crew, mission success, and safety of flight. The pilot's primary responsibility is controlling and operating the vehicle. The pilot may also assist in the deployment and retrieval of satellites using the **robot arm**.

Mission specialists have overall responsibility for the shuttle systems, crew activity planning, consumables usage, and payload operations. If the payload includes scientific experiments, these may be part of the mission specialist's responsibility or this responsibility may be shared with the payload specialist. One of the most important responsibilities of mission specialists is extravehicular activities (EVAs).

Astronaut Training. After an applicant is accepted into the astronaut program, he or she becomes an astronaut candidate. Astronaut candidates train at NASA's Johnson Space Center (JSC). While there, they attend classes on shuttle systems, and also study basic science and technology, including the mathematics of space flight, geology, meteorology, guidance systems and navigation, oceanography, orbital dynamics, astronomy, physics, and materials processing. Astronaut candidates are also trained in both land and sea survival techniques. Astronaut candidates are required to be certified on scuba gear because some of the training takes place in a large tank of water at the JSC. The tank provides neutral buoyancy, which simulates the free-fall conditions of space flight.

The first experiences with free-fall (NASA prefers the term **microgravity**) can be disconcerting and sometimes provoke extreme nausea. So astronaut candidates receive intensive training in a specially designed and equipped military KC-135 jet cargo plane (nicknamed the "Vomit Comet"). By launching itself into a parabolic arc, the aircraft can create free-fall conditions in its padded cargo bay for about 20 seconds. This maneuver can be repeated as many as 40 times during a training flight.

Pilot astronauts are also required to maintain flying proficiency by flying at least 15 hours per month. NASA has a small fleet of two-seat T-38 jets that the astronauts can use. NASA also has a fleet of four Shuttle Training Aircraft. These are Gulfstream II business jets that can be set up to perform like the Orbiter during landing.

The heart of the training program for candidate astronauts is a set of simulators and mock-ups of shuttle systems. Candidates begin with the single systems trainer where astronauts learn about the shuttle systems. Then candidates move to the more complex Shuttle Mission Simulators where they receive training in all aspects of shuttle vehicle systems and tasks. Several other simulators and mock-ups can be used to train the candidates in operating the remote arm; getting into or out of the orbiter hatches including emergency exits, launches, and landings; and working with mission control.

The current class of astronaut candidates (Group 18) includes 13 men and three women. Eleven of the class of 2000 are active military personnel. Ten of the group are training as mission specialists and the remainder as pilots. All have advanced academic degrees. Six hold the degree of Doctor of Philosophy and one is a medical doctor. Two of the civilian mission specialist candidates are certified private pilots.

SPACE SHUTTLE *CHALLENGER*

"Space flight participants" were individuals selected by a specific program such as the Space Flight Participant Program. Christa McAuliffe, selected as a Teacher in Space Participant, is the only space flight participant to be included as a member of a shuttle crew. She was listed as a payload specialist on STS 51-L, along with the commander, Mr. Francis. R. Scobee, the pilot, Commander Michael J. Smith, mission specialists Dr. Ronald E. McNair, Judith A. Resnik, and Lieutenant Colonel Ellison S. Onizuka and payload specialist Mr. Gregory B. Jarvis. The STS 51-L crew died on January 28, 1986 when the shuttle *Challenger* exploded shortly after launch. As of 2001, NASA does not have an active Space Flight Participant Program.

robot arm a sophisticated device that is standard equipment on space shuttles and on the International Space Station; used to deploy and retrieve satellites or perform other functions

microgravity the apparent weightless condition of objects in free fall

Astronaut candidates are trained extensively before spaceflight so that they can adjust to zero-gravity conditions.

The Future

The United States has partnered with Japan, Canada, Russia, and the European Space Agency to build and operate the human-tended International Space Station. Using the International Space Station as a starting point, humans may eventually return to the Moon and travel to Mars. There will be a continuing need for highly trained professional astronauts from all the nations in the partnership as well as other nations. If NASA's plans and goals come to fruition, space exploration and the astronaut corps will become more diverse and international.

NASA currently accepts applications for the Astronaut Candidate Program at any time. Classes are formed from the existing pool of applicants whenever need arises. This usually works out to be every two years. Candidates are selected for both pilot and mission specialist categories. Both civilian and military personnel are eligible for either category. However, the extensive flight experience in high-performance aircraft required of pilot candidates eliminates most civilians from consideration. Civilians may apply directly to NASA at any time. Military personnel apply through their service branch.

Mission specialists and pilot astronaut candidates must have a minimum of a bachelor's degree from an accredited institution in engineering, biological science, physical science, or mathematics. Competition is intense, however, so an applicant with only the minimum requirement is not likely to be selected. Applicants for the position of mission specialist must have, in addition to the education requirements, at least three years of professional experience. In reality, an advanced degree and several years of pro-

fessional experience are necessary to be considered for candidacy. Mission specialists must be physically fit and be able to pass a physical examination. Their height must be between 58.5 and 76 inches.

Pilot astronaut applicants must meet several additional requirements including at least 1,000 hours as pilot-in-command time in jet aircraft; flight test experience, the ability to pass a tougher physical exam, and height between 64 and 76 inches.

Increasing Diversity Among Astronauts. Sally Ride became the first American woman in space in 1983. The twenty-first century holds out the promise of greater participation by women and minorities in space exploration. On July 23, 1999, when STS-93 lifted off, Air Force Colonel Eileen Collins became the first woman to command a space mission. Currently, 25 percent of NASA astronauts are women. NASA is currently launching a program to encourage more women and minority Air Force ROTC cadets to consider astronaut training. The current corps of astronauts reflects the diversity of American culture.

Age may also no longer be the limiting factor it once was. When Alan Shepard blasted off in a Mercury rocket as the first American in space, he had everything an astronaut was supposed to have. He had perfect eyesight, he was in top physical condition, and he had many hours of experience as a test pilot. He was also 37 years old, just three years under the cut-off age for astronauts. The late-twentieth- and early twenty-first-century astronaut corps looks quite different. When Story Musgrove blasted off in 1996, he was a balding ex-marine with six academic degrees. At 61, he was also the oldest person up to that time to have ventured into space. The astronaut corps includes six astronauts who are 50 years or older. The average age is 42, two years over the cut-off age faced by Alan Shepard.

There is no single person who can be chosen to represent the current astronaut corps, although Kalpana Chawla, a mission specialist, comes close. She was born in Karnal, India, and attended college in India before moving to the United States and graduating from the University of Texas with a master of science degree in aerospace engineering. She then graduated from the University of Colorado in 1988 with a doctorate of philosophy in aerospace engineering. Chawla is a certified pilot, holding a flight instructor's certificate, in addition to a commercial pilot's license for single and multi-engine land and sea planes.

Or perhaps Michael P. Anderson (Lieutenant Colonel USAF) can represent the current astronaut corps. He was born in Plattsburgh, New York and graduated from high school in Cheney, Washington. Anderson received a bachelor of science degree in physics and astronomy from the University of Washington and a master of science degree in physics from Creighton University in 1990.

As society changes, the astronaut corps will continue to become more diverse. The astronaut corps began with seven white males, all test pilots. In the early twenty-first century, the active astronaut corps is older and more diverse, including many women and minorities. Although moderate physical fitness is still a requirement, the emphasis is on advanced technical degrees and problem-solving ability. A doctorate in mathematics would be a

good first step toward a career as an astronaut. SEE ALSO Spaceflight, History of; Spaceflight, Mathematics of; Space Exploration.

Elliot Richmond

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Astronomer

Increased computerization in telescope operations means that fewer astronomers stay up late into the night, peering through lenses.

An astronomer is a person who studies everything above Earth's atmosphere. The field of astronomy is so broad that no one person can hope to be fluent in all aspects. Most astronomers specialize in one of several branches of astronomy. Astrophysicists study the origin and evolution of stars. Planetary geologists study the formation of planets. Cosmologists study the origin, evolution, and ultimate fate of the universe.

Astronomers rarely have the opportunity to study astronomical objects directly. They must depend on various forms of electromagnetic radiation received from distant objects. However, much information can be extracted from the radiation through careful analysis: chemical composition, temperature, motion through space, rotation speed, magnetic field strength, and other physical features.

In order to measure how bright an object in space is or measure its actual size, the distance to the object must be determined accurately. For example, a star could be bright because it is intrinsically a bright star, or it could be bright because it is very close. However, distance is one of the most difficult things to measure. To determine distances in space, astronomers depend on triangles.

Astronomers also use triangles to determine the distance to nearby stars. As Earth goes around the Sun, nearby stars shift their angular position by a small amount against the background of more distant stars. This angle, called a "parallax," can be measured accurately by astronomers. It is the vertex angle of a long isosceles triangle whose base is the diameter of Earth's orbit. Thus, using simple trigonometry, astronomers can determine the height of the triangle and thus the distance to the star.

Astronomers typically work at universities that own and operate major telescopes. To work in astronomy at this level, a doctorate is required along with several years of post-doctorate work. Astronomers in training take many courses in physics, engineering, computer science, and basic astronomy.

Since working in astronomy requires an understanding of both quantum mechanics and general relativity, the mathematics requirements are difficult. Astronomers take many courses in advanced mathematics.

There are very few jobs for astronomers. Even jobs as astronomical technicians are hard to get and are usually reserved for advanced graduate students in the field. However, astronomy is one of the very few scientific disciplines where amateurs can make a significant contribution. Astronomers observe **double stars**, **variable stars**, search for **asteroids**, and search for comets. There are official channels where the information collected by amateurs can be transferred to scientists. SEE ALSO Cosmos; Solar System Geometry, History of; Solar System Geometry, Modern Understandings of; Telescope; Universe, Geometry of.

Elliot Richmond

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DO ASTRONOMERS STILL PEER THROUGH A TELESCOPE?

It is no longer necessary to stay up all night to be an astronomer. Astronomers rarely "look through" a telescope. Many large telescopes, such as the Hubble Space Telescope, can be operated by remote control. Most images are captured by light-sensitive solid state devices attached to computers.

double star a binary star; two stars orbiting a common center of gravity

variable star a star whose brightness noticeably varies over time

asteroid a small object or "minor planet" orbiting the Sun, usually in the space between Mars and Jupiter

Astronomy, Measurements in

Astronomical measurement began before recorded history. Early astronomy was concerned mainly with establishing the calendar, which was of great importance to the first agricultural societies, who needed to accurately predict when to plant and harvest crops. The earliest collection of astronomical data that has been found dates from the Babylonian Empire of the fifth century B.C.E. Astronomical records were kept on clay tablets and consisted of intricate sequences of numbers that were used to calculate the daily positions of the planets.

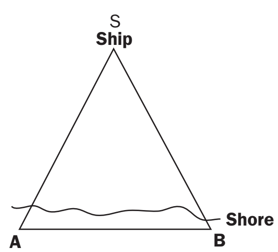
Much of what is known of early Greek mathematics was written almost a thousand years after the events occurred. A work known as the *Summary of Proclus*, written in the fifth century, refers to a lost history of geometry that was written around 350 B.C.E. by Eudemus, a pupil of Aristotle. Eudemus credited Thales with being the first **geometer**. According to this account, Thales was able to calculate the distance of a ship from the shore, although it is unclear how he determined this figure. However, the following is one possible method.

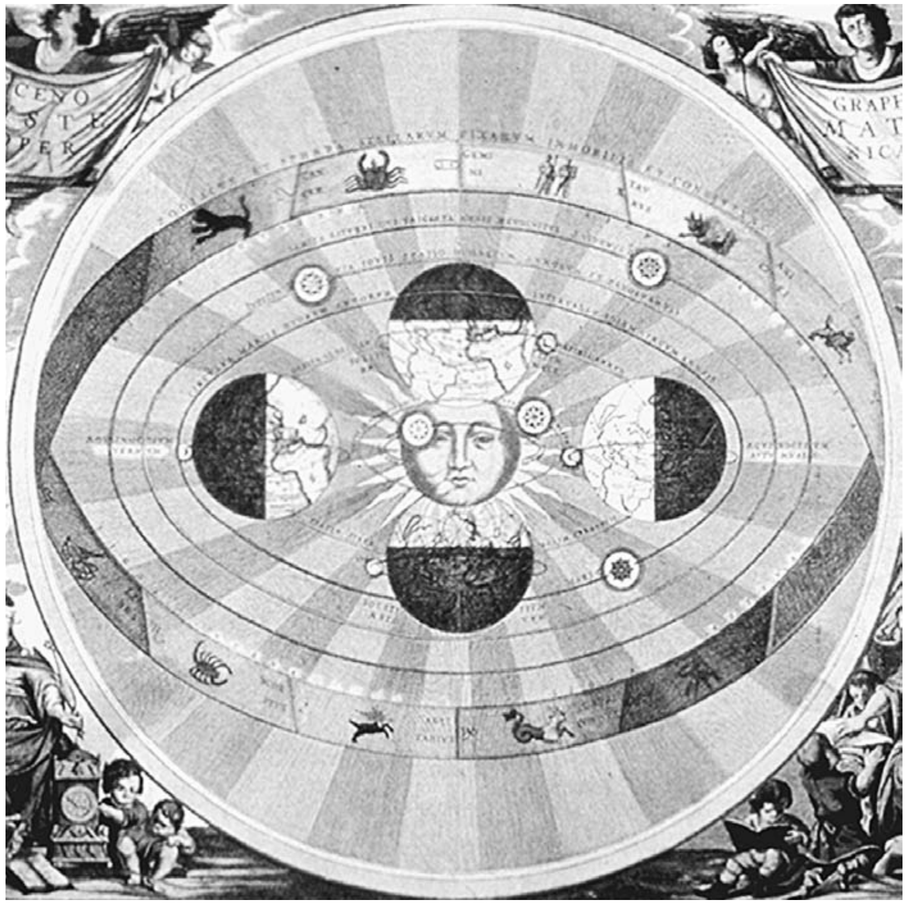
Let the ship be at a point *S* and the observers on the shore be at points *A* and *B*. They measure the angles through which they turn when first looking at each other and then at the ship, angle *ABS* and angle *BAS*. One observer now walks along the shore, counting the paces taken until reaching the other observer; this is the **baseline** of the observations. The distance to the ship can now be estimated by constructing a **scale drawing**. The longer the baseline, the more accurately the ship's distance can be estimated.

geometer a person who uses the principles of geometry to aid in making measurements

baseline the distance between two points used in parallax measurements or other triangulation techniques

scale drawing a drawing in which all of the dimensions are reduced by some constant factor so that the proportions are preserved





Copernicus's view of the Universe shows the Earth's revolutions around the Sun.

Early Attempts to Measure Astronomical Distances

A long baseline is needed to measure astronomical distances. On March 14, 190 B.C.E. there was an **eclipse** of the Sun, and Hipparchus had the angle of inclination of the Moon at the time of the eclipse measured at Istanbul and Alexandria. From this data he was able to estimate that the Moon's distance was 71 times Earth's **radius**. Given the imprecise instruments used in these measurements, this is fairly close to today's accepted value of 59 Earth radii.

eclipse occurrence when an object passes in front of another and blocks the view of the second object; most often used to refer to the phenomenon that occurs when the Moon passes in front of the Sun or when the Moon passes through Earth's shadow

radius the line segment originating at the center of a circle or sphere and terminating on the circle or sphere; also the measure of that line segment

longitude one of the imaginary great circles beginning at the poles and extending around Earth; the geographic position east or west of the prime meridian

subtend to extend past and mark off a chord or arc

circumference the distance around a circle

The challenge then was to calculate the size of Earth's radius. Earth's radius was first measured by Eratosthenes of Cyrene (276–195 B.C.E.), who read that at the summer solstice there was no shadow at midday in Syene. By observing the angle of a vertical pole at Alexandria on the same day and at the same time, Eratosthenes was able to estimate that the angle that the line of **longitude** from Alexandria to Syene **subtended** was an angle of just over 7°, which was $\frac{1}{50}$ of the circumference of Earth. Using the best available measurement for the distance between the two towns, Eratosthenes obtained a **circumference** for Earth of about 29,000 miles, as compared to our current average circumference of around 25,000 miles. Once the circumference was determined, the radius was easily calculated. Although the Greeks did not have telescopes, or other instru-

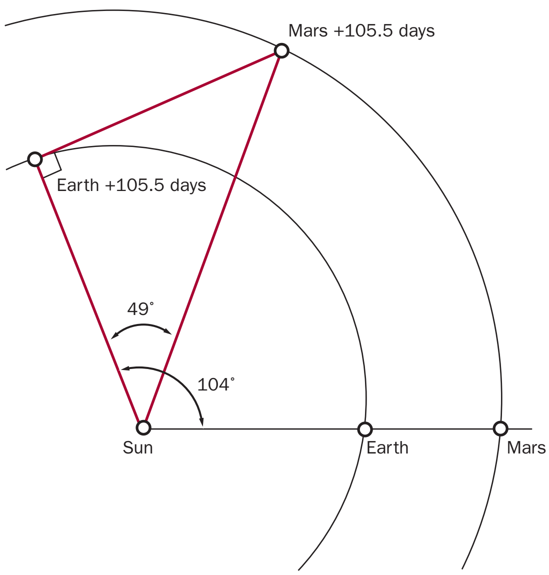
ments that would let them measure angles accurately, their measurements were fairly accurate.

Modern Astronomical Measurement

The modern period of astronomical measurement began with Nicolaus Copernicus (1473–1543), who reestablished the proposal that Earth is a **sphere**, an idea first advanced by Pythagoras in the sixth century B.C.E. He also concluded that Earth spun west to east and it was this movement that gave the effect of making the Sun, Moon, and stars rise in the east and set in the west. Copernicus placed the Sun at the center of the Universe with Earth and the planets revolving around the Sun. The stars were a fixed background far away from the Sun.

sphere the locus of points in three-dimensional space that are all equidistant from a single point called the center

Copernicus's theory was not readily accepted by the European society he was addressing. To prove Copernicus wrong, Tycho Brahe (1546–1601) undertook meticulous measurements of the planetary positions relative to the background stars, which are known as "sidereal measurements." However, one of Brahe's students, Johannes Kepler (1571–1630), examined this data and became a supporter of the Copernican theory. Kepler used Brahe's measurements of Mars to prove that Mars had an elliptical orbit around the Sun. By this time it was known that the Martian year was 687 days and that Earth's year was 365.25 days. Kepler determined that the average time for the Sun, Earth, and Mars to change from a straight-line position to a right-angled triangle was 105.5 days. During this time Earth has moved through $105.5 \div 365.25 \times 360° = 104°$, and Mars had moved through only $105.5 \div 687 \times 360° = 55°$.



Thus the right triangle contained an angle of $104° - 55° = 49°$. Because the value of the distance of Earth to the Sun was unknown, Kepler defined this measurement to be 1 astronomical unit (1 AU). Using **trigonometric ratios** in the right triangle proved that the orbit of Mars was 1.524 AU.

trigonometric ratio a ratio formed from the lengths of the sides of right triangles

The Significance of Determining Earth's Circumference. In 1617, the Dutch scientist Willebrord Snell completed a new assessment of the cir-

cumference of Earth and calculated it to be 24,024 miles. With an accurate value for the circumference of Earth, it was consequently possible to determine the distances between places on the different continents. Long baselines were now available to measure the distances to the planets, but there was also a need for accurate instruments for measuring angles. By 1672, the French Academy had telescopes whose scales were able to measure just a few seconds of **arc** (a second is $\frac{1}{3600}$th of a degree and $\frac{1}{1,296,000}$ part of a circle). An expedition was sent to Cayenne in French Guyana to measure the position of Mars at the same time the measurement was also made in Paris. Using these measurements, the distance of Mars from Earth was calculated. More importantly, the use of Kepler's planetary laws enabled astronomers to find the distance from each of the then known planets to the Sun, as well as the radius of Earth's orbit.

arc a continuous portion of a circle; the portion of a circle between two line segments originating at the center of the circle

Once the radius of Earth's orbit was known, it was possible to relate the Astronomical Unit to our everyday units of measure. Kepler had shown that planets do not rotate about the Sun in circles, but trace out **ellipses**. The Astronomical Unit is now defined as the length of the semi-major axis of Earth to the Sun, or 149,597,892 km (kilometers).

ellipse one of the conic sections, it is defined as the locus of all points such that the sum of the distances from two points called the foci is constant

Using Parallax to Measure Distances

Although there is a limit to the baseline that can be used on Earth, a journey into space is not needed to get longer baselines. If observations of distant objects are made six months apart, the **diameter** of Earth's orbit can be used as the baseline. Because the French had calculated the radius of Earth's orbit, the diameter was also known. However, another problem arises: How do you measure the angles to a distant star or planet with respect to Earth's diameter? This problem is overcome by using a concept known as **parallax**.

diameter the chord formed by an arc of one-half of a circle

parallax the apparent motion of a nearby object when viewed against the background of more distant objects due to a change in the observer's position

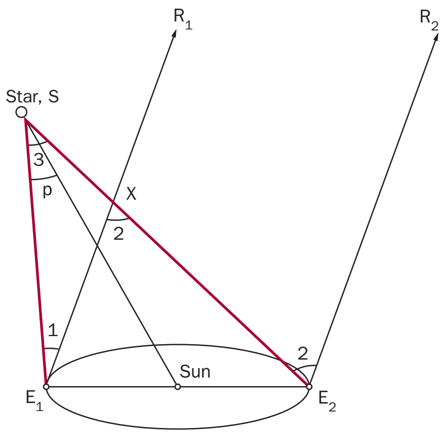
To understand parallax, hold up a finger with one arm fully extended and one eye closed. Look at your finger and also the distant background; now swap the eye that is closed. You should notice that your finger appears to move against the distant background. This is the idea behind parallax measurements.

By carefully observing the stars, it can be noted that although some are very distant, they appear to maintain the same **relative** position to each other; this is the distant background. In contrast, stars that appear closer to us seem to move during the year relative to the background stars.

relative defined in terms of or in relation to other quantities

Using the figure below, the angle between the star of interest and a reference star may be observed. This is the angle SE_1R_1. Exactly six months later, the angle between the reference star and the star of interest forms the angle SE_2R_2. Because the reference star is so distant, the lines E_1R_1 and E_2R_2 can be considered to be parallel. Therefore angle E_1XE_2 = angle SE_2R_2, and the angle E_1SE_2 ($\angle 3 + \angle 1 = \angle 2$) may be calculated. This angle is known as the parallax of a star. The measurement that astronomers use is called the "**stellar** parallax" and is half the angle E_1SE_2.

stellar having to do with stars



Because the angles involved in parallax measurements are very small their **radian** measurement can be used to determine the distance of the star from the Sun using the formula:

$$\text{Distance from the Sun to a star} = \frac{1}{p}\ \text{AU}.$$

radian an angle measure approximately equal to 57.3 degrees, it is the angle that subtends an arc of a circle equal to one radius

Light Years and Parsecs

The large numbers involved with the nearest distances of stars prompted astronomers to find alternative units of measure. Two such units are used: the light year and the parsec. The "light year" is the distance a photon of light would travel in vacuum in one year, and is a distance of 9.46×10^{12} km, or 63,200 AU. The "parsec" was first described in 1913 and is the distance that corresponds with a *par*allax of one *sec*ond of arc. The parsec is a distance of 206,265 AU, which is 3.26 light years or 3.086×10^{13} km.✶

✶Astronomers prefer to cite distances in parsecs.

Modern instruments on Earth can measure a parallax of up to one hundredth of a second—anything smaller than this is affected by Earth's atmosphere. This restricts our measurement of astronomical distances on Earth to 100 parsecs. There are only 5,000 stars out of the millions visible that are within the 100 parsec range. In 1989, the European Space Agency launched the satellite Hipparcos, which could measure angles of one thousandth of a second, a distance of 1000 parsecs. Hipparcos allowed astronomers to accurately measure the distance of many more thousands of stars.

WHO FIRST USED PARALLAX?

The first parallax measurement was completed in 1838 by F. W. Bessel, who found that the star 61 Cygni had a parallax of 0.3 seconds (1.45×10^{-6} radians), placing it at a distance of 687,500 AU. However, the errors involved with such small angles can cause great changes in distance. It is now known that 61 Cygni is almost twice that calculated distance. By 1890 the approximate distances to some 20 to 30 stars were known.

Measuring the Brightness of Stars

Around 129 B.C.E. Hipparchus introduced a scale of brightness to help distinguish one star from another. By his scale, a star of a magnitude of 6 is the faintest star the unaided eye can see, and stars that are 100 times as bright or greater have a magnitude of 1. In the early twenty-first century, scientific instruments are available that can measure the brightness of stars with a precision undreamed of by Hipparchus, who only had a personal judgment to determine relative brightness.

In 1856, Norman Pogson proposed that the difference of 5—the difference between stars of a magnitude of 1 and 6—be set at exactly 100 times magnitude. The difference in brightness between stars that differ by one mag-

nitude of brightness is thus $\sqrt[5]{100}$—that is, one is 2.152 brighter than the other. The ability to accurately measure the intensity of a star is important because a star's brightness drops off inversely to the square of the distance from a star. The brightness of a star is also dependent on its temperature, and the temperature will have an effect on the **spectrum** the star emits. If two stars with identical spectra are observed, and the distance of one of the stars through parallax measurement is known, their brightness can be compared. The variance in brightness is attributable to the difference in distance. Using the **inverse square law**, the distance of the star whose distance was previously unknown can be determined.

spectrum the range of frequencies of light emitted or absorbed by an object

inverse square law a given physical quantity varies with the distance from the source inversely as the square of the distance

Stars can give off radiation not only in the visible spectrum but also as radio waves, x-rays, and gamma rays. All of these different parts of the electromagnetic spectrum can be used in conjunction with the techniques already discussed to make astronomical measurements. SEE ALSO Distance, Measuring; Pythagoras; Solar System Geometry, History of; Solar System Geometry, Modern Understandings of; Trigonometry.

Phillip Nissen

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Athletics, Technology in

The differences in skill among athletes who compete at the highest levels of their sports are actually very small. Mere hundredths of a second are likely to separate those standing on the winner's platform from the also-rans after the Olympic 100-meter dash or 100-meter freestyle swimming competition. On any given day, the margin of victory of the world's top-ranked professional tennis player over, say, the 10th-ranked player can be just a few key points at critical moments in a handful of key games. The difference between being ranked number one and number ten is likely to be not just raw ability but the consistency with which a player exercises that ability from game to game, even point to point, during a match.

metallurgy the study of the properties of metals; the chemistry of metals and alloys

WHAT WAS IT LIKE IN THE OLD DAYS?

At the turn of the twentieth century, men dressed in long white pants whereas women wore long, bulky tennis dresses to play tennis. Today, male players wear shorts and shirts, and the women favor short, sleeveless dresses or tops with short skirts. Moreover, early tennis players used rackets made of wood, whereas graphite is now commonly used.

In this highly competitive environment, athletes look for any edge that will help them win. To achieve that end, they have enlisted the aid of science and technology. World-class bicyclists, for example, are as likely to be as knowledgeable about the latest advances in **metallurgy** as they are about their sport so that they can design aerodynamic, lightweight bicycles that will help them shave seconds off their times. They also carefully select their helmets and clothing to minimize wind resistance, again in an effort to eliminate any factor that would slow them down.

Athletes who rely on repetitive motions—the stride of a runner, the serve of a tennis player, the stroke through the water of a swimmer—use any technology that will help make those motions more consistent and eliminate wasted energy. The increasingly high-tech tools used include com-

puter simulations, advanced video cameras, wind tunnels, and sophisticated mathematical and physics models that enable athletes to break down their physical motions into their component parts in order to look for flaws or ways to improve. In a sense, a person does not have to choose between being an athlete and a scientist, for increasingly the two fields are merging in exciting ways.

Aerodynamics in Sports Technology

Although virtually any sport could be used to illustrate this new role of high technology—fencing, swimming, golf, cycling—tennis provides a good example. In the twenty-first century, world-class tennis players (and their coaches and trainers) must have a basic understanding of the laws of **aerodynamics** in order to truly understand the sport and gain an edge over competitors whose abilities match their own.

aerodynamics the study of what makes things fly; the engineering discipline specializing in aircraft design

The National Aeronautics and Space Administration (NASA) has developed a research program called "Aerodynamics and Sports Technology." The goal of the project is to examine tennis—which involves making a ball fly within the boundaries of an enclosed space—from an aerodynamic point of view, much as aircraft designers do. Researchers seek to answer basic questions. How does a tennis ball fly in the first place? How do its flight and direction change? What is its speed, and how and where does its speed change? How much does a tennis ball **spin**, and how does the spin of different types of shots differ from player to player? What happens to the ball when it hits the court? How does the ball interact with the air during its flight? The results of scientific research examining these questions provide a good example of how mathematics, physics, computer technology, and **biomechanics** can contribute to athletic performance.

spin to rotate on an axis or turn around

biomechanics the study of biological systems using engineering principles

Ball Speed, Ball Spin, and Biomechanics. One of the questions that interests sports scientists is ball speed. The usefulness of a radar gun is limited because it cannot measure changes in the ball's speed at different points in its flight. To measure ball speed, scientists had to develop photographic techniques that could freeze the movement of a ball traveling up to 120 miles an hour. Regular video does not work very well, for it records images at a speed of 30 frames per second at a relatively slow shutter speed. For this reason, television replays of tennis shots tend to be blurry, for the ball is simply moving too fast and the typical video camera is too slow.

To solve this problem, sports scientists make use of cameras that have a high-speed shutter. While the camera still records action at 30 frames per second, a high-speed shutter allows the camera to take a picture with a duration of perhaps $\frac{1}{1,000}$th of a second—or even a mere $\frac{1}{10,000}$th of a second. This shutter speed enables sports scientists to get clear pictures of the ball as it moves through the air and thus allows them to measure accurately its speed, and the change in its speed, during flight.

A second problem is measuring the ball's spin, and in particular the various types of spin, such as **topspin** and **underspin**. The difficulty with normal cameras, even those with very high shutter speeds, is that the ball can spin several times in $\frac{1}{30}$th of a second over a distance of several feet. To solve this problem, scientists enlisted a new technology called "high-speed digital recording."

topspin spin placed on a baseball, tennis ball, bowling ball, or other object so that the axis of rotation is horizontal and perpendicular to the line of flight and the top of the object is rotating in the same direction as the motion of the object

underspin spin placed on a baseball, tennis ball, bowling ball, or other object so that the axis of rotation is horizontal and perpendicular to the line of flight and the top of the object is rotating in the opposite direction from the motion of the object

At a biomechanics laboratory, a researcher watches as an athlete prepares to release a pitch with motion sensors attached to his body. The computers in the foreground record the data generated by the sensors as the athlete moves through his pitch.

Instead of recording at 30 frames per second, these new cameras can record from 250 to 2,000 frames per second. Combined with high-speed shutters, they theoretically could take 2,000 pictures per second, each with a duration of $\frac{1}{10,000}$th of a second, of a tennis ball during its flight. However, practically, investigators have found that 250 frames per second provide as much data as they need. With this type of equipment, scientists for the first time are able to actually see the spin of a ball as it moves through the air. They now know that during a serve a tennis ball can spin at a rate of up to 1,000 revolutions per minute.

Armed with high-speed digital recording cameras, however, sports scientists can also focus the camera not on the ball but on the player, thus freezing the movements of the player as he or she strokes the ball. Say, for example, that a player is having trouble with her serve. Nothing seems to be wrong with the player's toss, swing, or footwork, but for some reason she's "lost" her serve. High-speed digital recording might provide answers.

Video taken of the player's serve can be used to create sophisticated computer diagrams that show the precise path and speed of the player's racket, allowing the player and her coach to break the serve down into each tiny movement. Furthermore, multiple diagrams can be superimposed one over the other, exposing subtle changes in movement, racket speed, the angle of the racket head, where the racket strikes the ball, the position of the player's body and feet, and so forth.

With this information, players and coaches can modify a player's serve (or other stroke), look for tiny changes in the player's serve over time, examine differences between first serves and second serves (which are typically slower than first serves), and find a service technique that best fits that player's size, strength, and the strategy he or she wants to adopt depending on the unique skills of an opponent. Similar tools could be used by a fencer, baseball batter, swimmer, golfer, sprinter, or any other type of athlete.

UNLOCKING THE SECRETS TO A GREAT SERVE

One discovery made using high-speed digital video analysis was that Pete Sampras, winner of seven Wimbledon titles and known for his devastating serve, maintains a consistent racket speed throughout the course of his serve. This consistency enables him to maintain a highly fluid motion that is consistent from serve to serve.

Wind Tunnels and Computational Fluid Dynamics. Airplane designers and auto manufacturers have long studied what happens when a plane or car moves through the air, which creates resistance and drag. They learned early on that their results remained the same, regardless of whether a plane or car was moving in space or if the air was blown over a stationary plane or car in a wind tunnel. Sports scientists enlist the aid of wind tunnels to model the aerodynamics of baseballs, golf balls, and tennis balls. In particular, they want to identify what they call "transition," or the change from "laminar" (smooth air flow) to "turbulence" (rough air flow).

Wind tunnels, however, can be large and cumbersome, often as large as, or larger than, a typical room. But the action of a wind tunnel can be modeled mathematically, meaning that a "virtual" wind tunnel can be created on a computer. This type of work, based on the physics of fluids, is called computational fluid dynamics (CFD). CFD investigators begin by programming into the computer all of their available data. In the case of tennis simulations, they need to know the tennis ball's size and the configuration of its seam (typically an hourglass shape). They also program in the ball's speed and rate of spin—which they can now measure accurately using the video cameras discussed earlier. Because of the immense amount of data, a CFD simulation can sometimes take weeks to run on the computer, though tennis simulations take about half a day. With this information, the computer is able to model how the flight path of a ball changes during its flight.

These and other high-tech tools will not turn a weekend tennis player, golfer, or swimmer into an Olympic-class athlete, nor will they replace drive, dedication, and the will to win. But as the world's best athletes search for an edge, they provide invaluable information that was unavailable less than a generation ago. They also provide yet another example of the practical applications of science and mathematics. SEE ALSO Computer Animation; Computer Simulations; Sports Data.

Michael J. O'Neal

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Average

See Central Tendency, Measures of.

Babbage, Charles

British Mathematician and Inventor
1791–1871

B

Charles Babbage was born in England in 1791. He lived during the **Industrial Revolution**, and his scientific, technological, and political endeavors contributed significantly to its effects.

Industrial Revolution beginning in Great Britain around 1730, a period in the eighteenth and nineteenth centuries when nations in Europe, Asia, and the Americas moved from agrarian-based to industry-based economies

Babbage was the son of a wealthy banker and attended Cambridge University. A brilliant man, he was elected to membership in the Royal Society before receiving his master's degree in 1817. He was appointed to the Lucasian Chair of Mathematics at Cambridge in 1828, a position also held by such great scientists as Sir Isaac Newton and today's Stephen Hawking.

As an authentic **Newtonian**, Babbage advocated the reduction of all things to numerical terms and believed that they could then be understood and controlled. He was particularly attracted to the use of **statistics**.

Newtonian a person who, like Isaac Newton, thinks the universe can be understood in terms of numbers and mathematical operations

statistics the branch of mathematics that analyzes and interprets sets of numerical data

Babbage is often regarded as the "father of computing." In 1823, with financial support from the British government, he began work on what he called the Difference Engine, a steam-powered machine that would calculate mathematical tables correct to twenty decimal places. He built prototypes that produced tables of **logarithms** correct to eight decimal places but was never successful in constructing a full-size version.

logarithm the power to which a certain number called the base is to be raised to produce a particular number

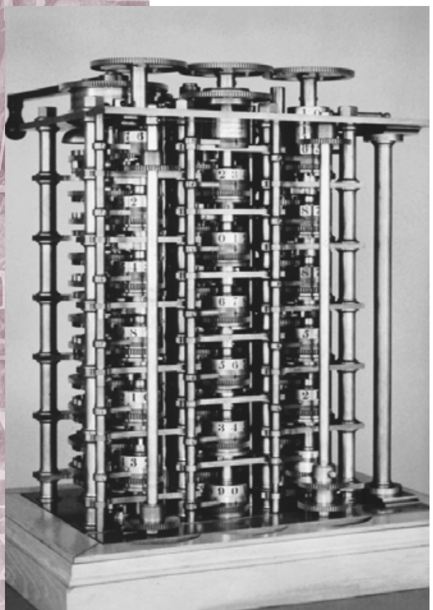
Instead, in 1833, Babbage became interested in designing and building an Analytical Engine. This device was to be a mechanical apparatus that could perform any mathematical calculation. It would be controlled by a "program" of instructions that the machine would read from punched paper cards. Although his Analytical Engine has never been constructed, Babbage's basic design was the foundation of modern digital computers.

Babbage was active in a variety of areas. Fascinated with rail travel, he performed research on railroad safety and efficiency, invented the **cowcatcher**, and promoted a standard gauge for train tracks. He established the modern postal system in Britain by developing uniform postal rates. His production of the first dependable actuarial tables of statistical life expectancies helped found the modern insurance industry.

cowcatcher a plow-shaped device attached to the front of a train to quickly remove obstacles on railroad tracks

Babbage invented, among many other devices, the **dynamometer**, better lights for lighthouses, and a speedometer. His ideas contributed to the

dynamometer a device that measures mechanical or electrical power



The full-size Difference Engine was to contain 25,000 precision-crafted metal parts and weigh over 2 tons. After 10 years and the exhaustion of government and personal finances, the partly built machine was abandoned.

growth of the machine tool industry. He also developed mathematical approaches to deciphering codes.

Concerned about the level of interest in science, Babbage published *Reflections on the Decline of Science in England* in 1830. He also helped create the British Association for the Advancement of Science, the Analytical Society, the Statistical Society, and the Royal Astronomical Society.

Babbage's book *On the Economy of Machinery and Manufactures* (1832) established the scientific study of manufacturing, known as operations research. It made an important contribution to political and social economic theory by regarding manufacturing as the primary component of economics. Quoted by Karl Marx in *Das Kapital*, its ideas were important in Marxist economic theory. His other writings included *Ninth Bridgewater Treatise* (1837), in which he attempted to harmonize his scientific and religious beliefs.

Although he was, for many years, a popular member of London society, he became ill-natured and unpopular in his old age. The honorary title of baron was offered to him, but he insisted instead on a life peerage—having all the privileges of a hereditary baron, including a seat in the House of Lords. It was never granted. He died in London in 1871. SEE ALSO MATHEMATICAL DEVICES, MECHANICAL.

J. William Moncrief

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Banneker, Benjamin

American Mathematician and Astronomer
1731–1806

Benjamin Banneker was one of the first African-American mathematicians.

Benjamin Banneker is best known for his work in mathematics and astronomy. According to W. Douglas Brown, Banneker was "the first American Negro to challenge the world by the independent power of his intellect."

A native of Baltimore County, Maryland, Benjamin Banneker was born on November 9, 1731, and spent most of his life on his father's farm located in what is now Ellicott City, Maryland. Although his father had been a slave, his mother was born free to a white English woman who came to America as an indentured servant and married a native African.

Banneker's family had sufficient means to afford schooling. The school was only open in the winter, and the pupils included a few whites and two or three black children. There Benjamin learned to read and do arithmetic to "double fractions." When he became old enough to help on his father's farm, he continued to teach himself.

In his early life, Benjamin constructed a wooden clock that was a reliable timepiece for over 20 years. It was the first striking clock of its kind

made completely in America. Benjamin quickly became known as the smartest mathematician for miles around. In 1791, Banneker was nominated by Secretary of State Thomas Jefferson and appointed by President George Washington to the commission to survey federal land for a national capital in Washington, D.C. He had an important role in the layout and design of the city, though his name does not appear on any contemporary documents.

Banneker devoted himself to the study of astronomy. In 1792, he produced his first almanac in which he recorded solar and lunar **eclipses**, tide tables, and positions of the Sun, Moon, and planets for each day of the year. The renowned work was given to Thomas Jefferson along with a letter from Banneker pleading for the rights of slaves held in the colonies. Jefferson sent the almanac to M. de Condorcet, secretary of the Academy of Sciences at Paris, praising the work. Thereafter, Banneker published yearly almanacs until his health declined in 1804.

eclipse the phenomenon that occurs when the Moon passes in front of the Sun or when the Moon passes through Earth's shadow

Benjamin Banneker died on October 9, 1806. ✶On the day of his funeral, fire consumed his house, which destroyed his laboratory.

Jacqueline Leonard

✶Benjamin Banneker's work and memory remains alive today through groups which bear his name.

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Bases

Although the number system commonly used for counting and measuring is based on the number 10 and is known as the decimal system, there are counting systems based on other numbers. For example, **base-2** and **base-60** number systems are also used for counting. Base-2, known as the **binary number** system, is used in electronic computers and other electrical devices. Time on a clock is partially measured in the base-60 system. Each hour is divided into 60 minutes and each minute is divided into 60 seconds. This entry will introduce the **base-10** and base-2 number systems.

base-2 a binary number system in which each place represents a power of 2 larger than the place to its right

base-60 a number system used by ancient Mesopotamian cultures for some calculations in which each place represents a power of 60 larger than the place to its right

binary number a base-two number; a number that uses only the binary digits 1 and 0

base-10 a number system in which each place represents a power of 10 larger than the place to its right

Base-10 Number System

Because humans have ten fingers, objects are naturally grouped in tens when counting. Counting a dozen apples with one's fingers consists of counting up to ten and then repeating the count, which results in one 10 plus two 1s. So all numbers in this base-10 system are made from just ten digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

For example, there are 365 days in a year. In the base-10 number system, the value of each of the three digits 3, 6, and 5 depends on their position. Starting from the left, the 3 stands for 3 hundreds; the 6 stands for 6 tens; the 5 stands for 5 ones, or units. So,

$$365 = (3 \times 100) + (6 \times 10) + (5 \times 1).$$

Using exponents, $100 = 10^2$, $10 = 10^1$, and $1 = 10^0$. So,

$$365 = (3 \times 10^2) + (6 \times 10^1) + (5 \times 10^0).$$

In a similar fashion, 2,030 is expressed as $(2 \times 10^3) + (0 \times 10^2) + (3 \times 10^1) + (0 \times 10^0)$.

Base-2 Number System

Whereas the base-10 number system is naturally suited to humans, base-2 is suited to computers and other devices that run on electricity. The electric current has two states—on and off. A computer is programmed to compute with groups of two using the binary number system.✶

✶The word "binary" means "comprised of two."

In base-10, a number is expressed in terms of the sum of multiples of 10: 10^0, 10^1, 10^2, and so on. But in base-2, a number is expressed in terms of the sum of multiples of 2: 2^0, 2^1, 2^2, and so on. This basically means that objects are grouped in twos. The following example shows how to express four base-10 digits in binary form.

2 (one group of two) $= (1 \times 2^1) + (0 \times 2^0) = 10$

3 (one group of two plus one) $= (1 \times 2^1) + (1 \times 2^0) = 11$

4 (two groups of two; same as one group of four) $= (1 \times 2^2) + (0 \times 2^1) + (0 \times 2^0) = 100$

5 (one group of four plus one) $= (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0) = 101$

The binary number 1011, for example, is equal to $(1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (1 \times 2^0)$, which in base-10 equals 11. So the binary (base-2) number 1011 and the decimal (base-10) number 1,011 represent totally different values. SEE ALSO Computers and the Binary System; Decimals; Powers and Exponents; Time, Measurement of.

Rafiq Ladhani

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Bernoulli Family

Swiss Mathematicians

Seventeenth and Eighteenth Centuries

The Bernoulli family was one of the world's most outstanding mathematical families. The members of the family who made the most significant contributions were two brothers, Jakob (1654–1705) and Johann (1667–1748), and Johann's son, Daniel (1700–1782).

Family History

Jakob and Johann were among the ten children of a spice merchant from Basel, Switzerland. Jakob was forced by his father to study theology but refused a church appointment when he completed his doctorate. Instead, he accepted a mathematics position at the University of Basel in 1687, a position he held for the remainder of his life.

Johann, required to study medicine, entered the University of Basel in 1683, where his brother Jakob was already a professor. The writings of Gottfried Leibniz (1646–1716), introducing the new field of **calculus**, had proved to be too difficult for most mathematicians to understand. Jakob, however, mastered their obscurity and taught Johann. The Bernoullis subsequently were among the first to recognize the great potential of calculus and to present numerous applications of this new mathematics, extending its usefulness and popularity.

calculus a method of dealing mathematically with variables that may be changing continuously with respect to each other

In 1691, Johann taught calculus to the prominent French mathematician L'Hôpital (1661–1704). Subsequently L'Hôpital published the first textbook on differential calculus, based entirely on Johann's notes but without giving him proper credit. Johann received his doctorate in medicine in 1694 and became professor of mathematics at the University of Gröningen in the Netherlands.

Problems in the Family

Both brothers were making important original contributions and had risen to the top ranks of mathematicians, but they became jealous of each other. Jakob could not accept Johann, whom he had taught, as an equal, and Johann was unable to be professionally gracious. They attacked each other's work publicly. When Jakob died in 1705, Johann returned to Basel to become professor of mathematics, a position he held until his death in 1748.

Jakob Bernoulli was first to use the term "integral" in calculus. He and Johann introduced the calculus of variation. He solved the equation now known as **Bernoulli's Equation** and applied the methods of calculus to the problems of bridge design.

Bernoulli's Equation a first order, nonlinear differential equation with many applications in fluid dynamics

Jakob's highly significant work, *Ars Conjectandi*, was published in 1713, after his death. It included the first thorough treatment of probability, a discussion of Bernoulli's law of large numbers, the theory of permutations and combinations, a derivation of the exponential series using the Bernoulli numbers, and a discussion of mathematical and moral predictability.

Jealousy and competition kept Johann Bernoulli (shown here) at odds with his brother, Jakob, and his own son, Daniel. But collectively, the Bernoullis made major scientific contributions through their work on topics such as differential equations and hydrodynamics.

Johann Bernoulli developed the theory of differential equations and discovered the Bernoulli series. His influential texts on integral calculus and differential calculus were published in the early 1740s. He applied the methods of calculus to numerous practical problems, including a number of aspects of navigation, optics, and kinetic energy. In addition to his influence on L'Hôpital, he taught the noted mathematician Leonhard Euler (1707–1783).

Continuing the Legacy

Johann's son Daniel was born in 1700. Forced by his father to study medicine, he received a medical degree in 1721. After publishing a discussion of

differential equations and the properties of flowing liquids in 1724, he accepted a position at the Academy of Sciences in St. Petersburg, Russia, where he taught medicine and physics. He returned to Basel in 1732 to teach anatomy and botany.

hydrodynamics the study of the behavior of moving fluids

kinetic theory of gases the idea that all gases are composed of widely separated particles (atoms and molecules) that exert only small forces on each other and that are in constant motion

In 1738, Daniel published *Hydrodynamica*, a thorough treatment of the properties of flowing fluids and the relationships among them. As a result, he is regarded as the founder of the science of **hydrodynamics**. He also treated the behavior of gases using mathematical probability, introducing the concepts that led to the **kinetic theory of gases**. He developed the mathematics and physics of vibrating strings and made numerous other contributions in a variety of fields. He accepted positions at Basel in physiology in 1743 and physics in 1750. He died in 1782.

A rather extraordinary incident occurred in 1735 when Johann and Daniel were awarded prizes by the Paris Academy of Sciences for separate work on planetary orbits. Johann became enraged at having to share the recognition with his son and banned him from his house. In 1738, when Daniel published *Hydrodynamica*, Johann published *Hydraulica*, which obviously plagiarized Daniel's work, apparently out of jealousy of his son's increasing reputation.

Other members of the Bernoulli family who were significant mathematicians included Johann's other two sons, Nikolaus (1695–1726), who died soon after accepting a position at the St. Petersburg Academy, and Johann II (1710–1790), who succeeded his father as professor of mathematics at Basel; Johann's nephew, Nicolaus (1687–1759), professor of mathematics at Padua; and Johann II's son, Johann III (1744–1807), who became a professor of mathematics at the Berlin Academy at the age of 19. Other grandsons and great grandsons made lesser contributions to mathematics. SEE ALSO CALCULUS; EULER, LEONHARD.

J. William Moncrief

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digital logic rules of logic as applied to systems that can exist in only discrete states (usually two)

machine code the set of instructions used to direct the internal operation of a computer or other information-processing system

binary existing in only two states, such as "off" or "on," "one" or "zero"

transistor an electronic device consisting of two different kinds of semiconductor material, which can be used as a switch or amplifier

Boole, George

British Mathematician
1815–1864

Digital logic is the set of rules applied to data (numbers, symbols, words) that are entered into a computer. These instructions direct computers and are called **machine code**. This code uses the **binary** digits 1 and 0 to switch **transistors** on or off. The basic operators that perform these tasks are AND, OR, and NOT. A hundred years before electronic computers were conceived, George Boole proposed this binary or digital logic.

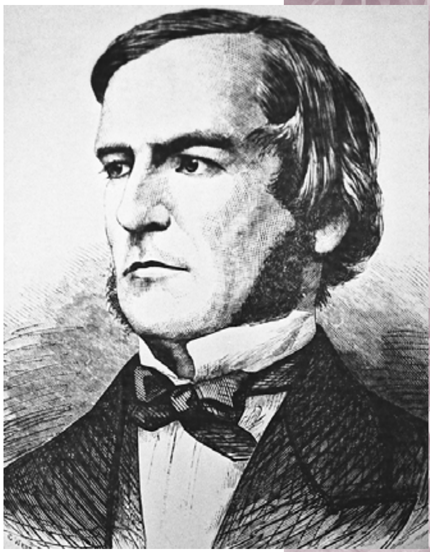
Born to working-class English parents, Boole later supported himself and his parents as a school instructor. Frustrated by the inferior mathematics texts

used to educate pupils, Boole was led to change the world of numbers. With only a basic science background, Boole labored 5 years to learn mathematics, eventually publishing in the *Cambridge Mathematical Journal.* His reputation was heightened in 1847 when he published *The Mathematical Analysis of Logic*, which introduced Boole's ideas on the two-valued (presence "1" or absence "0") system of algebra that represented logical operations.

For centuries, philosophers have studied logic. However, George Boole argued that logic could be taught with mathematics rather than with philosophy. Boole envisioned mathematical symbols, rather than words, as the truest representation of logic.

Even though he had not studied at a university, Boole was appointed in 1849 as mathematics professor at Queens (University) College in Ireland. Eventually, Boole was appointed to the position of mathematics chairman at Queens, where he gained a reputation as an outstanding teacher.

In 1854, Boole published *An Investigation into the Laws of Thought*, in which he combined algebra with logic. This concept became known as **Boolean algebra**. Because George Boole demonstrated that logic could be reduced to very simple algebraic systems, it was possible for Charles Babbage and his successors to design mechanical devices that could perform the necessary logical tasks. Today, Boolean algebra is the logic that computers use to perform everything they do.



George Boole helped establish the modern study of logic. His Boolean algebra is used to design computer circuits.

Boolean algebra a logic system developed by George Boole that deals with the theorems of undefined symbols and axioms concerning those symbols

Boolean Algebra

Boolean algebra is a branch of mathematics in which statements, ideas, numbers, and so forth are denoted by symbols (e.g., x, y, z) and can be acted upon by operators (e.g., AND or OR). The operator AND between two symbols (x AND y) is known as the union of x and y and refers to objects both in x and y. The operator OR represents the intersection of x and y (x OR y) and consists of objects either in x, in y, or in both x and y.

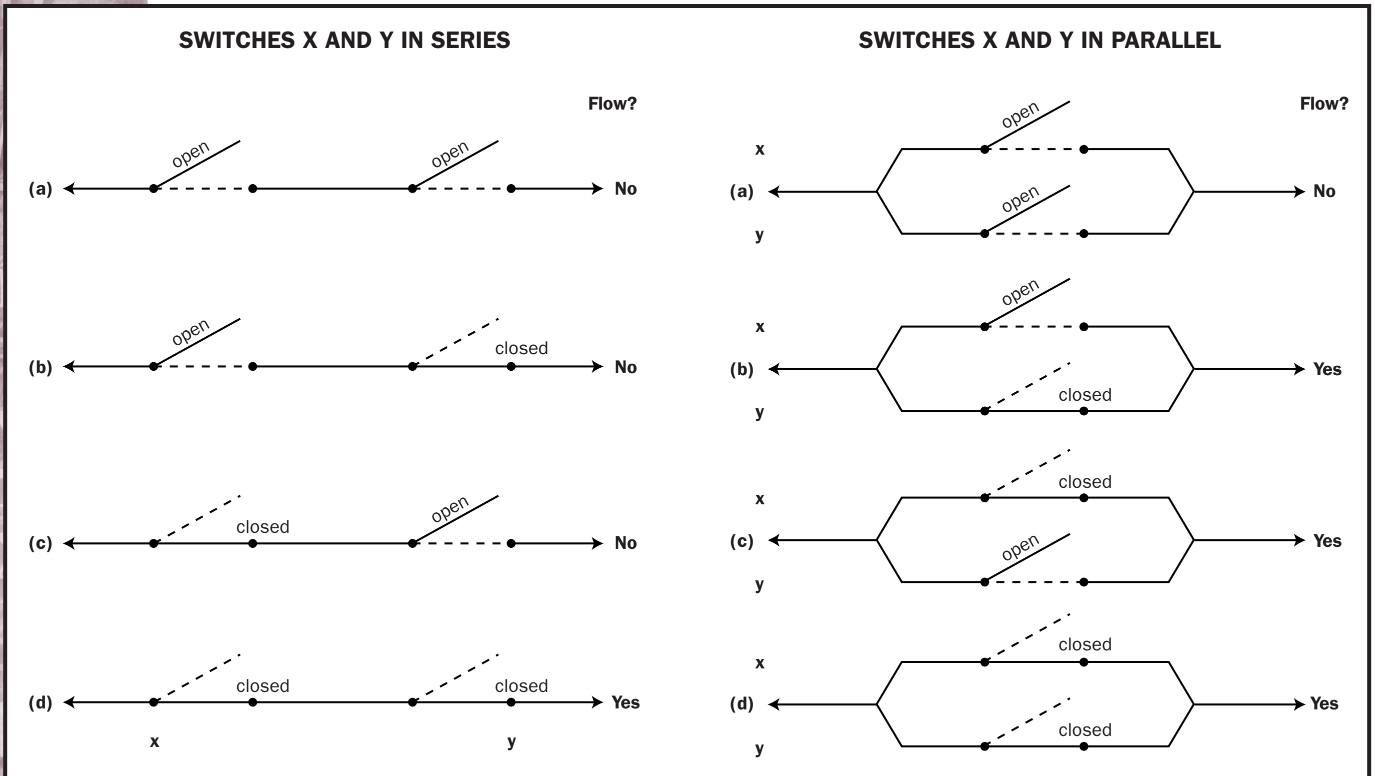
As an example of applying Boolean algebra in computers, let x and y denote two electronic circuits that are either closed (electricity flows) or open (electricity does not flow). The statement x AND y is represented by connecting the switches in series. (See series diagram.) The current will flow only if both x and y are closed. Similarly, a circuit with switches connected in parallel is represented by the statement x OR y. (See parallel diagram.) The current will flow if either x is closed, y is closed, or both x and y are closed.

Internet search engines use Boolean algebra in the form of AND and OR. For instance, searching "African OR American" produces articles that contain either or both these words, a result that is broad in scope. Searching for "African AND American" produces articles that contain both words, a result that is more narrow in scope.

Boole's Influence on Computers

Boole was a great mathematician, but he never forgot that mathematics came into existence to search for practical scientific solutions. Boolean algebra is the basis for subjects such as **information theory**, computer science, electrical-circuit theory, and artificial-intelligence research.

information theory the science that deals with how to separate information from noise in a signal or how to trace the flow of information through a complex system



Example of Boolean algebra as seen in electronic circuits.

Seventy-three years after Boole's death, an MIT student recognized the connection between electronic circuits and Boolean algebra. The student transferred two logic states to electronic circuits by assigning different voltage levels to each state. This connection was the step necessary to take Boole's theory to the practical design of computers. As a result, Boole is considered one of the founding fathers of computers and information technology. SEE ALSO Babbage, Charles; Computers, Evolution of Electronic; Mathematical Devices, Early; Mathematical Devices, Mechanical.

William Arthur Atkins (with Philip Edward Koth)

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Bouncing Ball, Measurement of a

geometric sequence a sequence of numbers in which each number is larger than the previous by some constant ratio

exponential an expression in which the variable appears as an exponent

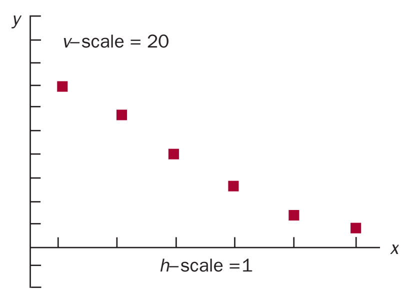
When a ball bounces, different mathematical models can describe what happens. If the ball bounces in place several times, a **geometric sequence** or **exponential** model describes the maximum height that the ball attains in

relation to the number of bounces. For any single bounce, a **quadratic** model describes the height of the ball at any point in time.

quadratic involving at least one term raised to the second power

Exponential Model: Maximum Height

To examine the maximum height a bouncing ball attains, ignore external factors such as air resistance. A ball bounced in place recovers a certain percentage of its original height. For example, suppose a ball that recovers 70 percent of its height is dropped from 200 feet. The maximum height it reaches after its first bounce is 70 percent of 200 feet, or 140 feet. After the second bounce, it reaches a height of 70 percent of 140 feet, or 98 feet. In similar fashion, the ball continues to rebound to a height that is 70 percent of the highest point of the previous bounce. The graph below illustrates these maximum heights.



Because each successive maximum height is found by multiplying the previous height by the same value, this is a geometric sequence. The maximum height can also be expressed as an exponential function, with the **domain** restricted to **whole numbers**. For this example, the maximum heights attained are shown on the next page, with values rounded to the nearest tenth of a foot.

domain the set of all values of a variable used in a function

whole numbers the positive integers and zero

Infinite Bouncing. Because the height of each successive bounce continues to be 70 percent of a positive number, the ball's bounce, in theory, will never reach a zero height. In practice, however, the ball loses energy and does eventually stop bouncing.

It is possible to calculate the total distance the ball travels in this theoretical world of endless bounces. Initially, the ball travels 200 feet. It then bounces up 140 feet and falls 140 feet, a total of 280 feet. It then bounces up 98 feet and falls 98 feet, a total of 196 feet. This pattern continues.

After the initial 200 feet, an infinite **geometric series** appears: $280 + 196 + 137.2 + 96.04 + \ldots$. Summing this series with the formula $S = \frac{a_1}{1 - r}$ results in $\frac{280}{1 - 0.7} = 933\frac{1}{3}$ feet. Adding the initial 200 feet, the total distance that the ball travels is $1133\frac{1}{3}$ feet.

geometric series a series in which each number is larger than the previous by some constant ratio; the sum of a geometric sequence

Quadratic Model: Height of Single Bounce

To consider the height of the ball at any given point in time, again assume a recovery of 70 percent, an initial height of 200 feet, and no air resistance. Using a formula from physics that relates an object's height to the length of time it has been falling, $h = 16t^2$ (16 is half the rate of acceleration due to gravity in feet per second squared), it can be determined how long it takes

REALITY CHECK

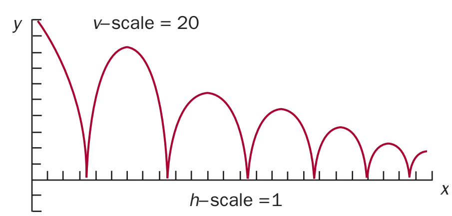
Both the geometric sequence (exponential model) and parabolic model discussed in this entry are theoretical. An interesting activity is to conduct an experiment, collect the data, and compare the experimental data with these theoretical models.

to fall from each height. Note that it takes the same amount of time to reach a height as it takes to fall from it.

BOUNCE	MAXIMUM HT (ft)	TM UP (SEC)	TM DOWN (SEC)	TM BETWEEN BOUNCES
0	200	0	3.54	3.54
1	200 x 0.7 = 140	2.96	2.96	5.92
2	200 x 0.7^2 = 98	2.47	2.47	4.95
3	200 x 0.7^3 = 68.6	2.07	2.07	4.14
4	200 x $0.7^4 \approx$ 48.0	1.73	1.73	3.46
5	200 x $0.7^5 \approx$ 33.6	1.45	1.45	2.90
x	200 x 0.7^x	$\sqrt{12.5 \times 0.7^x}$	$\sqrt{12.5 \times 0.7^x}$	$2\sqrt{12.5 \times 0.7^x}$

This relationship is a series of parabolas of decreasing height (because the maximum height decreases) and decreasing width (because the time between bounces decreases). Algebraically, if h represents the height and t represents the time, the first parabola can be expressed as $h = 200 - 16t^2$, where $0 \leq t < 3.54$.

Between the first and second bounces, the height can be expressed as $h = 140 - 16(t - 6.49)^2$, $3.54 \leq t < 9.45$. The 6.49 is when the highest point is achieved (halfway between the two bounces), and the 9.45 seconds is derived from adding 3.54 seconds to the time between the first and second bounces. It is at this time that the ball bounces again. One could continue deriving these equations in similar fashion. The figure below shows this series of parabolic arcs.



A Forward-Moving Bouncing Ball. Now assume that a ball, held at 200 feet, is thrown horizontally at the rate of 1 foot per second. The ball accelerates downward, acted upon by gravity, as it continues to travel horizontally, maintaining its horizontal rate of 1 foot per second. Because it accelerates downward at the same rate as a ball bounced in place, the heights of the two balls are identical at any point in time. Consequently, the graphs of their heights in relation to time are identical.

But consider graphs that would show the paths traveled by the balls. The path of the in-place ball is contained within a vertical line. The path of the thrown ball, however, forms parabolic **arcs**: as it accelerates downward, it continues to travel at a constant speed in a horizontal direction. In this example, given the horizontal component of 1 foot per second, and supposing that the units on the horizontal axis are labeled in feet instead of seconds, the graph of the thrown ball's path exactly matches the graph of its

arc a continuous portion of a circle; the portion of a circle between two line segments originating at the center of the circle

height over time. But if the horizontal speed is other than 1, the **scaling** must be adjusted. SEE ALSO Exponential Growth and Decay; Quadratic Formula and Equations.

Bob Horton

scaling the process of reducing or increasing a drawing or some physical process so that proper proportions are retained between the parts

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Brain, Human

The human brain has been compared to many things in an attempt to understand how it works. For instance, similarities have been pointed out between the connections in the brain and a large telephone switchboard. Since then, frequent parallels have been drawn between the brain and modern digital computers. As the twenty-first century begins, this comparison may be particularly appropriate because computers are being used in many applications that until recently were largely thought to be beyond the capability of machines and the exclusive province of the human brain.

The Computer Analogy

Computers are now being used in tasks from playing chess at a grand master level to visually recognizing what is being "seen" through video cameras in specific applications. The human brain's comparison to a computer is the best analogy thus far, although it is important to note that there are important differences.

Digital computers are built from **logic circuits** or "logic gates" that produce a predictable output based on the inputs. The inputs are generally one of two selected voltages, which are generally thought of as the binary values 0 (zero) and 1 (one). Gates may produce a logical 1 output if all of the inputs are 1 (an AND function), if any of the inputs are 1 (an OR function), or if the inputs can be inverted, producing a 1 only if the input is 0 (a NOT function).

logic circuits circuits used to perform logical operations and containing one or more logic elements: devices that maintain a state based on previous input to determine current and future output

From these simple functions more complex operations can be built by interconnecting logic gates, including circuits capable of adding numbers and memory devices capable of storing information. A digital computer may be built from millions of individual gates for its central processor and may use hundreds of millions or even thousands of millions of gates for its memory.

In the brain, there are special nerve cells known as neurons that function in ways that are strikingly similar to the logic gates in digital computers. Neurons possess structures for transferring signals that are lacking in most cells: a major fiber called the axon✶ and a number of smaller fiber branches called dendrites. A neuron gets input from the dendrites, as well as from the central cell body, and subsequently produces an output signal in the axon. This signal then provides inputs to other neurons, or to other cells such as muscle cells.

✶Whereas other types of cells are microscopic, the axon of the neuron may extend a meter or more in length and deliver its signal to hundreds or thousands of other cells.

Neurons are similar to computer logic gates because the output of an axon is a series of pulses of on and off signals, and never is a partial signal.

The inputs to the neuron determine if these pulses in the axon occur, and if so, at what speed they are repeated. The signal down an axon is an electrical signal, although the connection of signals between the axon and the dendrites of other neurons, known as the **synapse**, is achieved by a reaction of chemicals known as neurotransmitters.

synapse the narrow gap between the terminal of one neuron and the dendrites of the next

Unlike logic gates, which are thoroughly understood, the exact operation of the neuron is still not completely clear. Science has probes that can measure the activity in a single neuron, but research is still ongoing to determine exactly how the neuron responds to potentially thousands of simultaneous inputs. It is known that while some signals to the dendrites help cause a neuron to **fire**, others inhibit the firing process. But not all inputs contribute equally—some input signals to a neuron seem to have a stronger contribution than others. Repeated operation of a synapse may change its dendrite's contribution to the output of the cell, and this appears to be a major part of the memory process.

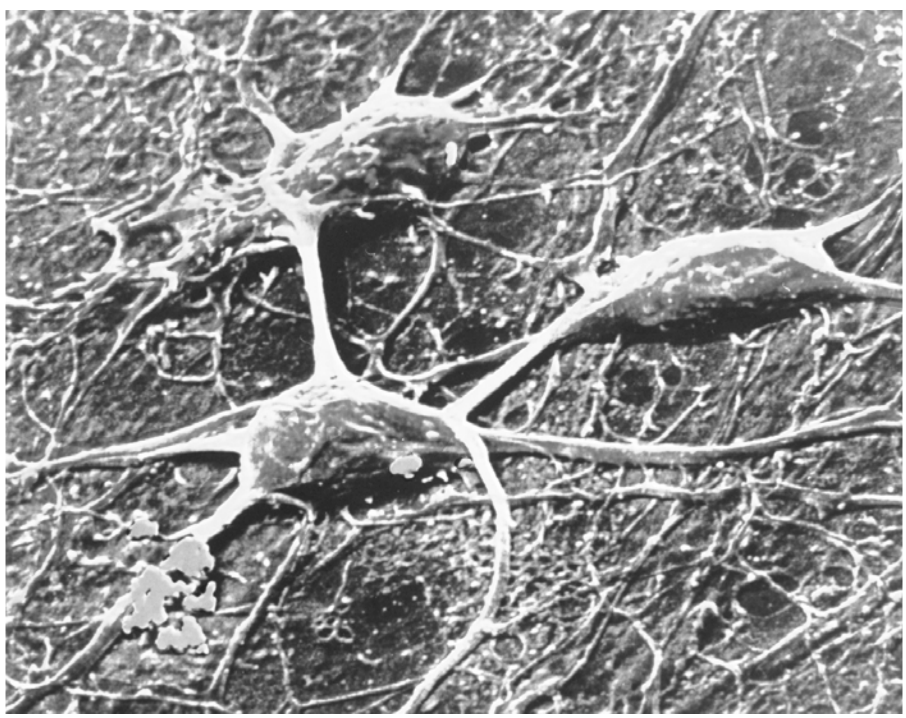
fire the reaction of a neuron when excited by the reception of a neurotransmitter

Speed and Processing

Whereas a computer operates at very high speeds, making perhaps 100 million or more sequential operations in one second, the chemical transfer of information at the synapse is relatively slow, occurring in about one-thousandth of a second. A computer, however, must process information through one or a small number of central processors and execute its function or program in a sequence of steps. In contrast, the brain processes information in a parallel fashion, with large numbers of neurons analyzing signals and producing outputs at once.

Modern supercomputers are just beginning to conduct **parallel processing**, with perhaps as many as a thousand central processors. It is not yet

parallel processing using at least two different computers or working at least two different central processing units in the same computer at the same time or "in parallel" to solve problems or to perform calculation



Three nerve cells and their dendrites are shown in this scanning electron micrograph. Information pulses through these cells and their branching fibers and is communicated to the other cells in the human body.

understood how to break many problems down into parallel operations, which the brain does automatically.

Digital computers operate by sequentially following a series of instructions in the computer memory, also known as a program, but the brain has no close equivalent to a written program. Its operation is based on linked networks of neurons that make up all of our mental processes, such as memory, cognition, emotions, intelligence, and personality.

Although it is frequently said that the average person uses only about 10 percent of their brain, it is hard to support this statement scientifically. The brain, by its very nature, is redundant. A failure of a single logic gate may render a digital computer nonfunctional, but the loss of one, or tens, or even of hundreds of brain cells may have little or no noticeable effect on the human brain. Whereas a small number of people show some remarkable traits, such as a photographic memory, it may be that most people record just as much in their memory, but their ability to access the information is weaker, requiring more stimulus before the brain recovers the memory.

Whereas a great deal is understood about digital computers, far less is understood about the human brain. There is still much to be learned about how the neurons of the brain are arranged and interconnect, and how the inputs from our senses are processed. Rapid progress is being made in this area, but it is certain to be an important field of science full of new discoveries for years to come. SEE ALSO Computers and the Binary System; Computers, Evolution of Electronic; Computers, Future.

Harry J. Kuhman

HOW MANY NEURONS AND SYNAPSES?

Research in the early twenty-first century indicates that there are about 10^{11} (one hundred billion) neurons in the human brain, plus or minus a factor of 10. The number of interconnections, or synapses, is much harder to determine, but current estimates are at least 10^{14} to 10^{15} interconnections.

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Bush, Vannevar

Electrical Engineer
1890–1974

Vannevar Bush is credited with the invention of some of the most advanced computers of his time.

Vannevar Bush is best known for mobilizing U.S. scientific research during World War II. In 1913, he completed his bachelor's and master's degrees in mathematics at Tufts College in Massachusetts, and in 1916, Harvard University and the Massachusetts Institute of Technology (MIT) jointly awarded him a doctorate in electrical engineering. After teaching at Tufts, he joined the electrical engineering department at MIT in 1919, where he would eventually be appointed dean in 1932.

At the time, electrical engineers were primarily concerned with the technical problems associated with delivering power over long distances. Bush, however, foresaw that in time the profession's role would be to develop increasingly sophisticated electrical devices for the home and for industry. In

1922, he was one of the founders of what became the Raytheon Corporation, a maker of electronics parts, and by the end of his career he held forty-nine electronics patents. His most famous inventions involved electromechanical computing devices, including analog computers.

By 1931, his most successful device, the Differential Analyzer, was up and running. It used a complicated system of cams and gears driven by steel shafts to generate practical solutions to complex physics and engineering problems. With the help of the Rockefeller Foundation, by 1935, he had developed the Rockefeller Differential Analyzer, the most powerful computer available until the arrival of early electronic computers in the mid-1940s. In a 1945 article in the *Atlantic Monthly*, he proposed a device that he called the Memex, an indexed machine for cross-referencing and retrieving information that foreshadowed the development of **hypertext** and the World Wide Web.

hypertext the text that contains hyperlinks, that is, links to other places in the same document or other documents or multimedia files

During World War II, Bush was the nation's driving force behind government and military sponsorship and funding of massive science projects, which until then had been funded primarily by industry and private foundations. In 1940, he was appointed chair of the National Defense Research Committee, formed to organize scientific research of interest to the military. A year later he took the helm of the Office of Scientific Research and Development (OSRD). There he used his academic, industrial, and government contacts to organize and fund American scientists and engineers in the war against the Axis powers. Because of this work, he is often referred to as the architect of the "military-industrial complex."

Under Bush, the OSRD played a lead role in two major projects. One was to enlist universities and industry in the development of microwave-based radar systems to replace the inferior long-wave radar systems developed by the Navy in the 1930s. The other was the Manhattan Project, which developed the atomic bomb. He thus set in motion the scientific efforts that culminated in the destruction of Hiroshima and Nagasaki in 1945 at the end of World War II.

By 1949, Bush was becoming disillusioned with the military-industrial complex he helped create. That year he published *Modern Arms and Free Men*, a widely read book in which he warned against the danger of the militarization of science. He died in Belmont, Massachusetts, on June 28, 1974. SEE ALSO Mathematical Devices, Mechanical.

Michael J. O'Neal

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Calculators

Throughout human history man has invented devices to make repetitive tasks, such as mathematical calculation, easier. For much of history, this was difficult. The advent of methods, which progressed to machines, to help with this dilemma made life much easier for many.

Early Calculators

The first aids in counting were notched sticks and counting cords, simple tools used to retain numerical information over time and distance. These devices did not perform any mathematical operations.

A major revolution came with the abacus. This device allowed both arithmetical operations and the ability to store a fixed sum, which was represented by a given pattern of beads. The abacus, which is still in use today, was fast and extremely useful in the hands of a trained user.

The French mathematician Blaise Pascal is credited with inventing the mechanical calculator in 1642, based on a stylus and wheel design. In 1820, based on Pascal's design, Thomas de Colmar invented the "Arithmometer." In his design, de Colmar took Pascal's wheels and turned them on their sides. They were made wider, and tally marks were put on the outside of the drum.

The first adding machine that printed results on tape was introduced around 1872. When first introduced, many of the adding machines were hand-driven, but later models had electric motors. One of the more impressive of the early adding machines was the Burroughs Class 1/Model 9. It was 19 inches deep, over a foot tall, and weighed more than 63 pounds.

In 1911, the Monroe LN was introduced. Previously adding machines had been separate from those that performed multiplication and division. The Monroe LN combined these two machines into one.

As the world entered the 1950s, new progress in electronic circuitry and data processing techniques began to pave the way for reasonably sized electronic calculators. At this time, electronic adding machines were common, but most designs were based on the technology of the mechanical calculator, with electricity used to power the drum or pinwheel movement. The calculators of the 1960s were desktop sized, utilized thousands of separate electronic components, and were costly to assemble.

In addition to being more convenient because of their smaller size, the pocket calculators of today are able to perform a much wider range of functions than adding machines of the past.

Electronic Calculators

The first mass-produced electronic calculators often had retail prices of nearly one thousand dollars. The high cost was, in part, because the technology was so new, and materials were at a premium.

A breakthrough came in 1964 when Victor Business Machines contracted with General Microelectronics to develop an integrated circuit-based desktop electronic calculator. Following this, Texas Instruments began to produce an integrated circuit-based, hand-held calculator called the "Cal-Tech." It performed the four basic arithmetical operations.

Modern Calculators

The calculators of today have come a long way from where they started. Today, calculators can perform a range of complicated functions as well as basic ones. Commonly, people use calculators to ease the burden of performing routine mathematical calculations, such as balancing checkbooks. Hand-held calculators now allow people to do calculations anywhere, with little risk of error in the results. SEE ALSO Abacus; Computers, Evolution of Electronic.

Brook E. Hall

Internet Resources

Before HP Calculators. The Museum of HP Calculators. June 2001. <http://www.hpmuseum.org>.

Calculus

As students first begin to study calculus in high school or college, many may be unsure about what calculus is. What are the fundamental concepts that underlie calculus? Who has been credited for the discovery of calculus and how is calculus used today?

What is Calculus?

Calculus was invented as a tool for solving problems. Prior to the development of calculus, there were a variety of different problems that could not be addressed using the mathematics that was available. For example, scientists did not know how to measure the speed of an object when that speed was changing over time. Also, a more effective method was desired for finding the area of a region that did not have straight edges. **Geometry**, **algebra**, and **trigonometry**, which were well understood, did not provide the necessary tools to adequately address these problems.

geometry the branch of mathematics that deals with the properties and relationships of points, lines, angles, surfaces, planes, and solids

algebra the branch of mathematics that deals with variables or unknowns representing the arithmetic numbers

trigonometry the branch of mathematics that studies triangles and trigonometric functions

velocity distance traveled per unit of time in a specific direction

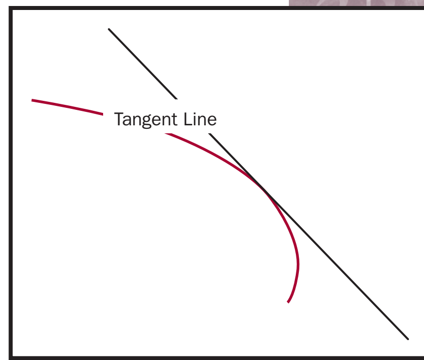
At the time in which calculus was developed, automobiles had not been invented. However, automobiles are an example of how calculus may be used to describe motion. When the driver pushes on the accelerator of a car, the speed of that car increases. The rate at which the car is moving, or the **velocity**, increases with respect to time. When the driver steps on the brakes, the speed of the car decreases. The velocity decreases with respect to time.

As a driver continues to press on the accelerator of a car, the velocity of that car continues to increase. "Acceleration" is a concept that is used to

describe how velocity changes over time. Velocity and acceleration are measured using a fundamental concept of calculus that is called the **derivative**.

Derivatives can be used to describe the motion of many different objects. For example, derivatives have been used to describe the orbits of the planets and the descent of space shuttles. Derivatives are also used in a variety of different fields. Electrical engineers use derivatives to describe the change in current within an electric circuit. Economists use derivatives to describe the profits and losses of a given business.

The concept of a derivative is also useful for finding a **tangent** line to a given curve at a specific point. A tangent line is a straight line that touches a curve at only one point when restricted to a very small region. An example of a tangent line to a curve is shown in the figure. The straight line and the curve touch at only one point. The straight line is the tangent line to the curve at that point.



Example of a tangent line to a curve.

Tangent lines are useful tools for understanding the angle at which light passes through a lens. Derivatives and tangent lines were useful tools in the development and continued improvement of the telescope. Derivatives are also used today by optometrists or eye doctors to develop more effective methods for correcting vision. Physicists use tangent lines to describe the direction in which an object is traveling, and chemists use tangent lines to predict the outcomes of chemical reactions. These are only a few examples of the many uses of tangent lines in science, engineering, and medicine.

Derivatives along with the concept of a tangent line can be used to find the maximum or minimum value for a given situation. For example, a business person may wish to determine how to maximize profit and minimize expense. Astronomers also use derivatives and the concept of a tangent line to find the maximum or minimum distance of Earth from the Sun.

The derivative is closely related to another important concept in calculus, the **integral**. The integral, much like the derivative, has many applications. For example, physicists use the integral to describe the compression of a spring. Engineers use the integral to find the "center of mass" or the point at which an object balances. Mathematicians use the integral to find the areas of surfaces, the lengths of curves, and the volumes of solids.

The basic concepts that underlie the integral can be described using two other mathematical concepts that are important to the study of calculus—"area" and "limit." Many students know that finding the area of a rectangle requires multiplying the base of the rectangle by the height of the rectangle. Finding the area of a shape that does not have all straight edges is more difficult.

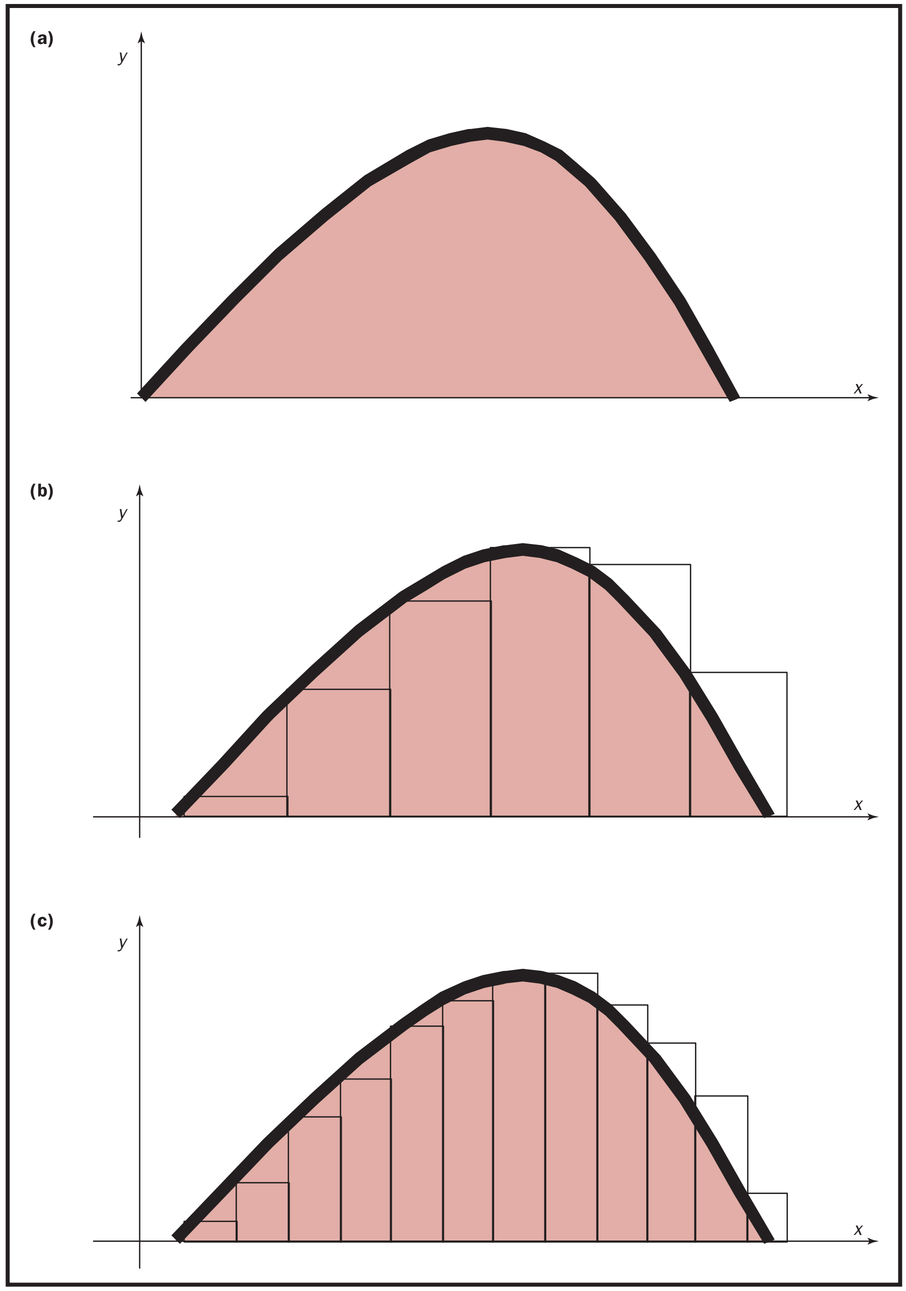
The area between the curve and the x-axis is colored in (a) of the figure on the following page. One way to estimate the area of the portion of the figure that is colored is to divide the region into rectangles as is shown in (b). Some of the rectangles contain less area than is contained in the colored region. Some of the rectangles contain more area than is contained in the colored region. To estimate the area of the colored region, the area of the six rectangles can be added together.

If a better estimate is desired, the colored region can be divided into more rectangles with smaller bases, as shown in (c). The areas of these

derivative the derivative of a function is the limit of the ratio of the change in the function; the change is produced by a small variation in the variable as the change in the variable is allowed to approach zero; an inverse operation to calculating an integral

tangent a line that intersects a curve at one and only one point in a local region

integral a mathematical operation similar to summation; the area between the curve of a function, the x-axis, and two bounds such as $x = a$ and $x = b$; an inverse operation to finding the derivative



The colored region in these three graphs represents the area between the curve and the *x*-axis. Summing the areas of smaller rectangles (c) provides a better estimate of the area than (b).

rectangles can then be added together to acquire a better approximation to the area of the colored region.

If an even better estimate of the colored region is desired, it can be divided into even more rectangles with smaller bases. This process of dividing the colored region into smaller and smaller rectangles can be continued. Eventually, the bases of the rectangles are so small that the lengths of these bases are getting close to zero.

The concept of allowing the bases of the rectangles to approach zero is based on the limit concept. The integral is a mathematically defined function that uses the limit concept to find the *exact* area beneath a curve by dividing the region into successively smaller rectangles and adding the areas of these rectangles. By extending the process described here to the study of

three-dimensional objects, it becomes clear that the integral is also a useful tool for determining the volume of a three-dimensional object that does not have all straight edges.

An interesting relationship in calculus is that the derivative and the integral are inverse processes. Much like subtraction reverses addition, differentiation (finding the derivative) reverses integration. The reverse of this statement, integration reverses differentiation, is also true. This relationship between derivatives and integrals is referred to as the "Fundamental Theorem of Calculus." The Fundamental Theorem of Calculus allows integrals to be used in motion problems and derivatives to be used in area problems.

Who Invented Calculus?

Pinpointing who invented calculus is a difficult task. The current content that comprises calculus has been the result of the efforts of numerous scientists. These scientists have come from a variety of different scientific backgrounds and represent many nations and both genders. History, however, typically recognizes the contributions of two scientists as having laid the foundations for modern calculus: Gottfried Wilhelm Leibniz (1646–1716) and Sir Isaac Newton (1642–1727).

Leibniz was born in Leipzig, Germany, and had a Ph.D. in law from the University of Altdorf. He had no formal training in mathematics. Leibniz taught himself mathematics by reading papers and journals. Newton was born in Woolsthorpe, England. He received his master's degree in mathematics from the University of Cambridge.

The question of who invented calculus was debated throughout Leibniz's and Newton's lives. Most scientists on the continent of Europe credited Leibniz as the inventor of calculus, whereas most scientists in England credited Newton as the inventor of calculus. History suggests that both of these men independently discovered the Fundamental Theorem of Calculus, which describes the relationship between derivatives and integrals.

The contributions of Leibniz and Newton have often been separated based on their area of concentration. Leibniz was primarily interested in examining methods for finding the area beneath a curve and extending these methods to the examination of volumes. This led him to detailed investigations of the integral concept.

Leibniz is also credited for creating a notation for the integral, $\int$. The integral symbol looks like an elongated "S." Because finding the area under a curve requires "summing" rectangles, Leibniz used the integral sign to indicate the summing process. Leibniz is also credited for developing a notation for finding a derivative. This notation is of the form $\frac{dy}{dx}$. Both of these symbols are still used in calculus today.

Newton was interested in the study of "fluxions." Fluxions refers to methods that are used to describe how things change over time. As discussed earlier, the motion of an object often changes over time and can be described using derivatives. Today, the study of fluxions is referred to as the study of calculus. Newton is also credited with finding many different applications of calculus to the physical world.

WHO WAS FIRST?

Gottfried Wilhelm Leibniz began investigating calculus 10 years after Sir Isaac Newton and may not have been aware of Newton's efforts. Yet Leibniz published his results 20 years before Newton published. So although Newton discovered many of the concepts in calculus before Leibniz, Leibniz was the first to make his own work public.

It is important to note that the ideas of Leibniz and Newton had built upon the ideas of many other scientists, including Kepler, Galileo, Cavalieri, Fermat, Descartes, Torricelli, Barrow, Gregory, and Huygens. Also, calculus continued to advance due to the efforts of the scientists who followed. These individuals included the Bernoulli brothers, L'Hôpital, Euler, Lagrange, Cauchy, Cantor, and Peano. In fact, the current formulation of the limit concept is credited to Louis Cauchy. Cauchy's definition of the limit concept appeared in a textbook in 1821, almost 100 years after the deaths of Leibniz and Newton.

Who Uses Calculus Today?

Calculus is used in a broad range of fields for a variety of purposes. Advancements have been made and continue to be made in the fields of medicine, research, and education that are supported by the methods of calculus. Everyone experiences the benefits of calculus in their daily lives. These benefits include the availability of television, the radio, the telephone, and the World Wide Web.

Calculus is also used in the design and construction of houses, buildings, bridges, and computers. A background in calculus is required in a number of different careers, including physics, chemistry, engineering, computer science, education, and business. Because calculus is important to so many different fields, it is considered to be an important subject of study for both high school and college students.

It is unlikely the Leibniz or Newton could have predicted the broad impact that their discovery would eventually have on the world around them. SEE ALSO Limit; Measurements, Irregular.

Barbara M. Moskal

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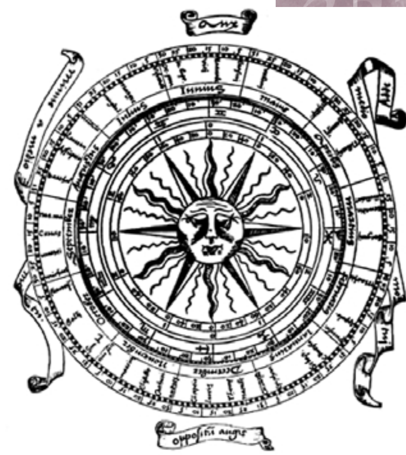
Calendar, Numbers in the

Calendars have always been based on the Sun, Earth, and Moon. Ancient people observed that the position of the Sun in the sky changed with the seasons. They also noticed that the stars seemed to change position in the night sky throughout the year. They saw that the Moon went through phases where it appeared to change its shape over the course of about 30 days. All of these astronomical phenomena happened regularly. As a result, people began to organize their lives around these periodic occurrences, thereby forming the basis for calendars.

Through the centuries, calendars have changed because of astronomy, politics, and religion. Over time, astronomers learned more about the precise movements of the Sun, Earth, and Moon. Politicians would often change the calendar for their own personal gain. In ancient Rome, officials would sometimes add days to the calendar to prolong their terms in office. Church officials also had specific ideas about when religious holidays should occur. Eventually, calendars evolved into the most widely used calendar today, the Gregorian calendar. This is a calendar based on the amount of time it takes Earth to travel around the Sun ($365\frac{1}{4}$ days). However, a year cannot be divided evenly into months, weeks, or days. So even with all of the technology available in the twenty-first century, the Gregorian calendar is still not a perfect calendar.



This undated calendar with Latin text tracks Venus as seen by Earth throughout the year. It depicts an unhappy Sun in the center with rings representing the Zodiac, months, dates, and Venus' phases.

Ancient Calendars

Some of the first calendars came from ancient Babylonia, Greece, and Egypt. Ancient Egypt and Greece date back to at least 3000 B.C.E. and Babylonia existed from the eighteenth through the sixth century B.C.E. The first written record of the division of a year, however, may have come from as far back as 3500 B.C.E. in Sumeria, which later became Babylonia.

The ancient Babylonians had a lunar calendar based on the phases of the Moon. Each month was 30 days long, the approximate time between full moons, and there were 12 months in the year, thus yielding a 360-day year (12 months $\times$ 30 days/month). However, the actual time between full moons is closer to 29.5 days, so a lunar year is 354 days (12 months $\times$ 29.5 days/month). But after only a few years, this 6-day difference would cause the seasons to occur in the wrong calendar months. To correct the calendar, the Babylonians added days or months to it when necessary. Calendars that make periodic adjustments such as this are called "lunisolar" calendars.

The ancient Greeks also used a lunisolar calendar, but with only 354 days, the same as an actual lunar year. The Greeks were also the first to adjust their calendar on a scientific basis rather than just adding days when it became necessary.

Around 3000 B.C.E, the Egyptians became the first culture to begin basing their calendar on a solar year, like the current Gregorian calendar. Although Egyptian calendars were 365 days long, astronomers eventually learned that the actual solar year had an extra one-quarter day. But an attempt to apply this knowledge to the calendar did not occur until 238 B.C.E., when Ptolemy III, King of Egypt, decreed that an extra day should be added to every fourth year to make up for the extra one-quarter day each year. However, his decree was largely ignored and the calendar did not contain a "leap year" until 46 B.C.E., when Julius Caesar, Emperor of Rome, created the Julian Calendar.

Caesar and the Calendar

In Rome, the Roman calendar was used from the seventh century B.C.E. to 46 B.C.E. It originally had 10 months and 304 days. The year began with March and ended with December. Later, in the seventh century, Januarius and Februarius were added to the end of the year.

By 46 B.C.E., the calendar had become so out-of-step with the actual seasons that Caesar decided to make 46 B.C.E. last for 445 days to make the calendar consistent with the seasons once more. He then fixed the year at 365 days with every fourth year being a leap year with 366 days. His new calendar was called the Julian calendar. In the Gregorian calendar today, the extra day in a leap year falls at the end of February because February was the last month of the year in the Roman calendar.

Many of the names of the months in the modern Gregorian calendar come from the Roman calendar. The fifth through tenth months of the calendar were named according to their order in the calendar—*Quintilis*, *Sextilis*, September, October, November, and December. For example, Quintilis means "fifth month" in Latin. Two of the months in the Julian calendar were later renamed—Quintilis became July, after Julius Caesar, and Sextilis became August, named after Caesar's successor, Augustus.

When Julius Caesar changed the calendar to 365 days, he did not spread the days evenly over the months as one might expect. Therefore, February continues to have 28 days (29 in a leap year), while all other months are 30 or 31 days.

The Gregorian Calendar

The Gregorian calendar was created in 1582 by Pope Gregory XIII to replace the Julian calendar. The Julian year was 11 minutes and 14 seconds longer than a solar year. Over time, it too, no longer matched the seasons and religious holidays were occurring in the wrong season. In 1582, the **vernal equinox**, which occurred on March 21 in 325 C.E., occurred 10 days early. Pope Gregory XIII consequently dropped 10 days from the calendar to make it occur again on March 21. In addition, he decided that every century year (1500, 1600, etc.) that was evenly divisible by 400 should be a leap year. All other century years remained common years (365 days).

vernal equinox the moment when the Sun crosses the celestial equator marking the first day of spring; occurs around March 22 for the northern hemisphere and September 21 for the southern hemisphere

The Gregorian calendar was slowly accepted by many European and western countries. As different countries adopted the calendar, it went through some interesting changes. For example, Britain* did not accept the calendar until 1752. Previously, the British began a new year on March 25. In order to begin 1752 on January 1, the period January 1 through March 24 of 1751 became January 1 through March 25 of 1752, and 1751 lasted only from March to December.

***When Britain switched from the Julian to the Gregorian calendar in 1752, the loss of 11 days caused by the date adjustment spurred riots in the streets.**

Other Calendars

In addition to the Gregorian calendar, many religious calendars exist. For example, the Islamic calendar is a lunar calendar of 354 days. Leap years occur every 2 or 3 years in a 30-day cycle. Many mathematical equations exist for converting dates from religious calendars to the Gregorian calendar.

Many people have proposed changing to a more standard calendar in which the months would have an equal number of days and calendar dates would fall on the same days of the week from year to year. But changing the calendar faces stiff opposition because national and religious holidays would have to be changed. SEE ALSO Time, Measurement of.

Kelly J. Martinson

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Carpenter

Whether constructing houses or building furniture or cabinets, carpenters spend a good portion of their time as mathematicians, particularly **geometers**. Before starting work on a project, they have to be able to calculate the correct amount of materials they need. Once work begins, they have to measure materials accurately, and calculate lengths, areas, angles, etc., to create a finished product.

Carpenters use a variety of mathematical tools, such as a right-angle ruler, which may seem unusual to those outside of the field.

geometer a person who uses the principles of geometry to aid in making measurements

Sometimes the mathematics that carpenters use is relatively easy. Using simple arithmetic, a carpenter can, for example, calculate the number of two-by-four studs needed in a wall of a given length when the studs are 16 inches apart, being sure to include the extra two-by-fours needed around doors and windows and at the top and bottom of the wall.

Sometimes, though, the mathematics of carpentry is more complicated. A carpenter building a staircase, for example, is faced with the difficult problem of making sure that each step is the same width, that the rise of each step is the same, and that the stairway fits into the space available without being too steep or too shallow. Similarly, in building a roof, a carpenter has to calculate the slope of the roof accurately, and then cut materials to make sure they conform to the slope and fit precisely.

Fortunately, carpenters have tools to help with these types of mathematical problems. One is a carpenter's square, which is a right-angle ruler with calibrations that measures angles. The other is a construction calculator, which is programmed to solve construction problems and gives measurements in eighths and sixteenths of inches rather than in decimals. SEE ALSO Geometry, Tools of.

Michael J. O'Neal

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Carroll, Lewis

British Mathematician, Writer, and Photographer
1832–1898

Lewis Carroll is the pen name of Charles Lutwidge Dodgson, who was born in Darebury, England, in 1832 and died in Guildford, England, in 1898. He taught mathematics at Christ Church College of Oxford University for most

Lewis Carroll produced mathematical works and logic puzzles but is better known for his tale of whimsy and satire, *Alice's Adventures in Wonderland.*

of his life and wrote a number of mathematics texts. His fame, however, rests in being the author of children's stories and poems, including *Alice's Adventures in Wonderland* (1865) and *Through the Looking Glass* (1872).

Dodgson's father was an Anglican minister who had excelled in mathematics at Christ Church College. As a child, Dodgson invented games and stories to entertain his ten brothers and sisters. He attended Richmond School and Rugby School before entering Christ Church College in 1851. He did particularly well in mathematics and classics and, after graduating in 1854 with first honors in mathematics, immediately became an instructor in mathematics at Christ Church, remaining in that position until 1881.

Dodgson was the author of a number of mathematics articles and books, including *Notes on the First Two Books of Euclid* (1860); *Euclid and His Modern Rivals* (1879); *A Syllabus of Plane Algebraic Geometry* (1860); *Curiosa Mathematica*, Part I (1888) and Part II (1894); and *Symbolic Logic*, Part I (1896) and Part II (unpublished until 1977).

Dodgson, or Carroll, is best remembered for the children's books that resulted from his efforts to entertain the children of the Dean of Christ Church, Henry George Liddell. One of Liddell's daughters, Alice, is immortalized as the heroine of one of the most popular children's books ever to be written.

J. William Moncrief

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Cartographer

A cartographer makes maps from information gathered during a survey. The mapping of an area begins by creating a network of points and measuring the distances and angles between them. The next step is to map all the details of the land, such as rivers and roads, between the accurately fixed points in the network. After measuring a baseline distance between two points, the cartographer measures the angles between the two points at the end of the baseline, and then measures a third point with electronic instruments that record how long it takes light or radio waves to travel between two points. The three points form a triangle, which allows the cartographer to calculate the length of the other two sides in a process called **triangulation**.

triangulation the process of determining the distance to an object by measuring the length of the base and two angles of a triangle

In addition to measuring the details of the land, cartographers also measure the heights of many points in the area that they are mapping. From a large number of these points, they can draw in the contours that show the relief of the land.

All of these techniques require knowledge of linear algebra, geometry, and trigonometry. Some of the linear algebra elements that are needed

include knowledge of **determinants**, eigenvalues and **eigenvectors**, **quadratic forms**, and **generalized inverses**. Knowledge of geometry is necessary for measuring different shapes and sizes in the field, and then plotting and drawing those objects. The use of trigonometry is also necessary, including the **law of cosines** for sides and for angles. SEE ALSO Angles of Elevation and Depression; Global Positioning System; Maps and Mapmaking; Trigonometry.

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Census

One of the oldest and most widespread use of counting and mathematics is the census.✶ The census is the official count of the people in a geographic area. The Constitution of the United States calls for a census to be taken every 10 years. Originally, the purpose of the census was to provide information for the periodical **reapportionment** of the House of Representatives. Today, census data is used for many reasons, including establishing where and how federal and state monies should be distributed.

The U.S. Constitution requires its citizens to be "enumerated" or physically counted. But the precise meaning of the word "enumerate" has been debated in recent years. The beginning of the twenty-first century still sees census officials and researchers recommending different methods for estimating the number of people living in an area rather than counting. Debate continues around estimation methods because they can give different results than traditional counting methods.

Problems with Undercounting

Under the original Constitution, Native Americans were not counted in a census. African Americans were counted, with 92 percent listed as slaves, and each slave counting as three-fifths of a man. Other groups, such as Asians, were not included at all at this point. Over the years, different groups gained recognition and were included in the census. But even in the twentieth century, some groups continued to be undercounted, such as those living in dense urban areas. With multiple family units often living at one address, the chance of finding an address for all individual families is less than for single-family residences.

Although counting methods today recognize all ethnicities and socioeconomic classes, concerns about undercounting are still an issue. After the 1990 census, at least three cases went before the United States Supreme Court regarding problems or disagreements with either the manner in which the census was completed or its results. In *Wisconsin v. City of New York* (1996), the charge was that the census had undercounted a population of New York by not including some members of certain minority groups. This alleged undercount benefited the State of Wisconsin by increasing the number of representatives from Wisconsin. The U.S. Secretary of Commerce

determinant a square matrix with a single numerical value determined by a unique set of mathematical operations performed on the entries

eigenvector if there exists a vector space such that a linear transformation onto itself produces a new vector equal to a scalar times the original vector, then that vector is called an eigenvector

quadratic form the form of a function written so that the independent variable is raised to the second power

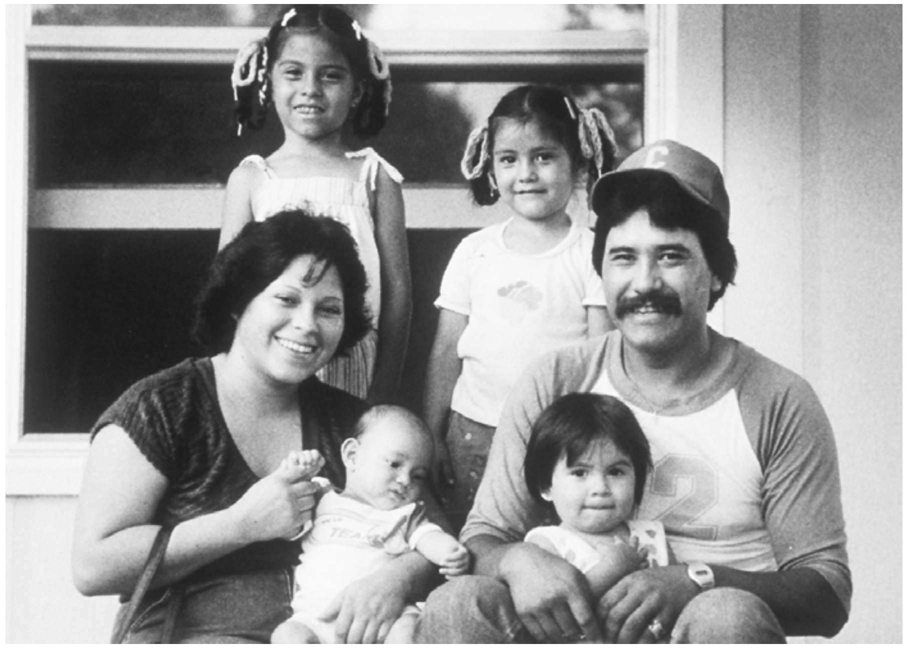
generalized inverse an extension of the concept of the inverse of a matrix to include matrices that are not square

law of cosines for a triangle with angles A, B, C and sides a, b, c, $a^2 = b^2 + c^2 - 2bc \cos A$

✶**The census has its origins in the Roman Empire, but the modern system has its roots in seventeenth-century Sweden.**

reapportionment the process of redistributing the seats of the U.S. House of Representatives, based on each state's proportion of the national population

The potential undercounting of certain minorities by the U.S. Census Bureau may lead to underrepresentation of minorities in Congress and unintended consequences such as a lack of federal funding for areas having large minority populations.



refused to make any statistical change to the data: this decision was upheld by the Supreme Court.

In two other cases brought to court in 1996, states were required to reapportion the congressional districts in their state. In doing so, districts highly irregular in shape were created in order to given voting strength to a minority group. The Supreme Court ruled these redistricting plans violated the equal protection clause of the United States Constitution.

The Debate over Estimation

Around 1770, a science known as statistics was defined. Statistics is a branch of mathematics dealing with the collection, analysis, interpretation and presentation of masses of numerical data. Statistics were so new in the late 1700s that the authors of the United States Constitution did not have any confidence in this "new" science, and did not use it for census purposes. By the end of the 1800s, however, statistical methods and knowledge increased, and the concept of estimating a population by sampling a small part of it was no longer a strange thought.

Today, mathematicians believe that using sampling and estimation methods will reduce undercounts. Some ideas presented at a 1999 conference included basing estimates on water demand, birth records, death records, IRS tax returns, master address files, and census blocks, all of which have been utilized with varying degrees of success. The National Academy of Sciences recommended using an estimation method combined with traditional enumeration to fulfill federal law requirements and increase the accuracy of the census.

But a political debate centers on how the census count will affect the congressional districts. By using traditional enumeration methods, the likelihood of minorities being undercounted is greater. Therefore, individuals

who were elected by majority constituents desire the census to remain the same. In contrast, individuals who would benefit from a larger participation of minorities in elections prefer the census to be conducted statistically which would therefore increase estimates of minorities.

The Next Census

The 2000 census, with all of its faults, was the most detailed in U.S. history. It included race self-identification questions as well as ethnicity identification. Although this is more information than ever requested before, some minority groups allege that the census remains incomplete. Some of the problems encountered with the 2000 census included non-English-speaking citizens who could not read the form, the inability to include more than one race for couples who are biracial, and the lack of a permanent address at which to be contacted. Additionally, some believe the Census Bureau does not have the right to ask detailed personal questions, such as income or race.

Every tenth year during census time, many of these same questions resurface. Why does the Census Bureau need to know such information? Why does the U.S. Code not allow mathematicians to take advantage of statistics to simplify and make the count more accurate? These questions will surely be addressed again before the 2010 census occurs. SEE ALSO Babbage, Charles; Data Collection and Interpretation; Statistical Analysis.

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Central Tendency, Measures of

Any set of data consisting of numbers has important measurements that can be used to describe the entire set. One of these measurements is a single number that is considered the most representative of the set. Such a number is called "a measure of central tendency." Three such measures are commonly used, and each has its own advantages. These measures are known as the mean, median, and mode.

For numerical data, the "mean" (sometimes called the arithmetic average or arithmetic mean because there are other measures in mathematics also called the mean) is the sum of the values in a data set divided by the total number of values. The "median" of a set of data is the middle piece of the data after the values have been sorted from smallest to largest (or largest to smallest). For a set of data with an odd number of values, the median is

THREE MEASURES OF CENTRAL TENDENCY

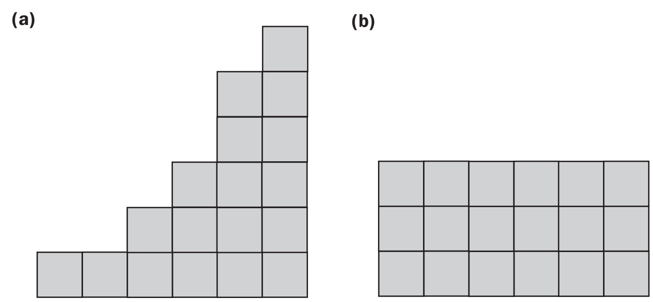
The three common measures of central tendency of a data set are as follows:

- Mean—the sum of the values divided by the number of values in the set;
- Median—the middle of the data when values are sorted from smallest to largest (or largest to smallest); and
- Mode—the value that occurs most often.

actually the middle data point. If a set of data has an even number of values, two numbers are considered to be in the middle of the data, and the median is halfway between these two middle data points. The "mode" is the value in the data set that appears most often. A set of data can have no mode, one mode, or two or more modes.

A Visual Example

Imagine that the following are data points in a set: 1, 1, 2, 3, 5, and 6. Now suppose that each number represents the number of hours six people exercise in a week. The goal is to select a single number that represents the central tendency for these values—in this case, the "average" number of hours of exercise in a week for this group of individuals. Next, visualize each of the values as a stack of cubes with a height equal to the value. For example, 1 would be represented by a single cube, 2 by a stack of two cubes, and so on. (See part (a) of the figure below.) What are the measures of central tendency for these data?



To measure the mean, think of the stacks as "sharing" cubes across the stacks so that they all have the same height. In this case, each of the stacks would be 3 cubes tall, and therefore the mean of these data is 3. In other words, $1 + 1 + 2 + 3 + 5 + 6 = 18$, and $18 \div 6 = 3$. (See part (b) of the figure.)

To visualize the median—the middle value in a ranked data distribution—the cubes are arranged in order from the shortest to the tallest. The median is the height of the middle stack of cubes. Or in this case, with an even number of stacks, the arithmetic average (that is, the arithmetic mean) of the two middle stacks is the median. The two middle stacks have 2 and 3 cubes each, and the mean, or halfway point, of 2 and 3 is $2\frac{1}{2}$ cubes $[(2 + 3) \div 2 = 5 \div 2 = 2\frac{1}{2}]$.

To determine the mode—the most frequently occurring value in a data set—examine the distribution and look for stacks with equal height. In part (a), the first two stacks on the left are 1 cube tall. Since no other stacks have the same number of cubes, 1 is the mode for this data set.

The Impact of Outliers on the Mean

While measures of central tendency are useful in their ability to represent a large amount of information, they should always be interpreted carefully. The mean is the measure of central tendency that is most frequently cited

in popular usage of **statistics**, but it can be misleading without an understanding of how it was derived.

statistics the branch of mathematics that analyzes and interprets sets of numerical data

Imagine being told that the average salary for an employee at a ten-employee company is $75,000. This appears to be a large amount of money for the typical employee. However, suppose that this average included the high salaries of the president and chief executive officer. Management makes significantly more than the eight staff-level employees at this particular company. By including the two much higher values in the calculation of the mean salary, these two divergent values, known as **outliers**, will influence, or **skew** the mean toward a higher value, thereby suggesting that most staff-level employees at the company make more money than they actually do. In fact, none of the eight staff members in this example makes more than $40,000. Thus, the mean may not be the measure of central tendency most representative of a set of data.

outliers extreme values in a data set

skew to cause lack of symmetry in the shape of a frequency distribution

Unlike the median and mode, which are much less influenced by extremely high or extremely low values in a set of data, the mean is easily influenced by such outliers. A divergence in a relatively small number of observations in a data set can change the mean value of the data set considerably. For instance, if the salaries of the president and chief executive officer in the previous example were removed, the mean salary for the other eight employees would decrease to $32,000.

The person who is calculating a measure of central tendency must therefore understand exactly what type of measure is desired; otherwise, the results and their interpretations can be misleading. SEE ALSO Data Collection and Interpretation; Statistical Analysis.

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Ceramicist

"Ceramicist" is actually a term for two different but related fields of work. One definition is a person who deals with the most common find of an ancient archeological site: pottery. With these items, called potsherds, a ceramicist can define the timetable and the different communities of the site.

This type of ceramicist uses simple geometry to measure variables such as the height and length of the pottery they find. They also use scale drawings to record each site's dimensions and statistics for checking analyses. **Probability** and **correlation** are used in sampling to clarify hypotheses.

probability the likelihood an event will occur when compared to other possible outcomes

correlation the process of establishing a mutual or reciprocal relation between two things or sets of things

Ceramicists make detailed records of the different types of whole pots and broken pieces they find. This is known as a corpus and includes details

Throwing a vase on a potter's wheel requires a ceramicist to estimate suitable dimensions throughout the throwing process.

of the size, shape, design, and decoration. From this record, it is possible to construct a picture of the pot.

Another type of ceramicist is a person who works with clay and glazes to make functional and decorative pieces of pottery. These ceramicists use geometric mathematics skills and calipers to measure the accuracy of their pieces. Knowledge of ratios is necessary for mixing the ingredients for the clay, enamel, and paints. A ceramicist will also use financial mathematics if they sell their work. SEE ALSO Archaeologist; Artists.

Marilyn Schwader

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Chaos

The word "chaos" is used in mathematics to mean something other than what it means in everyday English. In English, we speak of chaos as a state in which everything has gone awry, there is complete disorder, and there are no rules governing this behavior. In mathematics, chaotic systems are well defined and follow strict mathematical rules. Although chaotic systems are unpredictable, they do have certain patterns and structure, and they can be mathematically modeled, often by an equation or a system of equations.

Chaos theory is the study of systems that change over time and are inherently unpredictable. Some systems, such as Earth's orbit about the Sun, are very predictable over long periods of time. Other systems, such as the weather, are notoriously unpredictable. Chaos theory builds a mathematical framework to account for unpredictable systems, like the weather, which behave in a seemingly random fashion.

The last two decades of the twentieth century saw tremendous advances in our understanding of chaotic systems. Chaos theory has been applied✶ to the study of **fibrillation** in human hearts, the control of epileptic seizures, the analysis of how the brain processes information, the dynamics of weather, fluctuations in the stock market, the formation of galaxies, the flow of traffic during rush hour, and the stability of structures during earthquakes, to name a few. Wherever there are unstable or unpredictable systems, chaos scientists can be found attempting to reveal the underlying order in the chaos.

✶In 1993, Goldstar Company created a washing machine whose pulsator motion exploited chaos theory.

fibrillation a potentially fatal malfunction of heart muscle where the muscle rapidly and ineffectually twitches instead of pulsing regularly

Early History of Chaos Theory

Although chaos theory is very much a late-twentieth-century development, most chaos scientists consider the great nineteenth-century mathematician, Henri Poincaré, to be the true father of the discipline. The fundamental phenomenon common to all chaotic systems is sensitivity to **initial conditions**, and Poincaré was the first to articulate this sensitivity in his study of the so-called three-body problem.

initial conditions the values of variables at the beginning of an experiment or of a set at the beginning of a simulation; chaos theory reveals that small changes in initial conditions can produce widely divergent results

Poincaré had described this problem in 1890 when he attempted to calculate the orbits for three interacting celestial bodies. Poincaré found that Newton's equations for two celestial objects were useless after a short time. The orbits became so tangled that he gave up hope of trying to predict where they would go. No one quite knew what to do about Poincaré's problem and so it remained essentially a mystery for the next eight decades.

Henri Poincaré is known for his statement of the principle of "sensitivity to initial conditions," which says that small differences in the initial conditions lead to very great ones in the final phenomena, so that prediction becomes impossible.

Chaos in the Twentieth Century

In the 1950s **meteorologists** had great hopes that accurate weather prediction was about to become a reality. The advent of computers that could analyze vast amounts of data was seen as a great breakthrough in weather forecasting. Unfortunately, by the end of the decade, the optimism had begun to fade. Even with the rapid analysis of data from weather stations all over the world, forecasting did not seem to improve.

Meteorologist and mathematician Ed Lorenz of the Massachusetts Institute of Technology wondered why and began attempting to model the weather with equations programmed into his computer. The computer used the equations to simulate several months of weather, producing a graph that would rise and fall according to changes in the variables of Lorenz's weather equations.

At one point Lorenz wanted to take a closer look at a certain part of his graph, so he went back to an earlier point on the plot, inserted the values for that point from a printout from the computer, and waited to see a repetition of the graph he had seen before. At first he did, but after a few minutes he was startled to see a new plot **diverging** from the old plot, creating an entirely different version of the weather. After checking his computer for a malfunction and finding none, he discovered the cause of this unexpected behavior. When he had restarted the simulation, he had entered values from the printout that were rounded off to three decimal places, whereas the computer was using six place decimals for the simulation. This small variation in the initial values had produced a completely different version of the simulated weather.

meteorologist a person who studies the atmosphere in order to understand weather and climate

diverge to go in different directions from the same starting point

By accident, Lorenz had rediscovered Poincaré's sensitivity to initial conditions. He realized that this meant that unless one has infinitely precise knowledge of all the initial conditions in a weather system, one cannot accurately predict the weather very far into the future. Lorenz called this finding "The Butterfly Effect"✶ because his results implied that the flapping of a butterfly's wings in Brazil could stir up weather patterns that might ultimately result in a tornado in Texas.

✶The now-popular phrase "If a butterfly flaps its wings"—which many people use in connection with global ecology—actually came from the study of mathematical chaos.

Fish Population Studies. In 1970, the biologist Robert May was studying the growth of fish populations using a well-known mathematical model for populations with an **upper bound** to growth. This upper bound is sometimes called the "**carrying capacity**" of the environment. The model was the logistic difference equation $P(t + 1) = rP(t)(1 - P(t))$, where r is a number greater than 1 representing the growth rate, $P(t)$ is the population as a percentage of carrying capacity at time t, and the factor $(1 - P(t))$ represents the fact that as $P(t)$ gets closer to 1 (closer to 100 percent of the carrying capacity), the growth rate of the population slows to almost zero.

upper bound the maximum value of a function

carrying capacity in an ecosystem, the number of individuals of a species that can remain in a stable, sustainable relationship with the available resources

May had noticed that fish populations sometimes maintained the same population from one year to the next and sometimes would fluctuate from year to year. He decided to study what happened with the logistic equation when

iterator the mathematical operation producing the result used in iteration

parameter an independent variable, such as time, that can be used to rewrite an expression as two separate functions

oscillating moving back and forth

bifurcation value the numerical value near which small changes in the initial value of a variable can cause a function to take on widely different values or even completely different behaviors after several iterations

he varied the values of the parameter r, which essentially represent the growth rates of the population. The logistic difference equation is an **iterator**. This means that one inputs a certain initial population called $P(0)$ into the equation, does the calculation, and receives as the output $P(1)$, which is the population at the next time interval. Then this output value $P(1)$ becomes the new input into the equation, yielding a new output $P(2)$, and so on.

Using a simple hand calculator, May carried out hundreds of iterations (repetitions) using different values for the initial population, $P(0)$, and different values of the growth **parameter** r. He discovered that when the value of r was between 1 and 3, the iteration produced a string of numbers that would ultimately settle on a single value no matter what initial value was used. This implied a stable population year after year. When r was raised slightly above 3, however, the output of the iteration cycled back and forth between two values.

This remained true for all initial values until May raised the parameter beyond 3.45 and observed that the values began **oscillating** among four values, again independent of the initial population. At about 3.54, the oscillation doubled again to eight distinct values, then to sixteen when r was increased passed 3.556, then to thirty-two and sixty-four, and so on. When the parameter value reached about 3.56994, this "period-doubling cascade," as May would call it, ended and the values seemed to jump about randomly.

At this point May noticed that the values produced by an iteration for a given r value were no longer independent of the initial value. For $r = 3.6$, for example, if the initial value were 0.1, the twentieth iteration was about 0.7977, but if the initial value were 0.09, the twentieth iteration was about 0.8635. The discrepancy became greater with additional iterations. Once again Poincaré's sensitivity to initial conditions had been rediscovered. In this case, May had discovered chaos in the logistic difference equation.

Feigenbaum's Constant. May's period-doubling cascade became the subject of intense study by the physicist Mitchell Feigenbaum in the mid-1970s. Whereas May had concerned himself with the qualitative aspects of the logistic system, Feigenbaum wanted to understand the quantitative basis of this period-doubling route to chaos, as it would come to be called. Starting with a handheld calculator and eventually making use of more powerful computers, Feigenbaum discovered a completely unexpected pattern associated with the parameter (r) values at which each new period doubling occurred. He called these **bifurcation values**.

Taking the computations far beyond where May had taken them, Feigenbaum came up with exceptionally high accuracy for the bifurcation values. For example, the first six are $b_1 = 3$, $b_2 = 3.449490...$, $b_3 = 3.544090...$, $b_4 = 3.556441...$, $b_5 = 3.568759...$, and $b_6 = 3.569692....$ So, for example, when the parameter is raised past $b_5 = 3.568759$, the logistic equation's output changes from an oscillation among sixteen values to an oscillation among thirty-two values. Feigenbaum had the computer draw a graph showing parameter values on the horizontal axis and population values on the vertical axis. The result, known as the Feigenbaum plot, is now one of the icons of chaos theory.

The Feigenbaum plot shows how the bifurcations come increasingly closer as the parameter is increased from left to right. When the parameter

value passes 3.56994. . ., the period doublings are no longer regular. This is the onset of chaos.

As fascinating as this was, Feigenbaum found something even more amazing in these numbers. He found that if he calculated ratios of the form $(b_{k+1} - b_k)/(b_k - b_{k-1})$ for larger and larger values of k, these ratios would approach the number 4.669201609. . .. Furthermore, he found that if he used other functions, such as **sine** or **cosine**, completely unrelated to the logistic equation, the result would be the same. The number 4.669201609. . . is now known as Feigenbaum's constant and is considered to be one of the fundamental constants of mathematics.

The Link to Fractals. The Feigenbaum plot exhibits self-similarity, meaning that any one of the branches looks like the entire plot. Figures that exhibit self-similarity are called "**fractals**," a term invented by Benoit Mandelbrot in 1975. Just as Euclidean geometry is the geometry of Newtonian mechanics, fractal geometry is the geometry of chaos theory.

Geometrically, the dynamics of chaotic systems are described by figures called "attractors," of which the Feigenbaum plot is one example. Although these dynamics are unpredictable due to sensitivity to initial conditions, they are geometrically bound to a certain structure, the attractor. It is in this sense that chaos scientists are said to seek order in chaos. SEE ALSO Fractals.

Stephen Robinson

FEIGENBAUM'S UNIVERSAL CONSTANT

Although initially limited to studies of chaos theory, Feigenbaum's constant is now regarded as a universal constant of nature. Since the 1980s, scientists have found period-doubling bifurcations in experiments in hydrodynamics, electronics, laser physics, and acoustics that closely approximate this constant.

sine if a unit circle is drawn with its center at the origin and a line segment is drawn from the origin at angle theta so that the line segment intersects the circle at (x, y), then y is the sine of theta

cosine if a unit circle is drawn with its center at the origin and a line segment is drawn from the origin at angle theta so that the line segment intersects the circle at (x, y), then x is the cosine of theta

fractal a type of geometric figure possessing the properties of self-similarity (any part resembles a larger or smaller part at any scale) and a measure that increases without bound as the unit of measure approaches zero

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Cierva Codorniu, Juan de la

Spanish Aeronautical Engineer
1895–1936

Juan de la Cierva Codorniu was born in Murcia, Spain, in 1895. Today he is remembered as the inventor of the **autogiro**, a forerunner of the helicopter. For six years he attended the Escuela Especial de Ingenieros de

autogiro a rotating wing aircraft with a powered propellor to provide thrust and an unpowered rotor for lift; also spelled "autogyro"

Juan de la Cierva Codorniu's trip across the English Channel was a world-famous international flight.

Caminos, Canales y Puertos in Madrid, Spain, where he studied theoretical **aerodynamics**. Following this, he entered a competition to design military aircraft for the government and built a biplane bomber with an airfoil (the part of a plane that provides lift) that he designed mathematically. The plane was tested in May 1919, but it crashed when the pilot stalled it.

aerodynamics the study of what makes things fly; the engineering discipline specializing in aircraft design

fixed-wing aircraft an aircraft that obtains lift from the flow of air over a nonmovable wing

rotary-wing design an aircraft design that uses a rotating wing to produce lift; helicopter or autogyro

Cierva believed that **fixed-wing aircraft** were unsafe, so he experimented with a **rotary-wing design**, and the world's first working autogiro✶ flew 200 yards on January 19, 1923. Two days later the autogiro was unveiled to the public and made three flights, the longest of which was two and a half miles. In 1925, he founded the Cierva Autogiro Company in England and later collaborated with the Autogiro Company of America. On September 18, 1928, he flew one of his autogiros across the English Channel, and in 1930, he flew one from England to Spain. Autogiros were used during the 1930s for military liaison, mail delivery, and agricultural purposes.

✶The term "autogyration" means that a plane is equipped with a conventional engine and propeller that pull it forward through the air. This forward motion causes the rotor to gyrate automatically, like a windmill.

Aerodynamic Principles

As a student, Cierva had learned that four aerodynamic forces are involved in flight: lift, gravity, thrust, and drag. Lift allows the craft to ascend; gravity is the force that pulls it down. Thrust propels the craft forward; drag is the force that holds it back. For a craft to ascend, the lift must be greater than the force of gravity, and for it to accelerate, thrust must be greater than drag. When the craft is flying straight and level at a constant speed, all four aerodynamic forces are in **equilibrium**. The foundation of flight is based on Bernoulli's Principle. Bernoulli, an eighteenth-century Swiss scientist, stated that as the velocity of a fluid (such as air) increases, its pressure decreases, causing lift.

equilibrium a state of balance between opposing forces

In a fixed-wing aircraft, lift is provided by the wing, thrust by the propeller. Cierva, though, believed that the autogiro controlled these forces better than fixed-wing aircraft, which had a tendency in those days to stall, or lose lift suddenly. He also wanted to develop an aircraft that needed only a short takeoff run and could slowly land in small areas. The autogiro was a major step toward those goals. The body and tail assembly were similar to those of an airplane, and thrust was provided by an ordinary engine and propeller. Lift, however, was provided not by fixed wings but by large airfoils similar to helicopter blades, mounted horizontally above the craft and rotated by airflow that resulted from the craft's forward movement. After early unsuccessful attempts, Cierva came up with the idea of mounting the blades on hinges at a hub, allowing them to flap and thus respond differentially to aerodynamic and **centrifugal** forces as they rotated. SEE ALSO Bernoulli Family; Flight, Measurement of.

Michael J. O'Neal

centrifugal the outwardly directed force a spinning object exerts on its restraint; also the perceived force felt by persons in a rotating frame of reference

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Circles, Measurement of

The measurement of circles is closely connected to the measurement of angles. This is because the measurement has a rotational aspect; that is, one ray must be turned about the vertex until it coincides with a more static aspect. The circle is the perfect tool for measuring **planes** of angles, because any angle cuts off the same fractional part of every circle whose center is at the vertex of the angle.

plane generally considered an undefinable term, a plane is a flat surface extending in all directions without end, and that has no thickness

gradient a unit used for measuring angles, in which the circle is divided into 400 equal units, called gradients

radian an angle measure approximately equal to 57.3 degrees, it is the angle that subtends an arc of a circle equal to one radius

sine if a unit circle is drawn with its center at the origin and a line segment is drawn from the origin at angle theta so that the line segment intersects the circle at (x, y), then y is the sine of theta

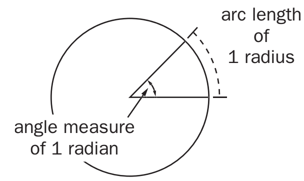
cosine if a unit circle is drawn with its center at the origin and a line segment is drawn from the origin at angle theta so that the line segment intersects the circle at (x, y), then x is the cosine of theta

Different Ways to Measure Circles

In degree measure, the circle is divided into 360 equal parts, while in **gradient** measure, 400 equal parts of a circle are used. Since there are four right angles in a circle, there are 100 gradients in a right angle, making gradients a better fit with the decimal system. Nevertheless, degree measure is the predominant system for measuring angles using fractional parts of a circle.

A different method for measuring angles based on a circle is **radian** measure. The importance of radian measure is that numerous results in calculus involving trigonometric functions—for example, **sine**, **cosine**, or tangent—require that the angle of the trigonometric function be measured in radians.

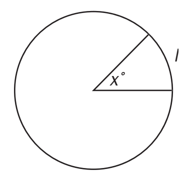
Using Radian Measure. To understand radian measure, it is necessary to understand that rather than using the fractional part of a circle cut off by the angle, the fractional part is converted to a distance. The portion of a circle cut off by an angle with its vertex at the center of the circle is called an arc of the circle. The distance used to measure an angle in radian measure is the length of that arc.



The first question is which circle to use. When using fractional parts of a circle, the particular circle used does not matter. However, when using arc length, the same angle cuts off a longer arc on larger circles than on smaller circles. So, when using arc length to measure an angle, the particular circle used does matter.

In order to simplify calculations, a circle with a radius of 1 unit is chosen. Then the conversion from fractional part of a circle to the length of arc follows from proportional reasoning. The ratio of the arc to the whole circle—that is, the fractional part of the circle—must be the same as the ratio of the length of the arc to the length, or **circumference**, of the whole circle. In symbols, if the number of degrees in the arc is x, the length of the arc is l, and the circumference of the circle is C, you can write: $\frac{x°}{360°} = \frac{l}{C}$.

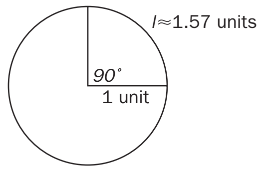
circumference the distance around the circle



The second question is how to determine the circumference, C. The ratio of the circumference of a circle to the length of its diameter, d, is defined to be π; that is, $\frac{C}{d} = \pi$. Multiplying both sides of this equation by d yields a formula: $C = \pi d$. Now since $d = 2r$, the previous formula can be rewritten as $C = \pi \cdot 2r = 2\pi r$. Finally, if $r = 1$, then $C = 2\pi r = 2\pi \cdot 1 = 2\pi$. Substituting this expression for C in the first ratio above, you get $\frac{x°}{360°} = \frac{l}{2\pi}$, and multiplying both sides of this equation by 2π gives $l = \frac{x}{360} \cdot 2\pi$ for the length of the arc cut off by an angle of $x°$ on a circle with a radius of 1 unit.

Using the last equation gives some equivalents. For example, if $x = 360°$, then $l = \frac{360}{360} \cdot 2\pi = 2\pi$. Therefore, $360° = 2\pi$ radians. Similarly, if $x = 180°$, $l = \frac{180}{360} \cdot 2\pi = \frac{1}{2} \cdot 2\pi = \pi$. This gives a basic conversion factor: $180° = \pi$ radians. An angle of one radian has the same measure as the central angle in a circle whose arc is the length of the radius. One radian is about $57°\ 17'45''$.

Other common equivalents which can be derived from the previous equation are: $90° = \frac{\pi}{2}$ radians, $60° = \frac{\pi}{3}$ radians, $45° = \frac{\pi}{4}$ radians, and $30° = \frac{\pi}{6}$ radians. In all of these conversions it is important to understand just what the radian measure means. For example, as shown below, to say that a $90°$ angle is equal to $\frac{\pi}{2}$ radians is the same as saying that, on a circle having a radius of 1 unit, a $90°$ angle with its vertex as the center of the circle cuts off an arc that is $\frac{\pi}{2} \approx 1.57$ units long, and so forth.

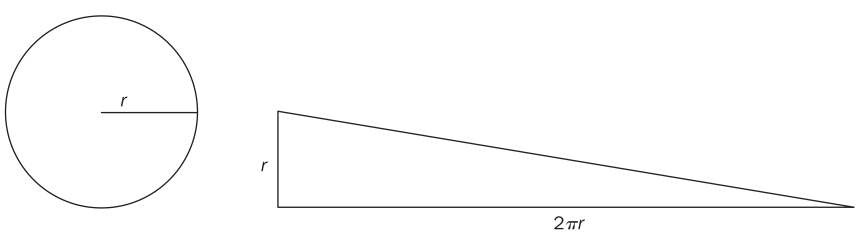


Historic Methods of Measuring Circles

Early tools for drawing circles almost certainly included attaching a drawing instrument to a length of cord, stretching the cord to the desired radius, holding the cord in place at the desired center, and swinging the drawing instrument in a circular path. However, by the time of Euclid (about 300 B.C.E.) a tool similar to the modern compasses, or dividers, was in use. However, the pair of compasses Euclid envisioned had an important difference from those in use today. Euclid used only collapsible compasses; that is, as soon as one point of the compasses was lifted from the paper, they closed. This prevented using the compasses as a set of dividers to transfer distances directly from one place to another in a geometric construction.

The term "collapsible compasses" refers not to the fact that the tool would close when lifted from the paper, but rather to the rule of construction imposed by the ancient Greek mathematicians that the tool was not to be used to transfer distances after being lifted from the paper. Interestingly, later mathematicians used some of Euclid's postulates and theorems to prove that his "collapsible compasses" are equivalent to the modern compasses.

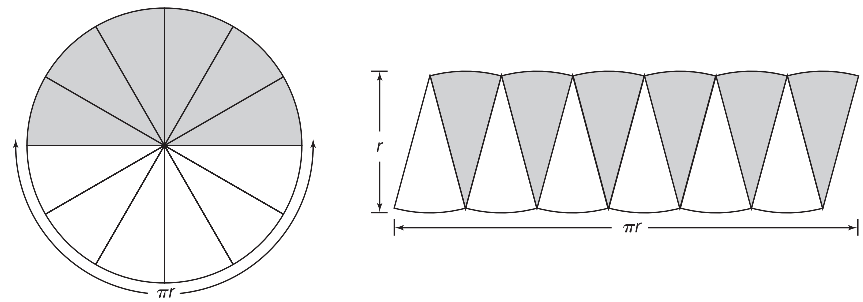
Circumference is a measure of the boundary of a circle. An equally important property is the area of a circle, which is a measure of the interior of the circle. In the third century B.C.E., Archimedes proved that the area of a circle is equal to the area of a right triangle whose legs are equal in length to the radius and the circumference of the circle. That is, since the area of a triangle is one-half the product of a base and the altitude to that base, and since in a right triangle, if one leg is used as the base, the other leg is the altitude to that base, Archimedes proved that the area of a circle is given by the formula $A = \frac{1}{2}r(2\pi r) = \pi r^2$.



It is difficult to know for certain what motivated this idea for Archimedes, but the diagram below shows one possibility. Notice that when a circle is cut into an even number of equal-sized sectors, or wedge shapes, these sectors can be rearranged to form a figure that is approximately a **parallelogram**. The length of the altitude of the approximated parallelogram would then be the same as the radius of the circle, and the length of the base of the parallelogram would be one-half the circumference of the circle. Then, the area

parallelogram a quadrilateral with opposite sides equal and opposite angles equal

of a parallelogram is the product of the lengths of its base and altitude, yielding the same formula for the area: $A = r(\frac{1}{2}C) = r(\frac{1}{2} \cdot 2\pi r) = \pi r^2$.



Squaring the Circle. One of the classic problems of ancient mathematics was the problem of finding a square exactly equal in area to a given circle. This problem has come to be known as the problem of "squaring the circle." The ancient Egyptians had such a procedure. They merely took eight-ninths of the square of the diameter, a formula that is equivilant to taking $\pi \approx 3.16$, but it is unclear whether they realized that this was just an approximation to the area of the circle.

The ancient Greek mathematicians spent much time and energy in seeking a solution to the problem of squaring the circle. They refined the problem by placing the restriction that the required square could be found by use of compasses and straight-edge only. A number of Greek mathematicians were able to produce constructions of squares whose areas exactly equaled a given circle, but none was able to do so under the restriction of compasses and straight-edge alone. However, in producing their solutions (which they knew full well did not satisfy the restriction), these ancient Greek mathematicians developed new and profound mathematics.

transcendental a real number that cannot be the root of a polynomial with rational coefficients

The problem of squaring the circle by means of compasses and straight-edge alone continued to intrigue mathematicians through the nineteenth century. Finally, in 1882, Ferdinand Lindemann put the age-old question to rest by proving that π is a special kind of real number called a **transcendental**. None of the transcendental numbers can be constructed using only a straight-edge and compass. In other words, it is impossible to find a square exactly equal in area to a given circle using only these tools. SEE ALSO Angles, Measurement of; Pi; Trigonometry.

Thomas G. Edwards

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City Planner

City planning involves not only the orientation of buildings and streets in a city but also the economic, social, and population conditions of the community. To develop available land or renovate existing property, city planners must understand architecture, surveying, economics, and even politics, in addition to many other variables.

City planners determine a city's need for buildings and services based on mathematical models that predict an increase in population. Population is calculated using census data, municipal registers (which record births, deaths, migration, marriages, and divorces), and statistical samples of the population as a whole. An expected increase in population will mean a need for more schools, hospitals, and businesses.

Maps and land surveys help the city planner determine where to locate new buildings or parks. A topographic map shows the elevation of the land—where fields, valleys, and waterways are located. An **areagraph** measures areas, in acres, to scale. Depending on a city's land use regulations and zoning codes, the city planner decides where to build residential homes, businesses, and public facilities.

areagraph a fine-scale rectangular grid used for determining the area of irregular plots

Cities were once laid out in a circular design with a public square or government building in the center and residential homes radiating out around it. Planners today use the same principle, locating residential areas near schools and churches, with shopping, business, and industrial facilities on the periphery of the community. Today, planners must also consider environmental regulations, pollution emissions, and waste disposal. SEE ALSO Census; City Planning; Population Mathematics.

Lorraine Savage

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City Planning

Cities were first established when nomadic people settled around food sources, religious sites, or waterways. Early people soon began to design the arrangement of personal dwellings, religious buildings, food stores, and government centers within their communities along trade routes that linked cities. In the fifth century B.C.E., the "Father of Town Planning," Greek architect Hippodamus of Miletus, designed towns in a geometric pattern that was not only functional but also had aesthetic balance. Likewise, the Romans arranged residences, temples, and forums in a symmetrical design.

City planning declined after the fall of the Roman Empire only to reemerge during the Renaissance with a revival of Greco-Roman styles. During this time, the popular design for cities featured **concentric** circles radiating out from a central point with straight streets connecting the circles, like spokes of a wheel. European cities such as Venice (Italy), London (England), and Mannheim

concentric sets of circles or other geometric objects sharing the same center

Rapidly growing cities like Seattle require careful planning to avoid traffic and housing problems.

(Germany) featured this design with wide streets surrounding a public square or government building. A famous example of this circumferential pattern is the extensive redesign of Paris by Baron George Eugène Haussmann in 1848. In addition to aesthetics, Haussmann took into consideration population density, sanitation, and recreational space.

In the New World, some cities, specifically New York, were designed in a grid or block pattern which allowed for easy expansion. However, during the **Industrial Revolution**, this design led to overcrowding, congestion, and a shortage of resources. City planners realized they needed to consider population growth, economics, income opportunities, environmental impact, and transportation when designing a city. With the establishment of municipal planning commissions, zoning ordinances, and subdivision regulations, city planning became standardized.

Industrial Revolution beginning in Great Britain around 1730, a period in the eighteenth and nineteenth centuries when nations in Europe, Asia, and the Americas moved from agrarian-based to industry-based economies

Perhaps the most important factor in city planning is predicting population growth. The population of a city will determine its need for housing, resources such as food and utilities, jobs, schools, hospitals, fire protection, social services and entertainment. Population trends are based on census data, comparisons with other cities' census data, municipal registrations, statistical models, and current **demographics**. Variables used in equations to predict population growth include birth and death rates, surviving birth rates, migration out of or into a city, education level and economic level. The current age of a city's population is important to know, because young couples are likely to have children who in turn will need schools, whereas a city with mostly retirees may need more healthcare facilities. To avoid overcrowding and dwindling resources, city planners must assess their city's ability to accommodate a severe influx of new residents or businesses.

demographics statistical data about people—including age, income, and gender—that are often used in marketing

Neighborhood density, or population per square mile, is also a concern. Land area density is based on the design of residential dwelling and the square footage needed per family. Single family dwellings typically house the fewest people. Duplexes, condominiums, tenements, apartment complexes, and high-rise apartment buildings house considerably more. The greater the population density, the greater the need for grocery stores, shop-

ping centers, gas stations, and so on, to accommodate residents. The number of schools an area needs is based on the population of couples at childbearing age. The number of hospitals needed is calculated based on the projected number of patient days per thousand residents. Fire protection and water supply are also based on population.

When preparing new land to be developed, the city planner consults a variety of maps. Land surveys and topography maps denote the elevation and features of the land to scale, whether it is fields, hills, valleys, floodplains, or mountains. An **areagraph** measures areas in acres on a scale. A **hydrograph** shows characteristics of water sources. City planners need to calculate a density versus acres proportion to determine how much land should be used for residential, business, and public facilities.

areagraph a fine-scale rectangular grid used for determining the area of irregular plots

hydrograph a tabular or graphical display of stream flow or water runoff

Depending on land use regulations, planners decide on the orientation of buildings, parking spaces, and streets. Houses can be arranged perpendicular, adjacent, or parallel to streets. The Sun's orientation throughout the day, based on geographic **latitude**, helps the planner determine shade areas and wind direction. In addition, knowing the incline of the land is important for proper drainage and installation of driveways.

latitude the number of degrees on Earth's surface north or south of the equator; the equator is latitude zero

The location and size of city streets are determined by traffic flow during peak hours. Planners consult highway capacity tables to determine the width of streets, percentage distribution of cars leaving the city versus entering, use of public transportation, residential and business traffic, available parking, and traffic light schedules. Pedestrian traffic on sidewalks and the percentage of the street used by pedestrians and bicycles are also estimated to avoid congestion.

When deciding where buildings should be located, the planner calculates the maximum distance of facilities in a radius around the residents. Churches and schools are typically within walking distance from residents. Major shopping outlets and hospitals are located on the outskirts of town. Industrial facilities are usually on the periphery of a community due to their freight access, noise level, and environmental safety aspect.

City planners also need to know the real estate appraisal of the land they are developing. If the property is already owned, the planners need to offer the owners a fair price. If planners are developing vacant land, they will want to know the value of the new structure they are building. Retail districts, residential, and industrial land each have varying values. Appraisals are based on market value, cost approach, or anticipated income value of the land.

Another concern in city planning is the character and reputation of the city. Abundant trees and sunlight, social events, shops, and cultural institutions make a city attractive to tourists and visitors. A city with few social services, inadequate sanitation, no community events, many homeless people, and tall skyscrapers that block out light may not be as attractive.

The future of city planning will depend on new technologies to more accurately predict population changes, economic development, and improvements in transportation and public administration. SEE ALSO Census; Population Mathematics.

Lorraine Savage

HOW DID THE COMET GET ITS NAME?

The term "comet" comes from the Greek word *kome,* which means "hair." As a comet nears the Sun, it warms up and partially evaporates. The evaporated material then streams away from the Sun, and resembles long strands of hair.

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Comets, Predicting

When comets are first seen through a telescope, they are faint, "fuzzy" patches of light at the limit of visibility. A few years ago, most comets were discovered by amateur astronomers who looked for comets as a hobby.* To find comets takes enormous patience and many nights of scanning the sky through huge binoculars or wide angle telescopes. To save time, amateur comet hunters also memorize large areas of the sky, so that they can immediately recognize when something new appears. Recently, professional astronomers have discovered many comets. Professionals normally do not spend much time looking for comets, because time on modern research telescopes is typically considered too valuable to spend on projects that are as uncertain as comet seeking. However, many research projects involve capturing long exposure electronic images of large areas of the sky for other purposes, and comets often appear in these images.

***If you find a new comet, it will bear your name!**

What Are Comets?

Aristotle thought that comets were atmospheric phenomena, since they changed appearance. Aristotle's **cosmology** divided the universe into various spheres. The spheres below the Moon (Aristotle's **sublunary** spheres) were the only spheres that could change. The unchangeable heavens included the Moon and everything beyond. However, Tycho Brahe demonstrated that comets must be farther from Earth than the Moon. If the Moon is observed from different places on Earth, its position in the sky will change slightly due to **parallax**. By measuring the angular change in the position of the Moon and knowing the distance between the observing locations on Earth, the distance to the Moon can be measured. The distance between the observing positions is the base of an **isosceles triangle** with the **vertex** on the Moon. The appearance of the Great Comet of 1577 (which is now known as Halley's comet) gave astronomers the opportunity to measure its parallax. According to Tycho's parallax measurements, it was six or seven times farther than the Moon was from Earth.

cosmology the study of the origin and evolution of the universe

sublunary "below the moon"; term used by Aristotle and others to describe things that were nearer to Earth than the Moon and so not necessarily heavenly in origin or composition

parallax the apparent motion of a nearby object when viewed against the background of more distant objects due to a change in the observer's position

isosceles triangle a triangle with two sides and two angles equal

vertex the point on a triangle or polygon where two sides come together

A comet consists of a core of frozen material. This core is only a few kilometers across, far too small to be seen as more than a tiny point. The core contains frozen methane, ammonia, and water ice. Dust is mixed in with the ice, so the comet core resembles a "dirty" snowball. A comet has a temperature of only a few kelvins. However, as this frozen material nears the Sun it warms up and begins to sublimate (pass directly from solid to gas). The gas and dust then form a halo around the core. It is this halo that is initially seen by astronomers. As the comet gets closer still to the Sun and warms up even more, the rate of sublimation increases and the **solar wind** blows the material away from the comet, forming the long "tails" that are

solar wind a stream of particles and radiation constantly pouring out of the Sun at high velocities; partially responsible for the formation of the tails of comets

the hallmark of comets. The gas is blown straight away from the Sun and **ionized** by the solar radiation, forming a bluish, glowing ion tail. The dust that is released when the gas sublimates drifts more slowly away, forming the whitish or yellowish, curving dust tail.

ionized an atom that has lost one or more of its electrons and has become a charged particle

The Origin and Movement of Comets

Comets probably originate far out on the edges of the solar system in a cloud of material left over from the formation of the solar system. When this cloud of material, called the **Öort cloud**, is disturbed by some gravitational influence, such as a nearby star, clumps of material begin to fall in toward the Sun. Hundreds of thousands (or even millions) of years later, a few of these objects approach the Sun and begin to warm up. Eventually, the objects swing around the Sun and head back out into space.

Öort cloud a cloud of millions of comets and other material forming a spherical shell around the solar system far beyond the orbit of Neptune

Most comets have orbital periods of thousands or millions of years. These "long period" comets are unpredictable. Even when the orbit of one is calculated, in a thousand or more years it will be impossible to tell if the same comet has returned or not. This unpredictability does not hold for "short period" comets (periods of less than 200 years).

Comets can approach the Sun from any direction. This suggests that the Öort cloud is spherical. Occasionally, a comet will approach the Sun along a path that takes it close to one of the large planets in the solar system. These gravitational encounters can alter the path of the comet and convert a long period comet into a short period comet. Halley's comet is probably the most famous of the short period comets. It is expected to return to the inner solar system in 2061. We know this, because comets follow the same laws of physics as other objects in our solar system.

All comets obey the same rules of motion that hold for the planets. Johannes Kepler first proposed these rules in the early part of the seventeenth century. Kepler's rules are usually stated in the form of three **empirical laws**:

1. All planets move in **elliptical orbits** with the Sun at one **focus**;
2. A line drawn from the Sun to the planet sweeps out equal areas in equal times; and
3. The square of a planet's orbital period is proportional to the cube of its **semi-major axis**.

If the period (P) is measured in years and the semi-major axis (a) is given in **astronomical units**, Kepler's third law can be written as $P^2 = a^3$. (One astronomical unit is the semi-major axis of Earth's orbit, which is also the average distance from Earth to the Sun.)

empirical law a mathematical summary of experimental results

elliptical orbit a planet, comet, or satellite follows a curved path known as an ellipse when it is in the gravitational field of the Sun or another object; the Sun or other object is at one focus of the ellipse

focus one of the two points that define an ellipse; in a planetary orbit, the Sun is at one focus and nothing is at the other focus

semi-major axis one-half of the long axis of an ellipse; also equal to the average distance of a planet or any satellite from the object it is orbiting

astronomical unit the average distance of Earth from the Sun; the semi-major axis of Earth's orbit

The Significance of Halley's Comet

Kepler's laws are now known to be a direct consequence of Isaac Newton's laws of motion and his law of universal gravitation. In 1705, the British astronomer Edmund Halley realized that Kepler's laws (as extended by Newton) could be applied to comets. He deduced that a comet seen in 1682 was orbiting the Sun like a planet and had probably been seen many times before. By examining previous sightings of what he thought to be the same object, he calculated its orbit and determined its period was 76 years and its semi-major axis was 18 astronomical units. This orbit took the comet past

Using Kepler's laws of motion, astronomer Edmund Halley predicted that a comet seen orbiting the Sun in 1682 would reappear in 1758. Although he did not live to see his prediction come true, this comet now bears his name in his honor. The comet's next appearance will be in 2060 or 2061.

coma the cloud of gas that first surrounds the nucleus of a comet as it begins to warm up

✶***Giotto* was named after an Italian artist who saw the comet in 1301 and painted it into a nativity scene.**

the orbit of Neptune, to the edge of the solar system where it cannot be seen by telescopes. Halley predicted that the same comet would return to the inner solar system again in 1758. Although Halley did not live to see the return of the comet, his prediction was correct: That particular comet was seen again in 1758. His successful prediction was a triumph of Isaac Newton's theories of gravity and motion. The comet was named in Halley's honor, starting a tradition of naming comets after their discoverers.

Since astronomers knew the period of Halley's comet, they were able to determine when it had previously appeared in the inner solar system and became visible. By putting together records from the Chinese and other ancient peoples, astronomers have determined that Halley's comet has been observed at every passage since 240 B.C.E. Halley is usually a spectacular sight, with its tail stretching out at least one astronomical unit. The tail can stretch many degrees across the sky.

In a real sense, a comet falls towards the Sun, accelerating as it descends. Its movement is most rapid as it passes near the Sun (as predicted by Kepler's second law). Then it moves away from the Sun, but the Sun's gravity is trying to pull it back. So it slows down as it moves away. At its maximum distance from the Sun, it is moving most slowly. So Halley's comet actually spends most of its time far from the Sun out past the orbit of Neptune.

The most recent appearance of Halley's comet, in 1986, was not ideal for viewing from Earth. However, a fleet of spacecraft was able to visit the comet. A Soviet spacecraft, *Vega 2*, traveled through the comet's **coma** and returned accurate information about the position of the nucleus. This allowed the *Giotto* spacecraft launched by the European Space Agency (ESA) to navigate within 600 km of the nucleus.✶ Moving this close was risky, because dust from the comet could (and did) damage the spacecraft. However, before the spacecraft was damaged, it was able to send pictures back to Earth. One picture returned by the *Giotto* spacecraft showed the actual core of the comet shrouded in dust.

The nucleus was larger than expected. It was about 15 km long and 10 km wide. It was also extremely dark, about the color of powdered charcoal, due to the thick layer of dust on its surface. The dust remains behind as the ice sublimates. Since *Giotto* passed very close to the nucleus of Halley's comet, ESA engineers were able to calculate its mass. It has a surprisingly low mass and a low density of about 0.1 g/cm^3. This figure is probably because much of the frozen gas has already escaped, leaving behind a very "fluffy" residue. The comet will continue to lose material every time it swings close to the Sun. It loses as much as 30 tons per second. This loss of material will completely destroy the comet in about 40,000 years or after 5,000 orbits.

Although astronomers know Halley's comet will return sometime in 2060 or 2061 to the vicinity of Earth, the moment it first becomes visible cannot be predicted accurately. The orbits of all comets are somewhat unpredictable. This is due in part to the gravitational forces of the planets that act on them as they pass through the solar system. Also, each time a comet passes by the Sun, jets of evaporating gas spew out of the surface. These jets act like rockets and change the orbit of the comet slightly. As a result, no one knows exactly when Halley's comet will return. So keep your eyes open

WHAT ARE EMPIRICAL LAWS?

Empirical laws are mathematical summaries of observations. They do not attempt to explain things, but they are very good at predicting the results of experiments.

in 2061. Who knows, you may be the first person to see its return! SEE ALSO ASTRONOMY, MEASUREMENTS IN; SOLAR SYSTEM GEOMETRY, MODERN UNDERSTANDINGS OF.

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Communication Methods

Just 150 years ago, a routine telephone call was not possible, images were delivered in person, and newspapers brought headlines of information that may have been days or weeks old. There was no television or radio coverage of world events. There was no World Wide Web for the world to share its collective knowledge.

Yet today, the ability to communicate with our business partners, colleagues, friends, and loved ones over long distances and at any time of the day is taken for granted. Via phone calls, e-mail, or fax, we transmit pictures, voice or text to almost anyone and anywhere at anytime. Communication has become faster, cheaper, and more commonplace. Mathematics is used to encode, transmit, and decode data in a practical and secure way. Mathematics drives the new technology and makes the communication revolution possible.

With advances in digital communications, telecommuting has reduced the need for face-to-face meetings and written correspondence.

The Challenge of Long-Distance Communication

Communication involves the sharing of ideas, information, sounds, or images. Communicating over distance requires three stages: encoding, transmitting, and decoding. For instance, about 1,500 years ago, the Incas—a highly organized, methodical culture without any written language—used **quipus** to communicate numeric information. A quipu was a method of encoding numbers for record keeping, and it involved an intricate set of knots tied into colored cotton cords that were connected in specific ways. The knots represented numbers stored in **base-10**, with the absence of a knot indicating zero, so that the user could distinguish, for example, between 48 and 408. Runners transmitted the information by carrying the quipu from one location to another. Whoever needed to look up the record, studied the knots and decoded the information.

quipus knotted cords used by the Incas and other Andean cultures to encode numeric and other information

base-10 a number system in which each place represents a power of 10 larger than the place to its right.

Methods of encoding, transmitting, and decoding have advanced with technology. We now can encode and decode with **binary numbers** and transmit via electricity, radio waves, and fiber-optic cables. All of these technologies require mathematics to make them viable.

binary number a base-2 number; a number that uses only the binary digits 1 and 0

When considering communication, key questions should be addressed.

1. How does information get encoded and decoded?
2. How can information be transmitted fast enough to make it practical?

3. How can information be sent without data loss or corruption?
4. How can information be sent securely?

The remainder of this article will discuss the first three questions. The answer to the fourth question involves **cryptography**, a deep and ancient subject with amazing recent mathematical advances whose treatment is beyond the scope of this article. Ironically, it is **number theory**, the most artistic, pure, abstract, and theoretical branch of mathematics, that provides the method of encryption for sending credit card numbers securely over the Internet.

cryptography the science of encrypting information for secure transmission

number theory the study of the properties of the natural numbers, including prime numbers, the number theorem, and Fermat's Last Theorem

The Encoding of Information: Analog versus Digital

There are two models for encoding information. With analog encoding, information is changed continuously from one physical form to another. For example, sound can be encoded into the vibrations of a needle, and stored as scratches on a vinyl disk. A **phonograph** decodes this information by reversing the process. The disadvantages of analog encoding are noise and inefficiency in the encoding process. Telegraphs, radio, television, cameras, tape recorders, and telephones were all originally analog inventions. There are now digital versions of all of these.

phonograph a device used to recover the information recorded in analog form as waves or wiggles in a spiral grove on a flat disc of vinyl, rubber, or some other substance

With digital encoding, information is transformed into a sequence of numbers. Written language can be encoded by using a specific number for every unique symbol in the language's alphabet. A picture can be encoded by splitting up the image into tiny pieces and using numbers to represent the color, intensity, and position of each piece. Sound can be encoded by splitting up the soundtrack into small pieces and using numbers to represent the frequency, intensity, and quality of the sound at each tiny fraction of a second. A similar process can be used with video. Compact disc (CD) and digital versatile disc (DVD) players are themselves digital devices.

Compared to analog encoding, digital encoding always misses information. However, if enough samples are taken, the missing pieces of information in between are not discernable. It is like looking outside through a screen door. If the screen is very fine, its presence is barely noticeable, even though it really covers a substantial percentage of one's view. Of course, if enough samples are not taken, fuzzy pictures, choppy recordings, or missing information may result.

Binary Numbers and Digital Encoding

Physically, it is easier to build transmission devices that send and receive only two different types of signals. Smoke signals and **Morse code** are early examples of this. Today, electricity can transmit either a high or a low voltage, where high represents 1 and low represents 0. Therefore, information tends to be encoded using only the numbers 0 and 1.

Morse code a binary code designed to allow text information to be transmitted by telegraph consisting of "dots" and "dashes"

A sequence of digits containing only 0s and 1s is called a binary number. For example, here are the base-10 numbers 0 through 9 represented in binary (in order).

0   1   10   11   100   101   110   111   1000   1001

Using this system, we can show 0 and 1 directly but since there is no symbol for 2, a new column is created that is thought of as the 2s column, writ-

ing 2 as 10. Every subsequent number is represented by adding one to the previous number and carrying to new columns whenever necessary. This is like base-10, but we carry when we reach 2 instead of 10. In other words, the columns represent powers of 2 instead of powers of 10, so the columns represent 1, 2, 4, 8 and so on. The binary number 100010011 therefore represents $2^0 + 2^1 + 2^4 + 2^8$ or $1 + 2 + 16 + 256 = 275$. Every base-10 number can be encoded into a binary number, and vice versa.

ASCII—The Binary Encoding of Text

How can binary numbers be used to transmit text? **ASCII** encoding (American Standard Code for Information Interchange, published in 1968, pronounced "askee") links each character that appears on a keyboard with a unique binary sequence of length eight called a **byte**. These characters include all upper and lower case letters, digits, various special symbols, and punctuation marks. For example, the ASCII value of "A" is 01000001; the ASCII value of "B" is 01000010; and the ASCII value of "5" is 00110101.

ASCII an acronym that stands for American Standard Code for Information Interchange; assigns a unique 8-bit binary number to every letter of the alphabet, the digits, and most keyboard symbols

byte a group of eight binary digits; represents a single character of text

ASCII BINARY CODING OF LETTER AND NUMBERS

0010 0000	space	0011 0000	0	0100 0000	@	0101 0000	P	0110 0000		0111 0000	p
0010 0001	!	0011 0001	1	0100 0001	A	0101 0001	Q	0110 0001	a	0111 0001	q
0010 0010	"	0011 0010	2	0100 0010	B	0101 0010	R	0110 0010	b	0111 0010	r
0010 0011	#	0011 0011	3	0100 0011	C	0101 0011	S	0110 0011	c	0111 0011	s
0010 0100	$	0011 0100	4	0100 0100	D	0101 0100	T	0110 0100	d	0111 0100	t
0010 0101	%	0011 0101	5	0100 0101	E	0101 0101	U	0110 0101	e	0111 0101	u
0010 0110	&	0011 0110	6	0100 0110	F	0101 0110	V	0110 0110	f	0111 0110	v
0010 0111	'	0011 0111	7	0100 0111	G	0101 0111	W	0110 0111	g	0111 0111	w
0010 1000	(	0011 1000	8	0100 1000	H	0101 1000	X	0110 1000	h	0111 0000	x
0010 1001	)	0011 1001	9	0100 1001	I	0101 1001	Y	0110 1001	i	0111 0001	y
0010 1010	*	0011 1010	:	0100 1010	J	0101 1010	Z	0110 1010	j	0111 0010	z
0010 1011	+	0011 1011	,	0100 1011	K	0101 1011	[	0110 1011	k	0111 0011	{
0010 1100	,	0011 1000	<	0100 1100	L	0101 1100	\	0110 1100	l	0111 0100	|
0010 1101	-	0011 1001	=	0100 1101	M	0101 1101	]	0110 1101	m	0111 0101	}
0010 1110	.	0011 1011	>	0100 1110	N	0101 1101	^	0110 1110	n	0111 0110	~
0010 1111	/	0011 1111	?	0100 1111	O	0101 1111	_	0110 1111	o	0111 1111	

Each byte consisting of eight binary digits (**bits** for short) can store $2^8 = 256$ different values, which is more than enough to store the various symbols on a keyboard. One bit can store two values, 0 or 1. Two bits can store 00, 01, 10, or 11. With three bits, eight possible sequences can be stored. With each new bit, the number of possibilities is doubled by adding a 0 in front of all the previous possibilities and by adding a 1. This implies that n bits can store $2n$ different configurations.

bit a single binary digit, 1 or 0

Unicode—The Binary Encoding of Written Characters

ASCII is acceptable for English but falls short when applied to the more diverse worldwide collection of languages and symbols. A more recent advance in written encoding is **Unicode**. Unicode is a universal, efficient, uniform, and unambiguous way to encode the world's written languages. It includes symbols for Hebrew, Arabic, and Chinese, as well as hundreds of standard business symbols. Unicode is a new standard that extends ASCII. It uses two bytes instead of one, or sixteen bits instead of eight. This gives Unicode the ability to encode a total of $2^{16} = 65{,}536$ different symbols.

Unicode a newer system than ASCII for assigning binary numbers to keyboard symbols that includes most other alphabets; uses 16-bit symbol sets

Compression—Huffman Encoding

compression reducing the size of a computer file by replacing long strings of identical bits with short instructions about the number of bits; the information is restored before the file is used

Huffman encoding a method of efficiently encoding digital information

algorithm a rule or procedure used to solve a mathematical problem

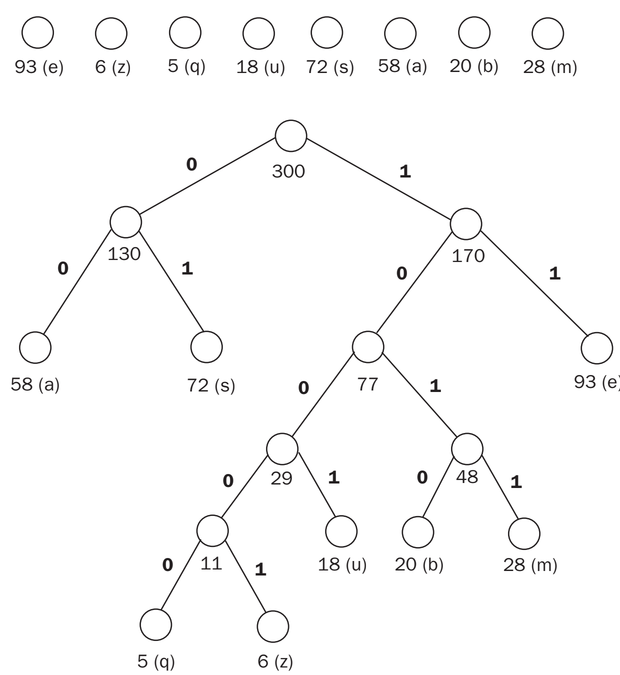
tree a collection of dots with edges connecting them that have no looping paths

Compression is a way to shorten messages without losing any information, thereby reducing the size and increasing the transmission speed of the message. The more commonly transmitted letters use shorter sequences while the less commonly transmitted letters use longer sequences. Morse code is a form of compression in analog encoding that uses short and long signals to represent letters. We now look at compression for digital encoding of text.

Each character of text is normally encoded with one byte. The inefficiency is that a q is encoded with the same eight bits as a t and yet the two letters occur with a different expected frequency. **Huffman encoding** takes into account the relative frequencies of the information being encoded.

For example, a message contains 93 *e's*, 6 *z's*, 5 *q's*, 18 *u's*, 72 *s's*, 58 *a's*, 20 *b's* and 28 *m's*. ASCII encoding requires eight bits per character, or $8 \times (93 + 6 + 5 + 18 + 72 + 58 + 20 + 28) = 2{,}400$ bits. Normal flat binary encoding will require three bits per character because there are eight different characters. This gives a total of $3 \times (93 + 6 + 5 + 18 + 72 + 58 + 20 + 28) = 900$ bits.

Huffman assigns the encoding in the following way: $e = 11$, $z = 10001$, $q = 10000$, $u = 1001$, $s = 01$, $a = 00$, $b = 1010$, $m = 1011$. This gives a total message size of $(2 \times 93 + 5 \times 6 + 5 \times 5 + 4 \times 18 + 2 \times 72 + 2 \times 58 + 4 \times 20 + 4 \times 28) = 765$, a 15 percent compression. This encoding is done with a "greedy" **algorithm** that builds a **tree** in order to determine the optimal encoding. The algorithm starts with a separate single node tree for each character labeled by its frequency. It repeatedly combines the two subtrees with the smallest weights, labeling the new tree with the sum of the labels on the two old trees, until there is just one tree. The initial trees and the final tree are shown below.



Error Correction—Hamming Codes

Error detection and correction allows us to ensure that information is transmitted without loss or corruption. When digital information is transmitted, it is easy for some of the bits to be corrupted because of lightning storms, surges in power supply, dirty lenses, and undependable transmission lines. How can we make sure that the actual data transmitted are received?

The answer is to add extra bits so that only some of the sequences are legal messages. This increases the size of the message and consequently slows down the transmission time, but allows more accurate transmission. For example, in the case of 4-bit messages, the sender can simply add an extra copy of the four bits making eight bits total. The receiver compares the two copies and requests a retransmission if the two copies differ. For example, the sequence 10001000 is legal but 10001001 would require a retransmission. The downside is that the correct 4-bit message cannot be determined.

Nonetheless, there are strategies that correct errors automatically without any need for retransmission. Assume that the message is broken up into groups of four bits and we want to correct any single bit error in a given group. This time two extra copies of the four bits are added, making the sum twelve bits total. Only sequences of twelve bits with three identical copies of the first four bits are legal. There are 2^{12} possible 12-bit messages and only 2^4 are legal. If there is a single difference between any two of the three copies, then we know that a single error has occurred and it can be corrected to be the same as the other two identical copies.

Why three copies? The three copies allow us to pinpoint where the error occurred. The idea is that *any two distinct legal messages must differ by at least three bits*. Hence an illegal 12-bit message that occurred due to a transmission error of a single bit must have come from a unique original legal message. The decoding is unambiguous. For example, if 100011001000 is received, then the correct message is 100010001000 and there is an error in the sixth bit transmitted.

What if more than one error occurs? If two or more differences exist, a retransmission can be requested. What if, by chance, many errors occur, resulting in a nonetheless legal message? That would be undetectable. This scheme was set up to correct only one error per 4-bit sequence. We could correct more than a single error if we were willing to add more redundant information and incur a proportionately larger increase in message and transmission time.

Hamming Codes. There is a more efficient way to correct single bit errors, and it is known as **Hamming codes**. The method is elegant and practical, but it is also complex. (For an explanation of why it works see Hamming's book, *Coding and Information Theory*.) Let $4 + r$ be the total number of bits sent. Hamming realized that each of the 2^4 legal messages must have $4 + r$ single-error illegal messages that can be created from it, and that if *any two distinct legal messages must differ by at least three bits*, then all of these illegal messages must be mutually distinct. This means that we need at least $(4 + r) \times 2^4$ distinct illegal messages and 24 distinct legal messages, totaling $(4 + r + 1) \times 2^4$ distinct messages. Since we have exactly $2^{4\times r}$ distinct binary sequences, we need $2^{4+r} = (4 + r + 1) \times 2^4$. The reader can check that r must be at least 3.

Hamming codes a method of error correction in digital information

Hamming demonstrated how to achieve this lower bound, thereby exhibiting the smallest possible size messages (7) to do single error-correction on 4-bit data messages. Here is how Hamming encoded 4 bits of data into 7 while allowing for the correction of a single error. He wrote his 4 data bits in positions 3, 5, 6, and 7. Then he computed the bits in positions 1, 2 and 4 depending on the number of ones in a particular set of the other positions. These extra bits are called **parity bits**. He then sets a given parity bit to 0 if the number of 1s in the appropriate positions is even, and to 1 if the number of 1s is odd. The sets of positions for the parity bits in positions 1, 2, and 4 are {3,5,7}, {3,6,7}, and {5,6,7} respectively.

parity bits extra bits inserted into digital signals that can be used to determine if the signal was accurately received

For example, if the message is: 0011, then we have _ _0_111. The {3,5,7} set gives 2 one's, an even number, so we set position 1 equal to 0. The {3,6,7} set is similar, so position 2 is also 0. The {5,6,7} set has an odd number of 1s so the fourth position is 1. The final encoded 7 bits is: 0001111.

To decode Hamming's scheme, the parity bits are recomputed, and if they match what was received, it is a legal message. If they do not match, then the sum of the positions with incorrect parity gives the position of the single-bit error. For example, if 0001111 is corrupted into 0001101, then the parity bit for position 1 is correct, while those for positions 2 and 4 are incorrect. This means that the error is in position 2 + 4 = 6. Hamming codes allow error detection and correction while adding the minimum amount of information to the data. SEE ALSO Analog and Digital; Bases; Compact Disc, DVD, and MP3 Technology; Computers and the Binary System; Cryptology.

Shai Simonson

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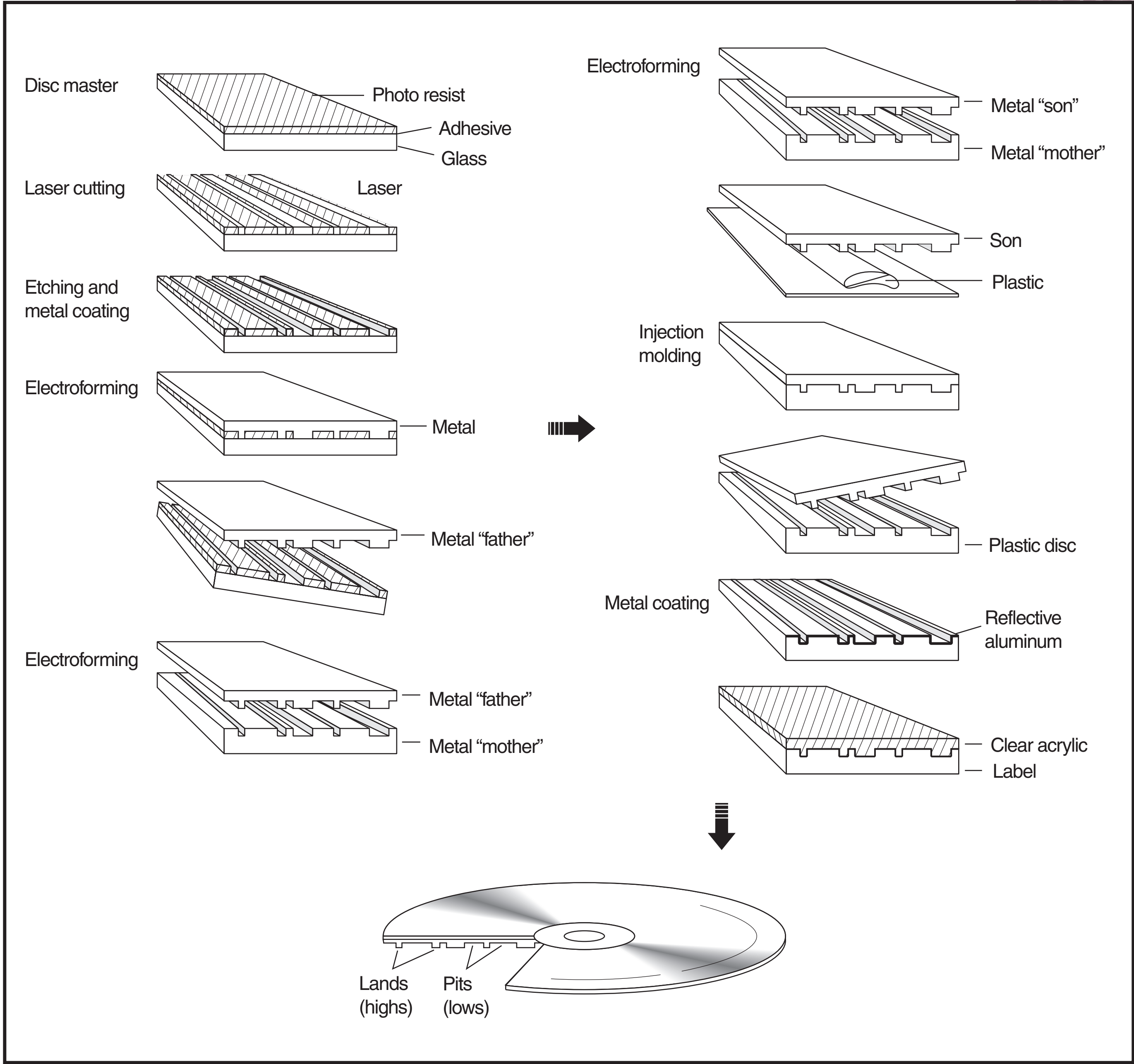
Compact Disc, DVD, and MP3 Technology

Compact discs (CDs), MP3s, and Digital Versatile Discs (DVDs) are currently some of the most popular ways to enjoy music and movies. Utilizing the latest in **digital** technology they have each been able to improve on the standards set by their predecessors.

digital describes information technology that uses discrete values of a physical quantity to transmit information



Compact Discs

A finished compact disc contains a series of tracks or indentations called "lands" and "pits." A CD player uses a laser beam to read these layers and convert the reflection first into an electrical signal and then into music.

A compact disc is a storage medium with digital data recorded on its surface. An optical beam reads the CD and reproduces the sound. Because the beam does not touch the CD, there is no wear and tear on the disc.

Compact discs store sound through a process known as Pulse Code Modulation (PCM). With this method, audio signals are sampled during short times intervals, and then converted into numerical data for storage in digital format.

To store audio in digital format requires large amounts of data. Optical discs were found to be ideally suited for this because they can store up to one million bits of data on an area as small as a pinhead.

How a CD Works. A CD can hold approximately 74 minutes of audio information. Not all of this information is music, though. Some of the space on a CD is used in error detection, synchronization, and display purposes.

Audio information on a CD is contained in the form of lands and pits (indentations). When it is read, a laser focuses on the lands and pits as the CD player's detector senses the differences in how they each reflect light. The detector then turns the reflected light into an electrical signal, which is then relayed to a circuit board and converted into sound.

Digital Versatile Discs

Digital versatile discs (DVDs) work in a manner very similar to CDs. Like a CD, a DVD is a disc layered with pits and lands. A DVD, though, can have more than one layer to it; a CD does not. Because of this, a DVD can hold about seven times more data than a CD.

How a DVD Works. A laser reads a DVD. The laser focuses on the bumps and flat areas of the disc, and a sensor in the DVD player senses differences in how these areas reflect light. These differences are then translated into video and audio signals, that the viewer watches as a movie.

Though DVDs have large storage capacities, uncompressed video data would never be able to fit on them. To put a movie on a DVD, it must first be encoded in the Moving Picture Experts Group (MPEG-2) format, which have standards for compressing moving pictures. Once a movie is in MPEG-2 format, it is stored on the disc. DVD players have a decoder in them that is able to uncompress the data quickly.

When an MPEG decoder uncompresses a movie, it must decide how it will encode the data. Each frame can be encoded in three ways: as an **intraframe**, **predicted frame**, or a **bidirectional frame**.

An intraframe contains the complete image data for that frame. This method provides the least **compression**. A predicted frame contains only data that relates to how the picture has changed from the previous one. In a bidirectional frame, data from the closest surrounding frames is looked at and averaged to calculate the position and color of each pixel.

The encoder decides to use different types of frames depending on what is being encoded. For example, if something is being encoded, and the frames stay similar from one to the next, predicted frames would be used.

MP3s

A Moving Picture Experts Group audio Layer-3 (MP3) file compresses music files into a size that can easily be downloaded by a computer. A 3-minute song from a CD requires 30 **megabytes** of data. To try and download this on a computer would take an immense amount of time. By compressing the data into a much smaller file, it can be obtained much quicker.

Music can be compressed by a factor of 10 to 12 as an MP3, and yet maintain the original CD-quality sound. The sound is compressed using a **compression algorithm**. This algorithm comes from a technique called **perceptual noise shaping**. With this technique the qualities of the human ear are taken into account. There are certain sounds the human ear can hear,

intraframe the compression applied to still images, interframe compression compares one image to the next and only stores the elements that have changed

predicted frame in compressed video, the next frame in a sequence of images; the information is based on what changed from the previous frame

bidirectional frame in compressed video, a frame between two other frames; the information is based on what changed from the previous frame as well as what will change in the next frame

compression reducing the size of a computer file by replacing long strings of identical bits with short instructions about the number of bits; the information is restored before the file is used

megabyte term used to refer to one million bytes of memory storage, where each byte consists of eight bits; the actual value is 1,048,576 (2^{20})

compression algorithm the procedure used, such as comparing one frame in a movie to the next, to compress and reduce the size of electronic files

perceptual noise shaping a process of improving signal-to-noise ratio by looking for the patterns made by the signal, such as speech

and sounds it can hear more than others. By recognizing sounds that will go unnoticed if removed, they can be omitted to make the file smaller.

CDs, MP3s, and DVDs have brought technology to the forefront of personal entertainment. By using the latest advances in technology they continue to improve what the public sees and hears. SEE ALSO Analog and Digital; Factors.

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Computer-Aided Design

Computers are sophisticated tools that are used in every facet of society to provide accuracy, flexibility, cost-savings, efficiency, record-keeping, vast storage, decision-making tools, and modeling. The use of computers to design two- or three-dimensional models of physical objects is known as computer-aided design. Designers in architecture, electronics, and aerospace or automotive engineering, for example, use computer-aided design (CAD) systems and software to prepare drawings and specifications that once could only be drawn or written by hand.

Before CAD, manufacturers and designers would have to build prototypes of automobiles, buildings, computer chips, and other products before the prototypes could be tested. CAD technology, however, allows users to rapidly produce a computer-generated prototype and then test and analyze the prototype under a variety of simulated conditions.

Manufacturers and designers also use CAD technology to study optimization, performance, and reliability problems in prototypes and in already existing designs. Drawings can be modified and enhanced until the desired results are achieved. CAD is very useful for prototyping because it allows designers to see problems before time and money is invested in an actual product.

A Short History of CAD System Development

The aerospace industry and mechanical engineering problems figured prominently in the early development of CAD system applications. The earliest uses of this technology demonstrated the need for fast computers with the ability to process large quantities of data. In early CAD systems, computer programs used numerical methods to simulate real events, evaluate and analyze structures, and calculate optimal performance levels.

In the 1950s the first CAD system—a graphic air defense system—was developed for the U.S. Air Force. In the 1960s CAD systems incorporated two-dimensional **geometry** to depict the height and width of calculated objects, and CAD became a tool for drafting applications. By the 1970s many

geometry the branch of mathematics that deals with the properties and relationships of points, lines, angles, surfaces, planes, and solids

Computer-aided design allows scale models to be built, tested, and refined before the product is actually manufactured.

automated design and drafting systems became available. The 1980s brought the first CAD system to be available on a personal computer rather than a mainframe computer. By the 1990s the use of CAD systems had expanded beyond engineering applications to the architecture industry. What once was a need for better database manipulation was now a need for three-dimensional geometry and algebraic manipulations.

Basics of CAD Systems

Architectural renderings and drawings can be modeled on computers as searching, selection, and computational tasks. Searching tasks are represented as sets of variables with acceptable values. The design software selects alternatives using a generate-and-test process.

Design also takes the form of optimization. In these instances rigid structural limits are imposed within a narrow design space. Computations are performed to predict performance and redesign plausible solutions.

Searching, selecting, and optimizing are iterative, but compilation is not. Compilation uses programmed algorithms to translate facets of a problem into machine language and arrive at a workable solution.

Mathematical Models. Numerical functions that are the basis for mathematical models are at the heart of CAD tasks. Mathematical models define the basic building blocks of three-dimensional (3-D) models. Points, curves, and lines are just a few of the predefined elements that are manipulated geometrically in CAD Systems.

DATA-INTENSIVE CAD SYSTEMS

Vast amounts of library data are used in CAD systems. These\ data can contain numerical methods, graphics, geometry, an interface, database manager, and a data structures toolbox.

Points are usually represented by either two or three values depending on whether a two-dimensional (2-D) or 3-D model is being used. Curves are one-dimensional entities that smoothly connect moving points. Lines are entities that have uniform endpoints. Many other geometric models are used in CAD systems to represent circles, arcs, different types of curves, polygons, and other fundamental entities that are used to construct images.

Many mathematical models exist to handle basic operations such as **interpolation** and root finding. Interpolation allows CAD programs to make "educated" guesses about a function's or object's location in a coordinate system from two known values. Curves and boundary points can be used to determine surface spaces—a feature that is very useful in design and manufacturing applications. **Numerical differentiation** approximates the derivative of an unknown function, and can be used in problems involving heat transfer, for example. Integration, a primary task in calculus, uses known and unknown values to solve equations that result in precise calculations for analytical approximations. Root finding techniques can be useful for computing changes in signs, determining convergence, and solving polynomial equations.

interpolation filling in; estimating unknown values of a function between known values

numerical differentiation approximating the mathematical process of differentiation using a digital computer

Displaying 3-D Objects. After mathematical modeling develops a rendering of a computer image, it is stored and displayed. Two methods are used to display two- and three-dimensional objects: raster and vector graphics.

Raster graphics save the image in a **matrix** as a series of dots or pixels. The pixels are stored as bits, and the compilation of bits is known as a **bitmap**. Bitmap images have a photographic type of quality with smooth lines, rich colors, and textures. Bitmap images can be manipulated at the pixel level, and their conversion to their binary representation requires a large amount of computer processing. Each pixel in a bitmap is highlighted and projected on a display screen. All pixels comprising an image appear as a continuous image. An electron beam traces the pixels of an image on the display screen.

matrix a rectangular array of data in rows and columns

bitmap representing a graphic image in the memory of a computer by storing information about the color and shade of each individual picture element (or pixel)

Vector graphics save images as collections of lines, circles, and intersecting angles. Usually the clipart found in most word processing packages serves as a good example of a vector graphic. In contrast to the photographic quality of bitmap images, vector graphics lack the realistic appeal and smooth surfaces of raster graphics. The lines that create the graphic image are stored as two points {x, y} in a matrix. An electron beam traces over the points several times and the pathway becomes illuminated, and this process produces the lines that make the images.

Future Directions for CAD

Virtual Reality (VR) is a 3-D visualization tool that is used to construct buildings, manipulate objects in real-time, and build prototypes. Certain features of virtual reality are likely to appear in future CAD packages once standards and protocols are developed. This scenario is desirable to reduce the duplicity involved in transferring geometric data between CAD and VR systems.

In addition to the inclusion of VR technology, further advances in CAD technology could include Object-Oriented principles that allow data, data

structures, and methods to be encapsulated and features that allow 3-D objects to gain access to the properties, member variables, and associated data through class designations. Another development for future CAD systems might include the use of databases containing building modules that can be shared by both VR and CAD systems.

Future developments in CAD systems might also yield more sophisticated systems for building complex, irregular forms. Such a development along with rapid prototyping, combinations of VR visualization techniques, and **artificial intelligence (AI)** might make CAD systems more responsive and more powerful development tools in manufacturing, design, and engineering environments.

artificial intelligence the field of research attempting the duplication of human thought process with digital computers or similar devices; also includes expert systems research

Other advances in CAD technology might involve sharable web-based design libraries and widespread use of agent-based design and problem-solving approaches. In agent-based approaches, the computer system would function like a knowledgeable partner that would work collaboratively to solve problem and address concerns. The agent would be a type of knowledge base that would store information and model planning and integration activities.

Distributed AI applications may be the main focus of the next generation of CAD tools because of their ability to codify and represent knowledge structures. These complex systems have the ability to "learn" or change their rule sets based on inconsistencies, inaccuracies, and incomplete data. Applying learning to CAD applications may help reduce the problems of planning and system malfunctions due to the introduction of inaccurate and incorrect data.

Other trends in CAD systems may include the application of formal optimization algorithms to manufacturing activities to improve design features of a product and reduce its costs. The application of **heuristics** or "rules of thumb" to scheduling and sequencing tasks may improve CAD systems in the near future. SEE ALSO Computer Simulations; Virtual Reality.

heuristics a procedure that serves to guide investigation but that has not been proven

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Computer Analyst

Computer analysts, also called systems analysts, plan and develop computer systems. Their work includes installing new systems or upgrading existing systems to meet changing business needs. They also computerize tasks that are still being done by hand or in some other less efficient way.

Computer analysts may either work for themselves or for computer and data processing companies. Also, they may work for government agencies,

manufacturing companies, insurance companies, financial businesses, and universities. A computer analyst normally needs at least a bachelor's degree in computer science or in a related area such as management information systems. An advanced degree further increases the chance of finding a good job.

Computer analysts study whatever problems a company might be having and then design new solutions using computer technology. This process includes meeting with people to discuss the problems and needs of their company. Analysts then use techniques, such as structured analysis, data modeling, mathematical modeling, and cost accounting, to plan systems. They may select both hardware and software to implement their plans.

Analysts then prepare reports, generally using charts and diagrams, to present the proposed system to the people who will use it. Analysts may prepare cost-benefit analyses to help management decide whether the system is acceptable. All of these tasks usually involve complex mathematics.

Once a system is chosen, the equipment is purchased and installed. Then the analyst must make sure that the entire system works correctly. The analyst prepares diagrams and instructions so that computer programmers can write the code necessary to make the parts operate as a system.

Some analysts are programmers, too. As the programs are written, analysts set up test runs of various parts of the system. After the new computer systems are in place, analysts may be asked to continue supporting the new systems for months or even years. SEE ALSO Computer Programmer.

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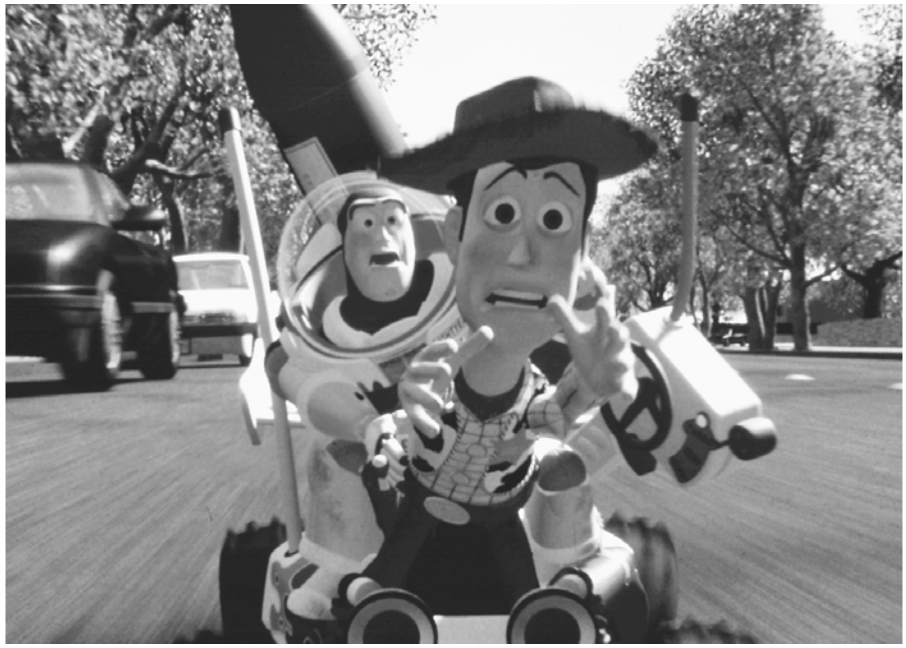
Computer Animation

Even a basic understanding of the technology, the complex mathematics (expedited by computer software), and the hours of manpower behind the making of a movie such as *Toy Story* (1995) or *A Bug's Life* (1998) could change the way a viewer watches computer-animated movies. Knowing more about computer animation helps one better appreciate scenes in which Buzz and Woody careen down the street, passing other cars and whizzing by trees in a background so near to real life that it is easy to forget that the entire movie is fully computer animated.

Animation

The word "animate" means "to give life to," and animating is the process of moving something that cannot move itself. Computer animation creates the illusion of movement through a succession of computer-generated still images. Fully appreciating the high-speed efficiency and the complexity of computer animation requires a basic understanding of how animation was achieved before the days of the computer.



This chase scene from the 1995 computer-animated movie *Toy Story* illustrates the use of single-point perspective, depth of field, blurring, and other techniques rooted in principles of mathematics, physics, and the science of human perception. Rendering the movie's 120,000 frames took 38 years of computing time distributed among many computers.

In traditional animation, sequential images were painted or hand-drawn on paper or plastic sheets called "cels." They were then filmed, one frame at a time, and played back at high speeds. This tricked the eye-brain response into perceiving movement of the characters and other objects displayed in the drawings, thus creating the illusion of movement.

The underlying process involved in preparing a computer-animated production has not changed much from the process of traditional hand-drawn animation. What is different is the speed with which it is done. It would take many, many pages of hand-written mathematical equations to illustrate the work that a computer does in a fraction of a second.

interpolate fill in; estimate unknown values of a function between known values

The Computer Animation Process

Computer animation begins with an idea, followed by the preliminary story. Next, the action scenes are sketched out in frames, with corresponding written explanations, thereby creating a storyboard. Then the detailed story is developed, the sound track is completed, and the key frames are identified. Another set of animators will later do the "in-betweening," which is the work of determining—through mathematical computations—a series of midpoint locations between key frames at which images must be **interpolated** to create more fluid movement.

Computer animation depends on a combination of scientifically based, mathematically calculated and produced steps. Computer animators focus on making three main determinations:

- how to make a single object on a two-dimensional screen look realistic;
- how to make the object's entire environment look real; and
- how to add realistic movement to the objects and scenes.

VIRTUAL HUMAN ACTORS

The 2001 computer-animated movie *Final Fantasy: The Spirits Within* used motion-capture analysis and other advanced animation software to simulate a full range of human emotions and movements. One-fifth of the production budget was devoted to making 60,000 strands of hair on the character of Dr. Aki Ross's head move naturally.

The clever animator knows the effects yielded by a range of "tricks of the trade"—all of which are based on scientific knowledge of perception.

Computer animation capitalizes on a number of principles of human perception in order to trick the viewer into believing that the images seen on a flat, two-dimensional screen are three-dimensional: that is, in addition to height and width, the picture also has depth. Whereas two-dimensional images are useful in providing certain types of information, such as simple directions found in billboards or signs, three-dimensional images provide a great deal more information, and the effort required to provide this information is considerably greater.

The Illusion of Surface. A three-dimensional image begins as a geometric shape, or a combination of shapes—squares, rectangles, **parallelograms**, circles, **rhomboids**, and especially triangles. An image of a human body might require a combination of thousands of shapes put together into a structure called the wireframe, which in its early development may be recognizable only as the symbol of whatever it will eventually depict. Next, the wireframe is given color and surface texture, and the effects of lighting, or reflectance, is determined. The wireframe surfaces must be lit from all possible angles to create the desired effect. The use of shading and shadow is especially important in enhancing the appearance of reality. Shading is what makes a ball look round or the folds in a blanket look soft and deep, whereas shadow produces the illusion of weight. The variations of these features, used according to mathematical models, can also give the impressions of warm and cold, soft and hard, smooth, scaly, and so forth.

parallelogram a quadrilateral with opposite sides equal and opposite angles equal

rhomboid a parallelogram whose sides are equal

The Illusion of Distance. Computer animators use the age-old knowledge of perception in creating an illusion of distance. Animators use "single point perspective," which is an everyday occurrence in which objects at a distance appear to **converge** to a single point just before they fade from view. By calculating the relative sizes of objects to make some appear nearer (larger) and others farther away (smaller), animators can mimic this natural principle of perception.

converge come together; to approach the same numerical value

Depth of field is used to diminish the size and clarity of objects at greater and greater distances from the viewer. In **anti-aliasing**, the computer adds grayed-out pixels (the smallest individual picture elements on the computer screen) to smooth out the naturally angular shapes that comprise computer images, thus creating the fuzzy illusion associated with distance. For example, trees at a distance appear to be smaller and fuzzier than at close range.

anti-aliasing introducing shades of gray or other intermediate shades around an image to make the edge appear to be smoother

The Illusion of Movement. An animator knows that the retina of the human eye can retain an image for about 1/24th of a second. So if images, or frames, are run in sequence before the viewer's eyes at a rate of twenty-four frames per second, the illusion of continuous movement and action will be produced. The addition of "blurring," produced by speeding up the exposure of images to thirty or even sixty frames per second, heightens the illusion of speed. An example is seeing a residual blur of a car that has just sped away. The complexity of the task, of course, is multiplied when groups of images must be animated to move together as a body and yet maintain each individual's unique appearance and behavior.

SOLVING PRINCESS DIANA'S DEATH

Much of the uncertainty regarding the 1997 automobile accident in Paris, France, that resulted in the death of Britain's Princess Diana was resolved through computer animation. The critical question of how fast the vehicle was traveling at the time of the accident, at first reported to have been between 55 and 140 miles per hour (mph), was later determined through applications of three-dimensional computer graphics to have been between 80 and 86 mph.

Computer graphics also allowed engineers to generate a realistic animation of the path of the car from its initial swerve to its final stop. By recreating the motion of the passengers in the car at the moment of impact and just following it, animation revealed that the princess hit the back of the driver's seat at approximately 50 mph. Had she been wearing her seatbelt, it is probable that she would have survived the accident.

The massive wildebeest stampede in *The Lion King* (1994), for example, used a computer to replicate the hand-animated wildebeest from many different angles, creating differences in appearance among them. Similarly, in the crowd scene in *The Hunchback of Notre Dame* (1996), a number of base characters were programmed into the computer. A program called CROWD then scrambled actions randomly so that the characters in the crowd were doing different things: laughing, clapping, jumping, and so forth.

The Future of Computer Animation

Although computer animation is often thought of in terms of video games and high-action animated movies, it is easy to overlook its many other applications. In fact, computer animation is used extensively in crime investigation, and applications are being developed steadily for fields such fluid dynamics, thermodynamics, meteorology (for the prediction of severe storms and other weather patterns), and the legal system (to reconstruct crime scenes and accidents).

Though remarkable strides have been made in the field of computer animation, technology is currently not able to keep up with the demands for even faster and more powerful computers. The movie industry, in particular, continues searching for ways to increase the speed of production. Although current technology has cut huge chunks of time out of production, still more power and speed are needed.

Computer graphics pioneer Matt Elson predicts that in the first half of the twenty-first century, three-dimensional characters will be created that will have the ability to "think" and to interact with their environments. These virtual characters will be able to understand and to respond to human speech and will be able to perform tasks and take care of many of the details of everyday human life. Elson predicts that when computer and software companies across the world pool their discoveries, a digital character with a "mind of its own" will become a "real" part of the everyday world of its human counterparts. SEE ALSO Computer Simulations; Computers, Future of; Virtual Reality.

Paula C. Davis

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Computer Graphic Artist

Computer graphic artists use computers to create designs that meet the needs of their clients. They may be self-employed, or may work for publishing companies, printing companies, art and design studios, advertising agencies, design firms, commercial art and reproduction firms, wholesale and retail trade establishments, and public relations firms. Computer graphic artists generally have a bachelor's degree in art or design.

Computer graphic artists design promotional displays and marketing brochures for a company's products and services. They also design company logos for products and businesses. A graphic artist may create visual designs of annual reports and other corporate documents. In addition, they design packaging for products such as food or toys.

Some computer graphic artists develop the overall layout and design of magazines or newspapers. Other artists develop the graphics and layout of Internet web sites. Some computer graphic artists produce the credits that appear before and after television programs.

Computer graphic artists use a variety of electronic tools to produce creative work for their clients.

All of these design tasks require precise mathematics in laying out the various sections of text and pictures. For example, a computer graphics artist working for a newspaper may have to equally space two 3.1-inch wide text columns with a 1.6-inch picture between them. They must account for margin widths, the overall size of the pages, and the type of printing machine that will be used.

Computer graphic artists use computers for designing, sketching, and image manipulation. A computer creates a reproduction of an object by executing a program that uses an **algorithm** based on a mathematical formulation of the object. The algorithm may transform a three-dimensional object so the computer can create the two-dimensional picture seen on the computer monitor screen. In the past, graphic artists performed the trigonometric operations on lists of coordinates by hand. Now, graphic artists use computers to speed up the calculations and to model more complex objects. SEE ALSO Computer Animation.

algorithm a rule or procedure used to solve a mathematical problem

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Computer Information Systems

The key word when dealing with computer information systems is "information." How many people want it? How can they access it? How can they be sure it is accurate? What role does mathematics play?

Large numbers of data records and the huge amount of information they contain require computers and mathematics to manage the data efficiently.

Millions of people want a variety of information. The greater the number of people who want the information, the more a computer is needed to store and retrieve it quickly, and the greater the chance that mathematics will be used to enable the entire process. It was estimated that in 1998, 100 million people around the world used the Internet. As the number of Internet users increases to one billion by 2005, terms such as Internet, Super Information Highway, the World Wide Web, or e-commerce (electronic commerce) will merely be a reference for the basic notions of computer information systems.

Computer information systems are several subsystems that rely heavily on mathematical processes to manage data or information: database

management, software engineering, telecommunications, systems analysis, and decision support systems. Statistics, set theory, **Boolean operators**, modular arithmetic, and **cryptography** are some of the mathematics that assist in dealing with computer information systems.✶

Boolean operators the set of operators used to perform operations on sets; includes the logical operators AND, OR, NOT

cryptography the science of encrypting information for secure transmission

✶**Boolean logic and Boolean algebra are named after the British mathematician George Boole (1815–1864).**

Statistics

Statistics is a branch of mathematics that involves gathering, summarizing, displaying, and interpreting data to assist in making intelligent decisions. Examples of the use of statistics in computer information systems include the following:

- gathering data on samples of manufactured products to determine the efficiency of the production process;
- predicting the outcome of presidential elections prior to the total count of votes;
- determining different rates for automobile insurance for male and female teenage drivers; and
- gathering data on automobile accidents that occur while driving using a cellular phone.

Consider the consequences if data are not gathered, summarized, or interpreted properly before decisions are made. For example, there was a major recall on tires produced by a certain manufacturer. Is it possible that data were not gathered under the proper conditions to have an advance warning about the potential for the tire tread separating from the rest of the tire?

Once the data are gathered, statistics is used to look for patterns, determine relationships, and create tables, graphs, and mathematical formulas to report data or predict additional data. The National Highway Traffic Safety Administration (NHTSA), for example, uses data to identify safety issues and monitor trends and the effectiveness of safety regulations.

Statistics is applied to the traffic safety data found with the primary database tools: the Fatal Analysis Reporting System (FARS), the National Sampling System (NASS), and crash reports collected by the states. For example, NHTSA used mathematics to calculate and compare the percentages of the overall 217,651 police-reported crashes in Oklahoma for the period of 1992 to 1994 with the 299 crashes in which cellular telephones were being used. It was revealed that 17 percent of the cellular phone–related accidents resulted from lack of attention, whereas only 9 percent of all accidents in the state were attributed to that circumstance. This information may influence drivers, lawmakers, and cellular phone manufacturers.

Set Theory and Boolean Operators

set theory the branch of mathematics that deals with the well-defined collections of objects known as sets; of particular interest are the infinite sets such as the rational numbers and the real numbers, a field investigated by Georg Cantor

How can people access information? First, the location of the data must be determined and then the data must be searched. Masses of data, words, pictures, and sounds have been collected by businesses, libraries, schools, and governments and made available on the Internet. For example, the information on the cellular phones and traffic accidents briefly described above was obtained on the Internet. The mathematics that helps computers to organize and retrieve information in efficient ways is called **set theory** and uses the mathematical concept known as Boolean operators.

To find the information about cellular phones, conduct a web search for "cellular phones." Entering "cellular phones" in the search window indicates that the set of data related to cellular phones is the data in which the searcher is interested. The web has been programmed by software engineers so that if "cellular phones" is entered as a search, it is understood that the set of web sites desired contain data about both terms—cellular *and* phones.

Web searches by default use "and" as the linking operator for search terms. Consequently, when "cellular phones" is entered in the search window, the search should return web sites that reference both terms. There is no guarantee that the search will result in the desired hits, but there is a good chance of getting useful information, although it may be intermixed with less useful data. For instance, a search for "cellular phones" may also yield a web site about *cellular* processes in science that were discussed in a *phone* conference.

To locate web sites that contain information about cellular phones and auto accidents, the searcher must employ the Boolean operator *and*. To investigate web sites about auto accidents regardless of whether they include information about cellular phones, the searcher would use the Boolean operator *or*. The results of the latter search should yield web sites about cellular phones, web sites about auto accidents, and web sites that contain both cellular phones and auto accidents.

In addition to these two rather typical Boolean operators (*and* and *or*), there is another seldom used operator: *and not*. If one wanted to search for web sites that contained information about "cellular" but not "phone," one could enter a web search for "cellular *and not* phone."

Knowing the basics of set theory and Boolean operators that underlie information management can increase the chances of getting a productive web search. In fact, the new web address that appears in the address window after a search will contain some shorthand for these Boolean operators. A search for "cellular *and* phone" will be reflected by "cellular+phone" in the address window. The shorthand for *and not* is a minus sign: "cellular−phone."

COMMON TERMS IN SET THEORY

Direct and indirect connections to the common terms in set theory include:

- *union*—combining information from all sets requested
- *intersection*—reporting only the common elements in all sets requested
- *complement*—everything that is available except elements in certain identified sets
- *empty or null sets*—sets that do not have any data in them
- *disjoint sets*—sets that do not have elements in common

Modular Arithmetic

There are at least two other concerns relating to computer information systems. How can the accuracy of the information be determined, and how can it be secured so that others cannot change it or remove it? The latter becomes particularly important when purchases are made on the Internet with credit cards or when private information is requested. But in this discussion only the former question will be addressed.

Information is often encoded in special ways to both ensure accuracy and to provide a quick way of communicating information electronically. For example, many common items bear special arrangements of vertical bars of various heights to represent such identifiers as zip codes, Universal Product Codes (UPCs), and International Standard Book Numbers (ISBN). Each of these numbers along with their special symbolic coding can be read quickly by a machine and thus can be quickly processed for computer information systems.

The Universal Product Code uses digital technology to uniquely identify products, services, and information. Today, everything from automobile parts to college application forms to insurance cards to utility bills use these bar codes.

Computer information systems use problem solving, which is a mathematical process, to detect the patterns in the bar codes. Many times, however, a special digit is included in the code merely as a "check digit." The check digit helps to signal when an error in the code is present.

For example, consider the following ISBN number for a certain book: 0-412-79790-9. The last digit here, 9, is a check digit. Try multiplying the digits of the ISBN number by the counting numbers, counting down from 10 to 1 and finding the sum of each product starting with the left-most digit:

$$10(0) + 9(4) + 8(1) + 7(2) + 6(7) + 5(9) + 4(7) + 3(9) + 2(0) + 1(9)$$

If the sum (in this case, 209) can be evenly divided by 11 (with a remainder of 0), then the ISBN number is correct. Now investigate the ISBN of this encyclopedia. SEE ALSO BOOLE, GEORGE; COMMUNICATION METHODS; CRYPTOLOGY; INTERNET DATA, RELIABILITY OF; STATISTICAL ANALYSIS.

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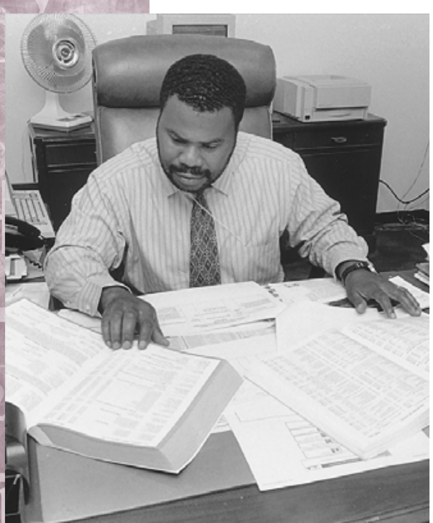
Computer Programmer



A computer programmer must gather and assess numerous data before designing a computer solution specific to the needs of the end user.

Although a computer can do multiple tasks at high rates of speed, the computer is only as smart, quick, and capable as the person who programmed it. Behind every well-running computer is a series of programs (instructions) that tell the computer what to do and how to do it. Computer programmers write these programs, and the efficiency with which the program is written determines how quickly instructions can be executed.

Programmers cannot write programs in plain English but have to use a language that the computer will understand. There are many different computer languages—such as Basic, C++, Perl, Visual Basic, Pascal (named in honor of Blaise Pascal's [1623–1662] achievements in mathematics), and Fortran—used in developing programs. These languages differ in their capabilities and commands, but they can all be broken down into machine code, which is a series of symbols and text.

Along with computer languages, computer programmers must know and understand mathematical formulas and operations. Programs use these operations, but in order for a command to be executed, it must first be entered correctly. An example of specific concepts used in If-Then loops are variables, comparing two numbers, and consecutive numbers. Logic and se-

quencing is also used. Some programmers deal with graphics and use dimensions, perspective, and scale to create their work.

A career in computer programming can be exciting and lucrative, but remember that the computer is only as smart as the programmer allows it to be. SEE ALSO Computers and the Binary System.

Elizabeth Sweeney

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Computer Simulations

Physical models such as crash tests are supplemented by computer models and simulations.

In simulations, people are confronted with real-life situations and can take risks without having to suffer the consequences of failure. People can experiment with dangerous chemicals on the computer screen, for example, and not be in danger from the actual chemicals. With laboratory simulations, there is no expensive lab equipment to buy and students do not have to wait as long for the results of an experiment. Simulations save time and money, reduce risks, and work well as aids in decision-making.

A simulation is an abstraction or imitation of a real-life situation or process. Computers can be used to simulate complicated processes such as the spread of a virus throughout a city, the long-term effects of urban renewal, or the economy of an entire country. In simulations, participants usually play a role in which they interact with other people or with elements of the simulated environment. A business management simulation, for example, might put participants into the role of executive of a mythical corporation, provide them with information about the corporation, and ask them to negotiate a new labor contract with a union bargaining team.

A well-designed simulation models the elements most important to the objective of the simulation, and it informs the participants about elements that have been simplified or eliminated. A complicated simulation might be too time-consuming for the intended audience, whereas an over-simplified simulation may fail completely to communicate its intended point.

One particular value of simulations is that they implement the problem-based learning method as directly and clearly as possible. In problem-based learning, one is taught to understand principles by being given a problem situation. Most simulations attempt to immerse participants in a problem.

Through simulations, participants are offered a laboratory in areas such as the social sciences and human relations as well as in areas related to the physical sciences, where laboratories have long been taken for granted. The great advantage of simulated immersion in a topic is that individuals are more likely to be able to apply to real life what they have practiced in simulated circumstances.

This raises the issue of the degree of realism captured by a simulation. Simulations* can vary greatly in the extent to which they fully reflect the realities of the situation they are intended to model. A common defect in poorly designed simulations is an overemphasis on chance factors in determining outcomes. Because many of the processes in the real world depend

***Although there is really no strict dividing line between a simulation and a model, a simulation is usually more complex and may contain models.**

random without order

at least partly on **random** events, a successful simulation includes an appropriate amount of random numbers to help best replicate a real-life situation.

Benefits of Simulations

Simulations typically have four main advantages over more conventional media and methodologies such as books, lectures, or tutorials:

(1) simulations tend to be more motivating;

(2) simulations enhance transfer of learning;

(3) simulations are usually more efficient; and

(4) simulations are one of the most flexible methodologies, applicable to all phases of instruction and adaptable to different educational philosophies.

It is well known and not surprising that simulations enhance motivation. Participants are expected to be more motivated by active involvement in a situation than by passive observation. Trying to fly a simulated airplane, for example, is more exciting than reading about flying. Trying to diagnose and treat a simulated patient is more exciting than attending a lecture about it.

Several motivational elements can found in most simulations, including fantasy, challenge, and relevance. Realistic fantasy (imagining oneself in an interesting activity) is a part of most simulations and is a function of the simulation storyline or scenario. Simulations that increase in difficulty as the participant progresses offer challenge as a motivation. Most learners consider simulations more relevant to their learning than lectures, books, or other passive methods, because they are engaging in the activity rather than just reading or hearing about it.

gaming playing games or relating to the theory of game playing

Many simulations include **gaming** techniques. These simulations are particularly popular in business education because the competition inherent in games is also a factor in the real world of business, including competition among firms, between management and labor, or among salespeople. Such simulation games have the potential to be more motivating than more conventional instruction, such as lectures and reading assignments.

Simulations are commonly used in scientific fields such as astrophysics, in which funding and logistical constraints may prohibit the actual testing of a new technology.

"Transfer of learning" refers to whether skills or knowledge learned in one situation are successfully applied in other situations. Simulations have good transfer of learning if what was learned during the simulation results in improved performance in the real situation. For example, a gardening simulation—in which one manipulates soil acidity, the exposure to sunlight, and the amount of watering—would have a better transfer of learning than reading a gardening book. The simulation gives one practice in gardening and the opportunity to try out different combinations of conditions and care. The book, however, only provides information and hints on how to so.

The term "transfer of learning" is often used in reference to quite different ideas. The term "near transfer" refers to applying what is learned to very similar circumstances. The term "far transfer" refers to applying what is learned to somewhat different circumstances, or generalization of what is learned. Simulations can be designed to optimize either type of transfer.

The idea of transfer of learning can be taken a step farther. Not only can one measure how effectively knowledge, skill, or information transfer from one situation to another, but one can also measure how efficient the initial learning experience is with respect to the transfer. This is best illustrated with a hypothetical example.

Suppose a teacher has two different classes for one chemistry course. The teacher gives one class a series of interesting and informative lectures dealing with a specific laboratory procedure. In the other class, the teacher provides a computer program with the same information and includes a simulation of the laboratory. On completing their respective forms of instruction, each class of chemistry students performs the procedure in a real laboratory. Both classes perform well on the actual laboratory experiment. On the basis of this information one could conclude that both instructional methods have the same transfer of learning.

However, if the lecture series took 10 hours and the average time to complete the simulation required only 5 hours, one might conclude that the simulation was more time efficient. That is, more transfer occurred per unit of learning time with the simulation than with the lectures. Although simulations do not guarantee time efficiency, there is evidence that well-designed simulations do foster it.

The last advantage of simulation is its flexibility. Simulations can satisfy more than one of the four phases of instruction. In fact, they usually satisfy at least two: either initially presenting material and guiding the learner through it (Phases 1 and 2) or guiding the learner through previously learned material and providing practice with it (Phases 2 and 3).

When simulations do provide initial instruction, they frequently do so by the scientific discovery learning or experimentation approach. Most other repetitive simulations are examples of this. Not all simulations teach in this way, however. Some provide extensive help or tutorial components that learners may access at any time and according to their own choice.

A simulation about road signs and driving laws might introduce the signs and rules, guide the learner in their use, and provide practice by letting the learner use the simulation over and over until familiar with the laws. Many simulations have this characteristic. If used once, the simulation presents information and guides the learner in its acquisition. If used repeatedly, it takes on the characteristics of a drill, and, like a drill, some simulations require the participant to continue until proficiency is demonstrated.

Spreadsheet Simulations

When one thinks of simulation software, one often thinks of expensive, complex software. But simulations that demonstrate manipulation of one or more **variables** can be created using only a spreadsheet. Spreadsheet simulations allow one to manipulate a variable and see the resulting numerical change or change in a graph.

variable a symbol, such as letters, that may assume any one of a set of values known as the domain

For example, a man on Earth weighing 160 pounds will have a different weight on Earth's moon or on another planet. This concept may be difficult to imagine. Using a spreadsheet, however, one can create

realistic examples and problems to help understand and apply the concept of gravity.

A spreadsheet simulation can be used to teach students how to make predictions based on the rules or laws they have discovered. For example, students can use atmospheric data to predict the weather or use **demographic** and historical voting data to predict the results of an election. Students have even used spreadsheet simulations to determine the probability of winning the largest bingo prize.

demographics statistical data about people—including age, income, and gender—that are often used in marketing

Spreadsheet simulations can also be used to explore mathematical relationships. One can create a spreadsheet to both calculate and plot the relationship between an unknown (*x*) and its coefficient and constants. As the values change, the equation is solved for (*x*), and the relationships between the coefficient, constants, and *x* are displayed in a graph that is automatically updated with each change.

Spreadsheets are useful for teaching math concepts such as surface area and volume problems and **polynomial** problems. Teachers can also use spreadsheets to teach math concepts in classes such as economics. SEE ALSO Computer-Aided Design; Computer Animation; Virtual Reality.

polynomial an expression with more than one term

Locord D. Wilson

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Computers and the Binary System

The basic architecture of today's electronic computers originated in a design proposed in the 1830s by Charles Babbage (1792–1871), a British professor of mathematics. The ideas that he wanted to embody in a huge steam-powered mechanical computing device were finally realized only in the mid-twentieth century, using relays and electronic circuits. Since then, computers have evolved from room-sized **mainframes** to desk-sized **minicomputers** to today's laptop and palm-sized **microcomputers** and the miniature digital **signal processors** found in many appliances.

mainframes large computers used by businesses and government agencies to process massive amounts of data; generally faster and more powerful than desktops but usually requiring specialized software

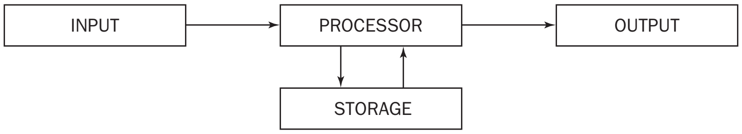
minicomputers a computer midway in size between a desktop computer and a main frame computer; most modern desktops are much more powerful than the older minicomputers and they have been phased out

microcomputers an older term used to designate small computers designed to sit on a desktop and to be used by one person; replaced by the term personal computer

signal processor a device designed to convert information from one form to another so that it can be sent or received

From the standpoint of the computer user, the most important aspects of the computer are still those recognized by Babbage: input, processing, storage, and output. Anyone who wants to use a computer for information processing is primarily concerned only with getting the information in, getting it processed, storing intermediate values, and using the results. Achieving these results depends on the interworkings of transistors, electric currents, coding systems, **binary numbers**, computer programs, programming languages, multiprocessing, and operating systems.

binary number a base-two number; a number that uses only the binary digits 1 and 0



Input, Processing, Storage, and Output

All electronic computers use the same fundamental architecture that Babbage recognized nearly 200 years ago. One or more input devices, such as a keyboard or mouse, handle entry of information (data) into the machine. A processor determines or computes new values from entered values and from other previously computed values. Storage elements retain entered and computed data values. Finally, output devices send entered and computed values to a printed page, an electronic display, a telephone line, a loudspeaker, or wherever they can be used by people or other machines.

In today's personal computers, the input devices may include a keyboard, mouse, joystick, trackball, light pen, touch screen, infrared detector, telephone line, microphone, scanner, magnetic diskette, and CD-ROM* disk. Input can take several forms—audio, video, text, numbers, and graphic images. New input devices are also continually being developed.

***CD-ROM stands for Compact Disk–Read-Only Memory.**

Processing reflects the computer's ability to combine and manipulate data to convert it into useful information. The main processor is an **integrated circuit** (IC) chip or set of chips, often called a CPU (Central Processing Unit). Other IC chips do some processing as part of the input and output devices.

integrated circuit a circuit with the transistors, resistors, and other circuit elements etched into the surface of a single chip of silicon

Storage, either on a short-term or long-term basis, is provided by IC chips that only store data and by magnetic disks (hard or fixed disks and removable diskettes), magnetic tapes, and writeable compact disks. Output, or the results of data manipulation, is provided by means of data terminal screens, printers, telephone lines, infrared ports, and loudspeakers.

Hardware

Input and output (I/O) devices described earlier are all part of the hardware of the computer—that is, they are realized in pieces made of plastic, metal, glass, and special materials (such as silicon and germanium) that can be seen and touched. I/O devices contain a wide variety of parts, large and small, many of which could be seen in a disassembled printer, mouse, or other I/O device.

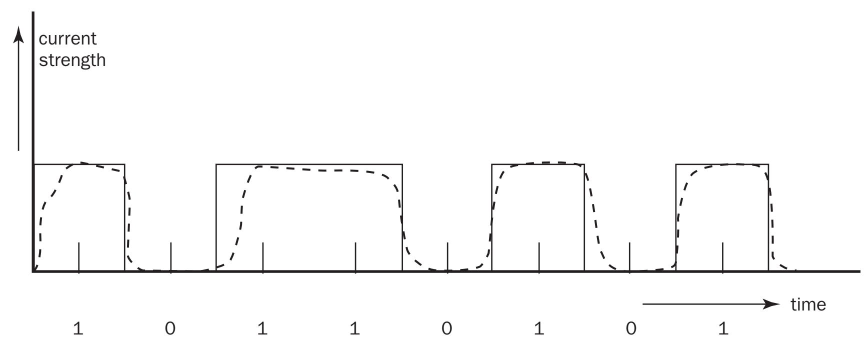
The basic electronic element in integrated circuit (IC) chips—whether used for processing or for storage—is the **transistor**. One IC chip may contain thousands or millions of transistors. The great power of the modern computer comes from being able to put enormous numbers of very tiny (miniaturized) transistors into a single plastic case, often no bigger than a pencil eraser.

transistor an electronic device consisting of two different kinds of semiconductor material, which can be used as a switch or amplifier

The basic function of a transistor is to control the strength of a small electrical current using a small electrical voltage. Transistors in a radio or audio player control the production of sound from radio waves, magnetic tapes, or compact disks. Transistors in a computer control electrical currents that represent information in the form of numbers and letters.

A radio or an audio player may produce sound imperfectly if there are electrical disturbances nearby, such as fluorescent lights or electric motors. These imperfections must be avoided in a computer if it is to produce correct results. Therefore, transistors in an integrated circuit chip are designed so that they control electrical currents by turning them on or off.

The strength of an electronic current varies over time, as shown by the dashed line in the figure below. A transistor in a computer can examine the current strength at fixed time intervals and decide at each time whether the current is on or off, as shown by the solid line in the figure. A current above a certain strength is considered to be on; anything below that level is treated the same as no current, and is considered to be off.



A signal (current) that has only two states—on or off—is called a binary signal. All modern computers use binary signals so that they will be insensitive to electrical disturbances. A binary signal representing a series of 0s and 1s is also called a **digital** signal. Electronic computers that use such signals are known as digital computers.

digital describes information technology that uses discrete values of a physical quantity to transmit information

digit one of the symbols used in a number system to represent the multipliers of each place

bit a single binary digit, 1 or 0

byte a group of eight binary digits; represents a single character of text

ASCII an acronym that stands for American Standard Code for Information Interchange; assigns a unique 8-bit binary number to every letter of the alphabet, the digits, and most keyboard symbols

Binary Coding

In a computer, a current that is considered to be either on or off at any time can represent a series of binary **digits**: 0s and 1s. When the current is off, the value is 0, and when it is on, the value is 1.

The two-digit binary numbering system is the basis of machine representation of numbers, letters, characters, and symbols. The smallest unit of information is the binary digit, or **bit**. Eight bits (known as a **byte**) is considered a complete unit for storage or computation.

By considering the binary digits in fixed-length groups, the electrical current can be interpreted to mean a series of decimal digits or letters. The coding scheme used in most small computers today is called the **ASCII** character set. By assigning a unique eight-digit binary number sequence (a byte) to each member in an alphabet of letters, decimal digits, and symbols, the ASCII set can represent each character on a keyboard by 0s and 1s.

Storage

Data values and commands being processed in a computer are represented as electrical currents that vary with time and are interpreted as a series of

Binary digits of the base-2 number system are perfectly suited to electronic computers because they represent the two electrical states of "on" or "off." All numbers, letters, and symbols can be represented by combinations of these binary digits (0s and 1s).

binary digits (0s and 1s). Data values and commands that are stored in integrated circuits are represented by currents and are "remembered" only as long as the circuits are powered.

In contrast, data values and commands that are stored for long periods must survive when power is off. They are also represented as a series of binary digits, but those digits are not realized as electrical currents varying over time. In a disk, diskette, or tape storage unit, binary digits are represented as magnetic spots on metallic coatings. Each spot is magnetized with its north pole in one direction or in the opposite direction. Special electromagnets write and read the spots under the control of the processor.

Processors and Calculation

A **processor** is an integrated circuit chip that combines transistors in such a way that it can do arithmetic, make comparisons, make decisions, and control the input, output, and storage elements of the computer. These operations are performed using data values that are entered into the computer, stored and processed in it, and sent as output from it.

processor an electronic device used to process a signal or to process a flow of information

Because the computer handles information in the form of letters and numbers (ultimately represented as binary numbers realized as electrical currents), it is natural to control its operation with a string of letters and numbers. For example, a certain series of letters and numbers fed to the processor on a particular circuit may cause it to fetch an item of data from a specified place in the storage unit. Such a series of letters and numbers is called a **command**.

command a particular instruction given to a computer, usually as part of a list of instructions comprising a program

Another command (a different series of letters and numbers) may cause two data values to be added together. Other commands can control movement of data values to and from storage, from input and to output, and can compare two data values and determine what command should be performed next.

Software and Programs

program a set of instructions given to a computer that allows it to perform tasks; software

A series of commands stored in the computer's storage unit is called a **program**. The same series of letters and numbers may be interpreted as commands if used in one way and interpreted as data values when used in another way. Because a computer can (1) store commands in the same way as data values and (2) modify its own commands based on the results of processing data, a computer can be called a "stored program machine." These two abilities give electronic computers great power.

Computer programs consisting of a series of commands are called software. Software is as essential as hardware in accomplishing the work that computers do. Software is realized as letters and numbers written or printed on a piece of paper or stored in magnetic form on disks and diskettes.

Multiprocessing and Interrupts

Several different command sequences, or programs, may be stored in a computer at one time. The processor may do all of the processing specified by one program and then do the processing specified by another. Under this arrangement, if the first program calls for some input or output task that takes considerable time (such as reading a tape), the processor might remain idle for quite a while. This time could be used to do some of the tasks specified by another program.

multiprocessing a computer that has two or more central processers which have common access to main storage

machine language electronic code the computer can utilize

Most computers allow for one program to be interrupted and another one to be started or resumed under specified circumstances. This capability is called **multiprocessing**. Multiprocessing allows one to start an operation, such as sending a message via e-mail, and then switch to another operation, such as word-processing, while the first operation is being completed.

Many modern computers contain numerous processors and can store huge quantities of data. Different processors may be doing different things at the same time, under the control of different programs, each referencing input, storage, and output units in turn or as needed.

WHAT ARE OPERATING SYSTEMS?

One of the most common kinds of programs is called an operating system. This is a computer program that controls the operation of a computer—particularly the running of other programs, including applications. Microsoft Windows is an operating system. Other widespread operating systems are Unix, Windows NT, and Linux, and the operating system for Apple Macintosh computers, generally known as Mac OS.

Machine Language, Programming Languages, and Applications

A series of commands consisting of letters and numbers (stored as binary digits, or bits) that the hardware processing unit can understand is said to be in **machine language**. Most processors understand very detailed commands, each of which does a small task. These can be combined in many varied ways to provide flexibility and hence great power.

For example, one command may call for fetching the data value stored at location A in the storage unit. The next command can ask data value stored at location B to be fetched. A third can cause the two values to be added together. A fourth would send the result to a display unit. The person creating the program probably thinks of this series of commands in terms of a single function, "display A+B," and is not interested in the individual commands that must be performed to cause this to happen.

Writing computer programs to do useful work has been greatly simplified by creating other programs that interpret instructions written in more user-friendly terms than machine language. Such programs are called **programming language processors**. Popular programming languages are BASIC, C, ADA, and PASCAL.

programming language processor a program designed to recognize and process other programs

Most people who use computers do not want to learn any programming language. Instead, they want to communicate with the computer in terms of the job they wish to do. Word processing is a typical example: A person does not want to write computer instructions just to compose a simple letter.

Programs that do particular types of data processing are called **applications** or application programs. Application programs have been written to do every kind of information processing imaginable: from playing solitaire, preparing tax returns, processing payrolls, and sending e-mail messages, to designing computer circuit boards, tracking airplanes, and controlling rockets. SEE ALSO Analog and Digital; Babbage, Charles; Bases; Communication Methods; Computers, Evolution of Electronic.

applications collections of general purpose software such as word processors and database programs used on modern personal computers

F. Arnold Romberg

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Computers, Evolution of Electronic

Electronic computers represent and process data through electronic signals and currents. These computers have evolved significantly over the past half-century. Today's computers are faster, smaller, more efficient, and more powerful than their room-sized predecessors.

Early Years

There are several opinions about the origin of the first electronic computer. Some believe the credit should go to the individual who provided the first written schematic for the electronic computer. Others credit the scientists who developed the first working model of a system. Many, however, credit the individuals who first patented the electronic computer.

The two earliest electronic computers were the ABC and the ENIAC. The Atanasoff-Berry Computer (ABC) was built at Iowa State University between 1937 and 1942 by John Vincent Atanasoff and Clifford Berry. The ABC—designed to produce solution sets for linear physics equations—performed **parallel processing**, separated memory and computing functions for calculations, refreshed memory, converted numbers from **base-10** to **base-2**, and carried out **binary** arithmetic. Atanasoff and Berry did not get a patent for their machine, and Atanasoff shared knowledge of ABC's construction and functions with John W. Mauchly, who visited with Atanasoff and viewed the ABC in 1941.

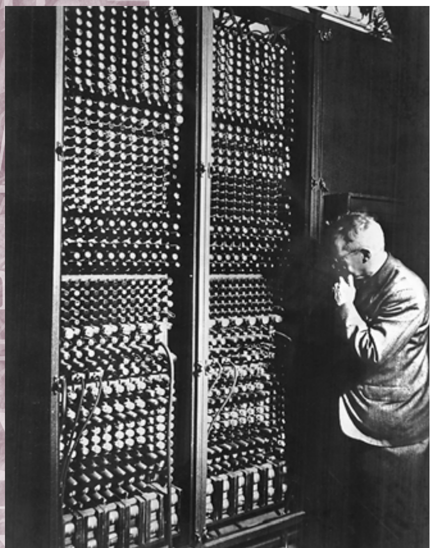
parallel processing using at least two different computers or working at least two different central processing units in the same computer at the same time or "in parallel" to solve problems or to perform calculation

base-10 a number system in which each place represents a power of 10 larger than the place to its right

base-2 a binary number system in which each place represents a power of 2 larger than the place to its right

binary existing in only two states, such as "off" or "on," "one" or "zero"

The ENIAC (Electronic Numerical Integrator and Computer) was created in 1946 by John W. Mauchly and J. Presper Eckert. ENIAC was the



The ENIAC computer required more than 17,000 vacuum tubes to perform ballistic calculations in decimal (base-10) notation. Although vacuum tubes were better than electromechanical relays, the tubes overheated and needed frequent replacements.

first large-scale electronic computer.* The 30-ton machine was built at the Moore School of Electrical Engineering on the University of Pennsylvania Campus with funding from the U.S. Army. The high-speed calculations it performed were used to accurately produce firing tables for artillery gunners.

But soon after ENIAC made its debut, its creators discovered some problems. ENIAC did not have enough internal storage to house numbers that it used in calculations. Another problem was ENIAC's difficulty changing programs or instructions. It took several hours to rewire ENIAC to perform different types of computational tasks.

ENIAC's successors were EDVAC (Electronic Discrete Variable Computer), EDSAC (Electronic Delay Storage Automatic Computer), and the IAS Computer. EDVAC, developed in 1949 by Maurice Wilkes and completed 3 years later, used binary numbers for arithmetic operations and digitally stored internal instructions. EDSAC, completed before EDVAC, had the ability to store programs internally. The IAS computer introduced the concept of **parallel operations**. It captured and moved digits in a number simultaneously.

Other Early Computers. Early electronic computing devices developed in the same era as the Atanasoff-Berry Computer and ENIAC also produced significant contributions to the evolution of the electronic computer. Such early devices included George Stibitz's Binary Calculator (1937), Konrad Zuse's Z1 (1941), and Howard Aiken's Mark-1 (1944). These computing devices used electromechanical relays to conduct electricity. Relays either allowed a current to exist in an opened or closed state. When the circuit was closed, the current flowed, but when it was opened, no current traveled through a circuit.

Another electromechanical computer named Colossus (1943) was used to crack secret German military codes during World War II. Alan Turing and other scientists developed Colossus at Bletchley Park, located northwest of London. Colossus functioned on the premise that it could solve every solvable mathematical or logical problem. Composed of vacuum tubes like ENIAC, Colossus read, scanned, and compared **ciphered** enemy messages with known Enigma codes until a match was found.

***Although ENIAC was billed as the first electronic computer, U.S. Federal Judge Earl Richard Lawson settled the dispute in 1973 and named the Atanasoff-Berry Computer the first electronic computer.**

parallel operations separating the parts of a problem and working on different parts at the same time

ciphered coded; encrypted

transistor an electronic devices consisting of two different kinds of semiconductor material, which can be used as a switch or amplifier

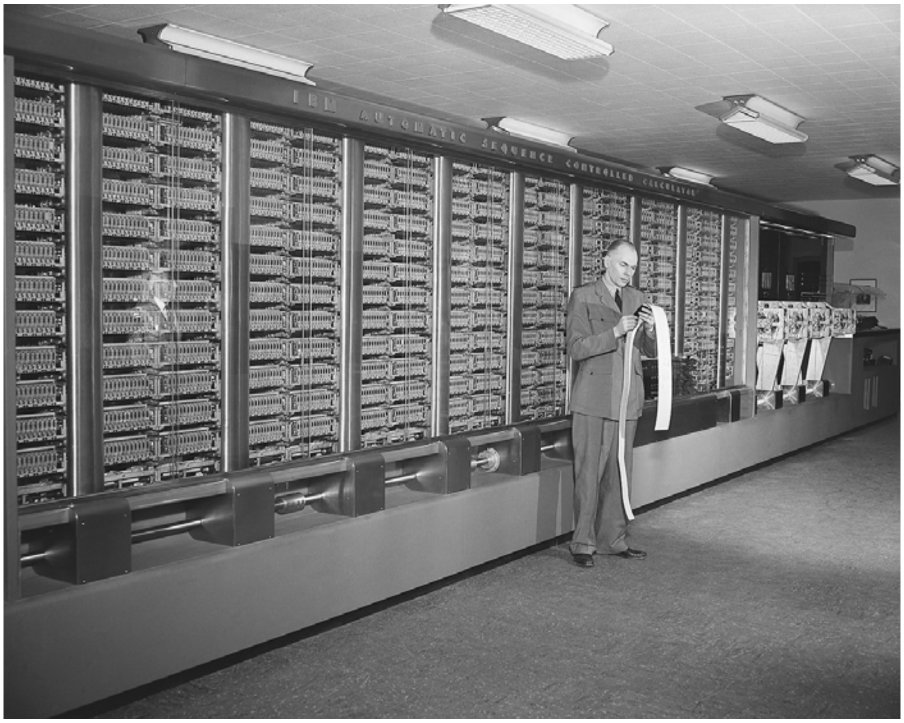
integrated circuit a circuit with the transistors, resistors, and other circuit elements etched into the surface of a single chip of silicon

Decreasing Size and Increasing Speed

The shortcomings of vacuum tubes prompted scientists to look for alternative materials to conduct electricity. Scientists also searched for ways to decrease the size of the computer while increasing its speed. The advent of the **transistor** and the **integrated circuit** chip produced the next major development in the evolution of the electronic computer.

The Transistor. The transistor, invented in 1947 by John Bardeen, W. H. Brattain, and W. B. Shockley at Bell Laboratories, made it possible to develop an electronic computer that was smaller and more efficient than its predecessors.

Yet two major drawbacks kept the transistor from being mass-produced: high cost and a lack of knowledge of its peculiarities. The transistor was further refined by Gordon Teal, who used silicon instead of



Howard Aiken's Mark-1 computer was used by the U.S. Navy during World War II. Its enormous size coupled with its high operating cost meant that its use was limited to government and major corporations.

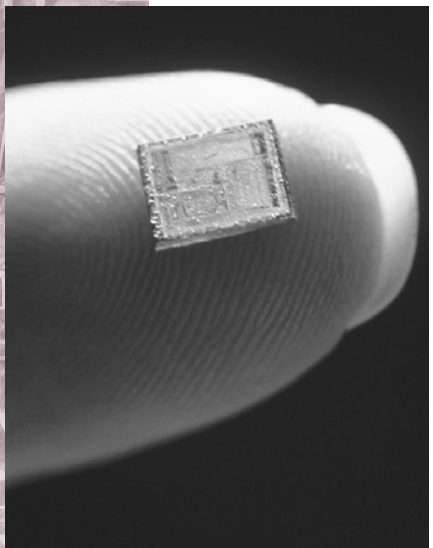
germanium to make the transistor. Silicon was cheaper and its use improved the speed of production and reduced manufacturing costs. New methods were devised to produce large silicon crystals and add the impurities that were necessary to make it usable.

These advances, along with decreasing manufacturing costs and encouragement by the U.S. military to build smaller electronic parts, helped reduce the size of electronic computers. The first computer that used transistors was TRADIC (Transistor Digital Computer). By the 1960s hundreds of computers used transistors to speed processing and reduce production costs.

The Integrated Circuit. The integrated circuit, developed by Jack St. Clair Kilby, provided a new means for incorporating transistors and other needed components on the same surface. The integrated circuit board allowed more information to be stored in a smaller area. Transistors also proved to be more reliable, less expensive, and faster than other technology of the time. The first working electronic computer that used semiconductor technology was built by Texas Instruments for the U.S. Air Force.

The early integrated circuit (IC) was soon replaced by an IC created by Robert Noyce. Noyce's version, completed in 1959, was adopted by Texas Instruments and mass-produced in 1962 under the nickname "chips." Chips saved space, miniaturized connections, and provided the speed necessary for sophisticated scientific and commercial applications.

Chips also changed the manufacturing process. Before the integrated circuit was invented, all components of an electronic circuit were produced as separate entities and then combined through wiring. Because chips are made of silicon, they have circuits that contain resistors, capacitors, and



Miniaturization has allowed microprocessors to be placed on a silicon microprocessor chip no bigger than a pencil eraser. The computing power in one of today's microchips is over 3,000 times greater than computing power of the microprocessors of the early 1970s.

transistors that can turn voltage on or off to represent the binary digits used by computers. SEE ALSO Bases; Boole, George; Computers and the Binary System; Computers, Personal; Cryptology; Turing, Alan.

Demetria Ennis-Cole

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Computers, Future of

Modern computers are found everywhere: homes, offices, businesses, hospitals, and schools, to name a few. Contemporary society has become so dependent on computers that many people become frustrated and unable to function when computers are "down." Because of this dependence, computers are regarded as essential tools for everything from navigation to entertainment.

Today's computers are smaller, faster, and cheaper than their predecessors. Some computers are the size of a deck of cards. Hand-held Personal Data Assistants and notebook computers or "ultra-lights" make users portable and give them the opportunity to work in a variety of places. These systems provide a wide range of connectivity and access to information on local, wide, and wireless networks. This gives users more convenience and more control over their time.

Future computers promise to be even faster than today's computers and smaller than a deck of cards. Perhaps they will become the size of coins and offer "smart" or **artificial intelligence** features like expert intelligence, neural network **pattern recognition** features, or natural language capabilities. These capabilities will allow users to more conveniently interact with systems and efficiently process large amounts of information from a variety of sources: fax, e-mail, Internet, and telephone. Already evident are some evolving cutting-edge applications for computer technology: wearable computers, DNA computers, virtual reality devices, quantum computers, and optical computers.

artificial intelligence the field of research attempting the duplication of human thought process with digital computers or similar devices; also includes expert systems research

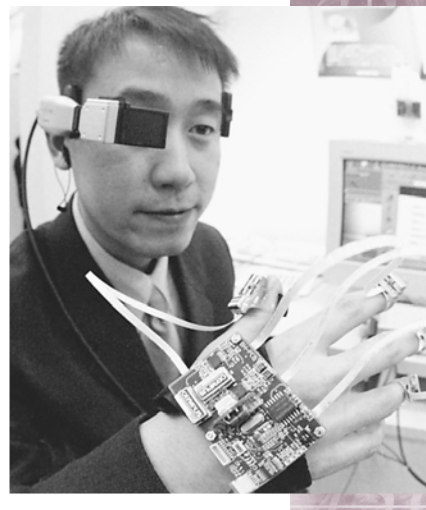
pattern recognition a process used by some artificial-intelligence systems to identify a variety of patterns, including visual patterns, information patterns buried in a noisy signal, and word patterns imbedded in text

Wearable Computers

Is a wearable computer in your future? With hardware shrinking and becoming more powerful and more able to execute instructions and perform computations in shorter timeframes, it is very possible that there will be widespread use of wearable systems in the future. A wearable is defined as a handless system with a data processor supported by a user's body rather than an external surface. The unit may have several components (camera, touch panel, screen, wrist-mounted keyboard, head-worn display, and so forth) that work together to bring technology to situational and environmental problems.

Assembly and repair environments are ideally suited for wearable technology because they deploy users with technical expertise to problem areas. Wearable computers allow users to keep their hands free at all times while providing access to technical specifications and detailed instructions for problem-solving and troubleshooting.

In the future, wearables may even be built into the fabric of clothing. Garments can be made using **conductive** and nonconductive textiles like organza and yarn, gripper snaps, and embroidered elements. Ordinary fabric can be connected to electronic components to add functionality and usability.



Wearable computing, such as hand-worn displays, may soon become commonplace. Wearable computers have been pilot-tested by many companies to determine whether the productivity of technicians increases when they are outfitted with wearables.

conductive having the ability to conduct or transmit

nucleotides the basic chemical unit in a molecule of nucleic acid

DNA-Based Computers

Can small molecules such as DNA be used as the basis for new computing devices? A biologist and mathematician named Leonard Adelman first linked genetics and computer technology in the mid-1990s. Adelman coded a problem using the four **nucleotides** that combine to form DNA and discovered that the DNA solution was accurate.

A DNA-based computer would be radically different from a conventional computer. Instead of storing data on silicon chips, converting data to binary notation (0s and 1s), and performing computations on the binary digits, DNA computing would rely on data found in patterns of molecules in a synthetic DNA strand. Each strand represents one possible answer to the problem. A set of strands is manufactured so that all conceivable answers are included. To winnow out a solution, the DNA computer subjects all the strands simultaneously to a series of chemical reactions that imitate mathematical computations.

The advantage of DNA computing is that it works in parallel, processing all possible answers simultaneously. An electronic computer can analyze only one potential answer at a time. The future holds great possibilities as DNA-based computers could be used to perform **parallel processing** applications, **DNA fingerprinting**, and the decoding of strategic information such as banking, military, and communications data.

parallel processing using at least two different computers or working at least two different central processing units in the same computer at the same time or "in parallel" to solve problems or to perform calculation

DNA fingerprinting the process of isolating and amplifying segments of DNA in order to uniquely identify the source of the DNA

tactile relating to the sense of touch

Virtual Reality Devices

Virtual reality (VR) immerses its user in a simulated world of possibilities and actions. In the virtual world, the user has the ability (through head-mounted displays, gloves, and body suits) to respond to **tactile** stimulation. Users manipulate objects, examine architectural renderings, and interact in

an environment before it becomes a physical reality. This is often very cost-effective, and it supports decision-making tasks. VR is often used in modeling situations, but its future holds promise in other areas: education, government, medicine, and personal uses.

In education, students and teachers may have the ability to interact inside virtual classrooms to explore ideas, construct knowledge structures, and conduct experiments without risk, fear of failure, or alienation. Government offices may use VR technology to improve services, provide better delivery of health care (model symptoms, progression, and prevention), and monitor environmental changes in air quality, wetlands, ozone layers, and other ecological areas (animal populations and forestry).

Medical areas could use VR to train interns and practicing physicians on new procedures and equipment; observe internal tissue production in three dimensions (3-D); collect and better analyze medical images; simulate surgical and invasive procedures; and empower therapists to use exposure therapy along with realistic models. VR technology could also be used to augment instructional games, 3-D movies, and real-time conferencing and communication efforts.

Quantum Computers

quantum theory the study of the interactions of matter with radiation on an atomic or smaller scale, whereby the granularity of energy and radiation becomes apparent

The first application of **quantum theory** and computers occurred in 1981 at Argonne National Laboratory. Quantum computers, like conventional computing systems, were proposed before supportive hardware existed. In 1985, a quantum parallel computer was proposed. Today, physicists and computer scientists still hope that the imprecision of subatomic particles can be used to solve problems that thus far remain unsolved.

The quantum computer would overcome some of the problems that have plagued conventional computers: namely, sequentially following rules and representing data as a series of switches corresponding to 0 or 1. By using subatomic particles, quantum computers will have the ability to represent a number of different states simultaneously. These particles will be manipulated by the rules of probability rather than absolute states or logic gates. Manipulating these small subatomic particles will allow researchers to solve larger, more complex problems, such as determining drug properties, performing complex computations, precisely predicting weather conditions, and helping chip designers create circuits that are presently impossibly complex.

Optical Computers

bit a single binary digit, 1 or 0

fiber-optic describes a long, thin strand of glass fiber; internal reflections in the fiber assure that light entering one end is transmitted to the other end with only small losses in intensity; used widely in transmitting digital information

As microprocessor chip designers reach physical limitations that prevent them from making chips faster, they are searching for other materials to conduct data through the electrical circuits of computer systems. If designers could harness photons to transmit data, faster microprocessor chips could become a reality.

This new frontier—optical computing—could allow computers to perform parallel processing tasks more efficiently and increase the speed and complexity of computers by allowing them to process billions of **bits** simultaneously. Optical computers might use **fiber-optic** cable, optical chips, or wireless optical networks to process and transmit data.

Fiber-optic cable is currently used in many establishments. It uses a laser to transmit billions of data bits through cables made of thin strands of glass coated in layers of plastic. Signals can be carried over a distance of 40 to 60 miles. A more recent development—optical chips—could cut the cost of optical communication by using Dense Wave Division Multiplexing technology to carry more information over a fiber. This would give users increased **bandwidth** for connecting to the Internet. Optical networks could be used to improve free-space optics, video delivery, and voice communications. SEE ALSO Computers, Evolution of Electronic; Computers and the Binary System; Virtual Reality.

bandwidth a range within a band of wavelengths or frequencies

Demetria Ennis-Cole

Internet Resources

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Computers, Personal

In 1976, the Cray-1 supercomputer was installed at Los Alamos National Laboratory in New Mexico. It was then the fastest computer in the world, performing 160 million **floating-point operations** per second. The computer cost 8.8 million dollars and generated so much heat that it required its own refrigeration system.

floating-point operations arithmetic operations on a number with a decimal point

By 2001, anyone could buy a computer that will fit on a desk, is 34 times as fast as the Cray-1, and has 32 times the memory. Moreover, it can compute, connect to other computers, play music CDs, and show DVD movies. This newer computer can also burn CDs and DVDs. All of this computing power could be purchased for less than $5,000. There has certainly been a revolution.

What began as a box of parts to be assembled and tediously programmed by electronics enthusiasts has become a tool easily used by office workers and schoolchildren. The rapid growth of computer software has enabled personal computers to play games and music, to be used as intelligent typewriters, to perform mathematical calculations, and, through online capabilities, to be used for email and to connect to the vast treasures of the Internet. Computers have become smaller, more powerful, faster, more user-friendly, and versatile enough to meet almost any need.

The Earliest Personal Computers

The January 1975 cover of *Popular Electronics* magazine showed a picture of the Altair 8800. This small computer was actually a kit that had to be assembled and programmed. It had no monitor (screen) and no keyboard, and it performed few functions. The kit was sold to computer hobbyists—that is, people who enjoyed building and programming the computers.

byte a group of eight binary digits; represents a single character of text

The Altair 8800 had 256 **bytes** of main memory, also called RAM (Random Access Memory). A major disadvantage of the Altair 8800 was that it had to be programmed in machine code, which is sequences of 0s and 1s that the computer understands. This machine code was entered using switches on the front panel of the Altair.

Beginner's All-Purpose Symbolic Instruction Code (BASIC) is a computer language that is easy to learn and was originally developed for use on large computers. BASIC converts English-like statements into machine code.

After seeing the 1975 *Popular Electronics* cover, a 19-year-old college student named Bill Gates and his programmer friend, Paul Allen, decided to write a form of BASIC that would run on a computer as small as the Altair. Gates and Allen believed there would be a demand for small computers and that these machines would need pre-written programs to run on them, called software. The two founded a company, called Microsoft, dedicated to developing software for small computers. By 1989 Microsoft's sales reached one billion dollars, and Bill Gates would eventually be considered the richest man in the world.

Another company dedicated to the small computer, Apple Computer, was founded on April 1, 1976, by Steve Jobs, Steve Wozniak, and Ron Wayne. The first Apple computer was only a bare circuit board (a thin plate on which electronic components are placed), and sold for $666.66 to hobbyists who would need to add a power supply, a monitor, and a keyboard. This machine, called the Apple I, sold about 200 units. Its 4,000 bytes of RAM (also called 4 kilobytes or 4K) greatly exceeded the Altair's 256 bytes.

More Advanced Personal Computers

The Apple II was introduced in 1977 and sold for between $1,300 and $3,000, depending on the options the buyer chose. A variety of software was available for this machine. Because it did not need to be programmed, people with little knowledge of computers or electronics could use it. The Apple II was a great commercial success. Over the next 16 years, there would be many models of the Apple II, with a total of over five million units shipped.

The first Apple II had 16K bytes of memory. The last Apple II, the Apple IIgs, had 256K bytes of RAM. It contained space for inserting additional memory so the amount of RAM could be expanded to 8 MB (8 megabytes, or 8 million bytes). Since the introduction of the Apple I, memory capacity had increased by a factor of 4,000. The Apple IIgs also had a color monitor, sound, a keyboard, and a **mouse**.

mouse a handheld pointing device used to manipulate an indicator on a screen

The market for personal computers greatly expanded in 1979 with the introduction of VisiCalc, a computerized spreadsheet. A spreadsheet is a program that allows a user to enter data into a table. The user can then

solve complicated "what if" problems by manipulating the data. For example, a person could change one number in a budget and see the effect it has on the entire budget. This easy and fast capability for financial analysis made personal computers an important business tool.

By the early 1980s, over one hundred companies, such as Texas Instruments, Commodore, Tandy, and Digital Equipment Corporation, were making personal computers. Personal computers varied widely in their memory, speed, and the function capability. Their prices also varied, starting at about one hundred dollars up to thousands of dollars. There was a personal computer for every need and price range.

In 1981, IBM, which had been a leader in developing large computers, developed the first IBM personal computer (PC). It was a complete personal computer system with a great variety of software. The first IBM PC had 16K bytes of memory, expandable to 256K bytes. Its starting price was $1,565 but could cost much more depending on options. In the next eighteen months, 136,000 PCs would be sold. In 1982, other companies began producing personal computers that looked like the IBM PC. These machines would be called "**clones**."

clones computers assembled of generic components designed to use a standard operation system

The same year, modems (standing for MOdulator/DEModulator) were introduced for personal computers. With a modem, a personal computer could transmit data to other computers, receive data, and access online databases over telephone lines. Modems allowed users of personal computers to communicate with each other through email and to connect to the Internet.

In 1983, Apple introduced a new computer called Lisa. Lisa was expensive and slow and did not sell well. However, Lisa incorporated a very important feature, the use of icons, that would influence later computers.

Before Lisa, computer users could only use commands or function keys to communicate with their computers. The introduction of pictures, called icons, which could be moved around on the computer screen as if they were objects on a desktop, allowed users to move a pointer on a screen using a mouse or trackball and click on a picture representing a command, file, or function.

This use of icons is called GUI (for **graphical user interface** and pronounced goo-ee). Using GUI, a person can control the computer without having to learn commands or use special keys. GUI was first developed at the Xerox Palo Alto Research Center in the 1970s but was popularized by Lisa's successor, the Apple Macintosh. The simplicity of using the "Mac" with its icons made it an enormous success.

graphical user interface a device designed to display information graphically on a screen; a modern computer interface system

Current Trends

The trend toward smaller and lighter personal computers began in 1981 with a suitcase-sized computer named the Osborne I. It was portable, although it weighed over 20 pounds, and had 64K bytes of RAM. The Osborne I cost $1,795 and was commercially successful. The Osborne Computer Company went bankrupt within 2 years, but portable computers became popular with the advent of laptops, notebooks, and handheld computers. Today, it is not technology but the size of the keys—which have to match adult fingers—that determines how small computers can become.

An important milestone in the history of personal computers is the development of flexible software. Such software makes it possible for non-

Steve Jobs, co-founder of Apple, poses with an iMac computer in 1999. The iMac comes with high speeds and many added features for a low price.

programmers to use programs other people have written, such as spreadsheets, word processors, and games. The great assortment of software available has made the personal computer more and more useful to individuals and has therefore increased the demand for computers. And as more people own and use personal computers, the market for a wider variety of software products also becomes greater.

Today, personal computers are almost as much a part of American life as the telephone and the automobile. They are expanding their influence in education and are an integral part of many classrooms. Businesses rely heavily on them. Personal computers have become as much a communication device as a computational one. Communication by e-mail is common and the World Wide Web houses billions of pages of information.

Computers have also become more prolific than ever before. Cities all over the world have Internet cafes, where people can buy time on personal computers. Many hotels provide access to personal computers or have facilities for connecting their guests' laptop computers. As computers become smaller, they are also becoming wireless, using radio frequencies for communication. For less than $5,000 you can hold 8 million bytes of storage in your hand. SEE ALSO Computers and Binary System; Computers, Evolution of Electronic; Computers, Future of; Numbers, Massive.

Loretta Anne Kelley

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Internet Resources

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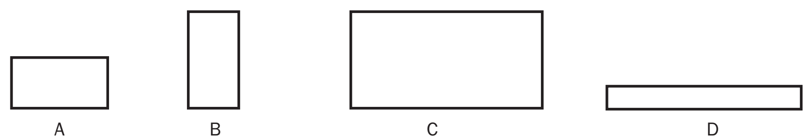
Congruency, Equality, and Similarity

What does it mean to say that one square is "equal" to another? It probably seems reasonable to say that two squares are equal if they have sides of the same length. If two squares have equal areas, they will also have sides of the same length. But although "equal areas mean equal sides" is true for squares, it is *not* true for most **geometric** figures.

geometric relating to the principles of geometry, a branch of mathematics dealing with the measurement, properties, and relationships of points, lines, angles, surfaces, and solids

Consider the rectangles shown below. The areas of A and B and D are all 2 square units, but it is not reasonable to say that rectangle A "is equal to" rectangle D, although their areas are equal.



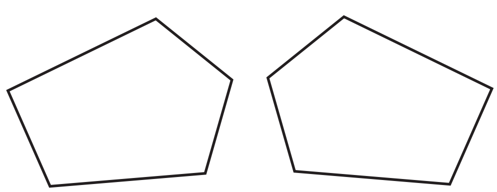
Geometry has a special mathematical language to describe some of these relationships. If Rectangle B is moved, turned on its side (rotation), and slid (translation), it would fit exactly in Rectangle A.

Two geometric figures are called **congruent** if they have the same size and the same shape. Two congruent figures can be made to coincide exactly. Rectangle A is congruent to Rectangle B. In mathematical notation, this is written as $A \cong B$.

congruent exactly the same everywhere; having exactly the same size and shape

Look at the two **polygons** below. Are they congruent? Can one of the polygons be slid (translated), turned (rotated), and flipped (reflected) so that it can fit exactly over the other? The answer is yes, and therefore these two shapes are congruent. Their corresponding, or matching, angles are congruent and so are their corresponding sides.

polygon a geometric figure bounded by line segments



Two	are congruent if:
Line segments	the measure of their lengths is the same
Circles	they have congruent radii
Angles	they have equal measure (degrees)
Polygons	their corresponding parts (sides and angles) are congruent

reflexive directed back or turning back on itself

symmetric to have exact correspondence in form or shape

transitive having the mathematical property that if the first expression in a series is equal to the second and the second is equal to the third, then the first is equal to the third

The congruence relationship ≅ is **reflexive** (A ≅ A, because any figure is congruent to itself), **symmetric** (because A ≅ B means that B ≅ A), and **transitive** (because A ≅ B ≅ F means that A ≅ F).

When an exact copy of a shape is made, the result is congruent shapes. Sometimes, instead of making an exact copy, a scale model, or a drawing that is smaller or larger than the original, is made. Blueprints, copies of photos, miniatures, enlargements are all examples of a relationship that is somewhat different from congruence.

Look back at the figure that shows Rectangle A and Rectangle C. The sides of Rectangle A measure 1 unit by 2 units. The sides of Rectangle C measure 2 units by 4 units. The corresponding angles of the two rectangles are congruent. Rectangle A could be enlarged to look like Rectangle C, or C could be shrunk to look like A.

The mathematical term for shrinking or enlarging a figure is dilation. The sides of these two rectangles are proportional: 1:2 = 2:4. When one figure can be made into another by dilation, the two figures are similar.

Two geometric figures are called *similar* if they have the same shape, so that their corresponding sides are proportional. Two similar figures may be of different sizes, but they always have the same shape.

Rectangle A is similar to Rectangle C. In mathematical notation, this is written as A ≈ C. Two congruent figures will always also be similar to each other, so A ≈ B.

Look again at the rectangles. No matter how Rectangle D is dilated, translated, rotated, or reflected, it cannot be made to have the same shape as A or C. So D is neither congruent nor similar to any other rectangle in this figure. SEE ALSO Photocopier.

Lucia McKay

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circumference the distance around the circle

plane generally considered an undefinable term, a plane is a flat surface extending in all directions without end, and that has no thickness

conic sections the curves generated by an imaginary plane slicing through an imaginary cone

ellipse one of the conic sections, it is defined as the locus of all points such that the sum of the distances from two points called the foci is constant

Conic Sections

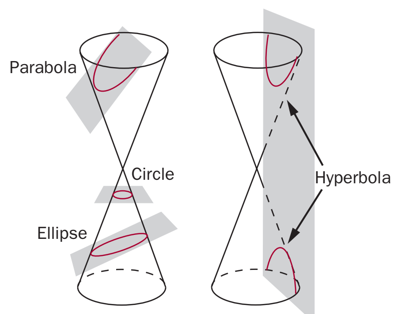
Imagine there are two cone-shaped paper drinking cups, each fastened to the other at its point, or vertex. The figure that would result is described mathematically as a right circular cone (sometimes called a double cone), which is formed by a straight line that moves around the **circumference** of a circle while passing through a fixed point (the vertex) that is not in the plane of the circle.

If a right circular cone is cut, or intersected, by a **plane** at different locations, the intersections form a family of plane curves called **conic sections** (see the figure). If the intersecting plane is parallel to the base of the cone, the intersection is a circle—which shrinks to a point when the plane has moved toward the cone's tip and finally passes through the vertex. If the intersecting plane is not parallel to the base, passes through only one half of the cone, and is not parallel to the side of the cone, then the intersection is an **ellipse**. If the plane intersects both halves of the cone, and is not par-

EQUATIONS FOR CONIC SECTION CURVES

Curve	General Equation	Notes	Example
Circle	$(x - h)^2 + (y - k)^2 = r^2$	center of circle = (h, k) radius = r	$x^2 + y^2 = 49$
Ellipse	$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$	length of major axis = $2a$ foci at c and $-c$ $b^2 = a^2 - c^2$ center = (h, k)	$x^2 + 25y^2 = 49$
Hyperbola	$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$	foci at c and $-c$ $a^2 + b^2 = c^2$ center = (h, k) equation of asymptotes: $y = \pm (b/a)x$	$\frac{x^2}{9} - \frac{y^2}{16} = 1$
Parabola	$a(x - h)^2 + k = y$	axis of symmetry: $x = h$ vertex = (h, k)	$(x - 7)^2 + 1 = y$

allel to the side of the cone, then the intersection is a curve that has two branches, called a **hyperbola**. If the plane intersects the cone so that the plane is parallel to the side of the cone, then the intersection is a curve called a **parabola**. The equations for the conic section curves can have the general forms summarized in the table.



These conic section curves—the circle, ellipse, hyperbola, and parabola—have been known and named for more than 2,000 years, and they occur in many applications. The path of a thrown ball, the arc of cables that support a bridge, and the arc of a fountain are examples of parabolas. The shape of a sonic boom, the path of comets, and the LORAN navigation system for ships involve the hyperbola. The paths traveled by the planets, dome-shaped ceilings, and the location of the listening points in a whispering gallery involve the ellipse. Of course, examples of the most familiar conic section—the circle—can be found everywhere: the shape of flowers, ripples in a pool, and water-worn stones. SEE ALSO Locus.

Lucia McKay

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Conservationist

If you love hiking, camping, being outdoors, and have a concern that future generations will not have the opportunity to enjoy our natural resources,

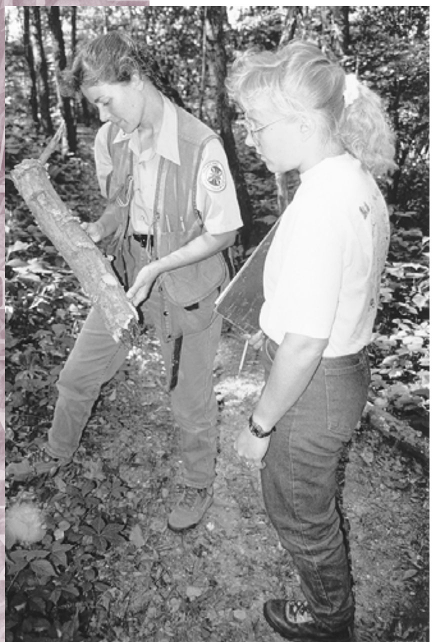
FOCI

For every point on an ellipse, the *sum* of the distances from that point to two points called the foci is the same. These two points, the foci, are on the major (longest) axis of the ellipse. The value of this constant sum is $2a$.

For every point on a hyperbola, the absolute value of the *difference* of the distances from that point to two points called the foci is the same. The absolute value of this constant difference is $2a$.

hyperbola a conic section; the locus of all points such that the absolute value of the difference in distance from two points called foci is a constant

parabola a conic section; the locus of all points such that the distance from a fixed point called the focus is equal to the perpendicular distance from a line



A conservationist must take careful measurements of many variables to estimate the supply of a natural resource so as to make an informed decision about whether conservation efforts should be increased.

then becoming a conservationist may be the career for you. Conservation is based on collecting, processing, and managing data regarding water, wildlife, land, vegetation, and the environment.

The need for conservationists has arisen because of the demands on natural resources. The formal education required to be a conservationist includes a college degree in the sciences and probably an advanced degree in an area of specialty. With an education geared toward conservation, there are a variety of jobs that one could attain. Data collection would be an appropriate job for someone who prefers to be outside interacting with nature. Another job might entail working with environmental policy, regulations, and legislation. Yet another could be working in higher education, sharing the field of environmental awareness.

Conservationists use mathematics when dealing with measurement, statistics, and technology. Collecting data regarding populations and quantities of land or resources is done first. Depending on the type of information being collected, computers can sometimes aid in the collection process. Information is entered into a computer to be organized and evaluated. With the increased popularity and necessity for this field, new programs and technology are emerging to make the details more complete and informative. Conservationists are looking for trends and continually assess and monitor natural resources to assist in developing strategies for their conservation. SEE ALSO Global Positioning System.

Elizabeth Sweeney

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Consistency

Mathematics contains a number of different systems, but each mathematical system, no matter how different it may be from another, has consistency as one of its goals. When a mathematical system is consistent, a statement and the opposite, or negation, of that same statement cannot both be proven true.

For example, in the familiar system of algebra, it is true that $a + 1 > a$. Even if a is a negative number, or 0, the statement is true. For example, $-3.5 + 1$ is -2.5, and -2.5 is greater than -3.5 (because -2.5 is to the right of -3.5 on a number line). Because this system is consistent, it is not possible to prove that $a + 1$ is less than a or equal to a.

Consistency is also important in the use of mathematical definitions and symbols. For example, $(5)^2$ equals 25 and $(-5)^2$ also equals 25. Is the square root of 25 equal to 5 or to -5? The answer is both: 5 is the square root of 25, and -5 is also the square root of 25.

However, for consistency, the symbol for square root, $\sqrt{\ }$, has been defined to mean only the positive square root, if one exists. So $\sqrt{25}$ has only one value (5) even though the equation $x^2 = 25$ has two roots, or solutions. In solving this equation, you must add the $\pm$ symbol to the radical symbol to show that both the positive and negative roots are desired.

$x^2 = 25$	Equation to be solved
$x = \pm\sqrt{25}$	Taking the square root of both sides
$x = 5$ or -5	The two solutions

The second step uses the rule that the square root can be taken of both sides of an equation, and the equation will still be true. Is it always true that, if $a^2 = b^2$, then $a = b$? Consider the following example.

$(-3)^2 = (3)^2$	This is true because $9 = 9$
$\sqrt{(-3)^2} = \sqrt{(3)^2}$	Taking the square root of both sides
$-3 = 3$	This is *not* true

What went wrong? If $a^2 = b^2$, then $a = b$ is consistently true *only* when a and b are both positive numbers. This means that this rule is true (and the system maintains consistency) if it is written as $a^2 = b^2$ means $a = b$ when $a, b \geq 0$.

When moving beyond the familiar systems of ordinary arithmetic and algebra, consistency poses some difficult challenges. For example, many mathematicians have wrestled with the sentence "This statement is not provable." If the statement is provable, it is false (a contradiction!); if it is true, it is not provable. The work of the mathematicians David Hilbert, Kurt Gödel, Douglas Hofstadter, and Raymond Smullyan delve into this puzzle. SEE ALSO Proof.

Lucia McKay

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Consumer Data

Consumers in the United States purchase between 5.5 and 6 trillion dollars worth of goods and services annually, making the U.S. economy by far the largest in the world. This economy is sometimes called a "free market" economy because, for the most part, the goods and services produced and the prices charged for those goods and services are determined by what consumers want and how much they are willing to pay for it. Except in rare cases of regulated industries, the government does not dictate to producers

how much of a product they can make or how much they may charge for the product. Using sophisticated research methodologies, industries determine the consumer demand for a given product and at what price they may sell that product to make the largest possible profit. Marketing experts then develop advertising campaigns to convince consumers to buy their company's product rather than the competition's. If two or more companies offer similar products of equal quality, consumers may make their buying decisions based upon the more appealing advertising campaign or upon which product is priced lower.

How Can Consumers Learn More About Products?

In an economy driven by consumer demand for goods and services, it is important for those consumers to have access to data that will assist them in making purchasing decisions. In the United States, numerous organizations have been formed to provide consumers with information about the quality and pricing of commodities and services offered by businesses. Consumers Union is a non-profit organization that publishes *Consumer Reports*, a monthly magazine that reports the results of testing done on various types of products, ranging from food items to electronic equipment to automobiles. The magazine articles rate the tested products on the basis of quality, price, and value. To decrease the possibility of bias in the magazine, *Consumer Reports* does not accept any advertising. Thus, its writers and editors are not influenced in their reporting to be less critical of a product simply because its producer pays the magazine huge amounts in advertising dollars. In addition to nonprofit groups such as Consumers Union, there are also a number of for-profit organizations that issue ratings of consumer goods and services. One of the largest, J. D. Powers and Associates of Agora Hills, California, does scientific surveys of consumers to determine their level of satisfaction with the products they use. When a company's product comes in at or near the top of a Powers survey, it will frequently tout this fact in their future advertising.

Understanding Unit Pricing

With so many sources of data available, it might seem that the American consumer should be able to make selections easily in the marketplace, but the fact that so many companies are vying for the consumer dollar can lead to a dizzying array of choices for the consumer to make. Most companies are not satisfied to market or price a given product in only one way. In addition to regular grocery stores, for example, there have arisen so-called convenience stores located near almost any neighborhood to make it easy for people to make a quick drive or walk to pick up one or two items of immediate need. There is a price for this convenience, however. A gallon of milk or a loaf of bread is typically more expensive at the convenience store than at the larger grocery stores. At the other end of the scale, there are huge "warehouse" stores that typically sell items in bulk at lower unit prices than the traditional grocery stores. Thus the idea of unit pricing is an important concept for consumers to understand.

The unit price for a product might be expressed, for example, in price per ounce. Thus a 16-ounce loaf of bread priced at $2.40 would have a unit price of $2.40/16oz., which equals 15 cents per ounce. Now suppose that

By comparing the unit price of a product when it is available in different amounts, and evaluating its perishability, a consumer can make purchases cost-effectively.

the same store has a 24-ounce loaf of the same brand of bread for $3.20. Which should the consumer buy? Based on unit price alone, the 24-ounce loaf is less expensive, since $3.20/24oz is only about 13 cents per ounce. On the other hand, suppose that this consumer cannot eat more than 16 ounces of bread in a week and that after a week the remaining bread has started to become stale and must be thrown away. Then the consumer has actually paid $3.20 for 16 ounces, which comes to 20 cents per ounce. Clearly, if one buys something in large quantities at a lower price per unit, it is only a bargain if the product will remain usable to the consumer over a longer period of time. Thus, buying light bulbs or canned foods in bulk makes sense if the unit price is lower than for smaller quantities, because light bulbs and canned food can be stored for long periods of time.

Another example is long-distance minutes. Mobile phone companies typically sell minutes of calling time in various quantities, usually at a lower "per minute" price for, say, 300 minutes per month than for 100 minutes per month. However, if the consumer seldom talks more than 100 minutes per month, the consumer is, in effect, throwing away the remaining 200 minutes, just like the grocery shopper throws away unused bread. The actual unit price per minute is then likely to be larger than the stated unit price. In such cases, the consumer is helped by knowing some simple consumer math, but must also be aware that mathematics applied in the absence of common sense can do more harm than good. SEE ALSO Data Collection and Interpretation.

Stephen Robinson

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ARE SALES REALLY GOOD DEALS?

The theory of discounting is to set an artificially high price, and then discount off this price. For example, examine how cotton sweaters arrive in a clothing store. Cotton is grown and sold to the sweater manufacturer (first markup). Sweaters are made, a brand name is attached, and they are sold to a distributor (second markup). The clothing store's buyers order the sweaters from distributors and put them on the shelves (third markup).

The sweaters are then typically offered at full retail price for several months. When the season closes, sweaters that did not sell are marked down. Assume the average total markup of a sweater from the farmer to the clothing store is 40 percent. If the sweaters are marked 30 percent-off prices still allow the clothing store a 10 percent profit margin on the sweaters. Thus, stores still profit when they have sales, with the volume of merchandise sold making up for the lower profit margin.

Whether these sales offer good values to the consumer depends upon the quality of products offered, the availability of the products from sources other than retail stores and the needs of the consumer.

Cooking, Measurements of

If asked to think of a measurement that relates to cooking, most people would immediately think of dashes of salt and teaspoons of sugar rather than ounces or pounds. In a household kitchen, a cook deals with small quanti-

Chefs must use the mathematics of proportions to increase the size of recipes so as to accommodate large numbers of people.

ties of ingredients like cups and tablespoons. In the kitchen of a restaurant or hotel, however, the recipes are more likely to require ounces and pounds. The reason for this is quite simple: restaurants must cook for a lot of people and cannot cook each meal individually.

Increasing The Size of a Recipe

To bake a cake in a household kitchen, a recipe will probably call for one-half cup of butter, usually packaged in a one-pound box containing four sticks. A recipe requiring a quarter-pound of butter will make a cake that will serve about ten people. If using that same recipe in a restaurant, a chef would need to provide for as many as two or three hundred people, and it would be unreasonable to have the chef make twenty individual cakes.

This problem can easily be solved using the mathematics of proportions. If one stick is used for ten servings, the chef simply needs to figure out the number of sticks needed for two hundred servings. Since two hundred servings is twenty times larger than ten servings, the chef would need to use twenty times the amount of butter for a single recipe, or twenty sticks of butter.

Unfortunately, there is a minor complication. When the chef goes to the refrigerator to get the butter, he or she will find butter in one-pound or fifty-five–pound blocks. Because they are dealing in such large volumes, chefs use cooking measurements like ounces and pounds that can easily be adjusted for servings both great and small. The original recipe requires one-half cup of butter or one-fourth of a pound. Twenty times that amount would be five pounds of butter, or five one-pound packages. A fifty-five–pound block of butter has the approximate dimensions of one foot by one foot by two feet, and amounts are sliced off using cheese cutters (a length of wire with wooden handles at each end).

What would the chef do to determine the amount of flour that would be required? If two cups of flour are needed for the recipe that serves ten, the chef could attempt to count out forty cups, but it would be too time consuming, easy to lose count, and generally unreasonable. Is it possible to figure out the flour necessary by converting forty cups of flour into pints, quarts, or gallons? Given that there are sixteen cups in a gallon, could the chef use two gallons (32 cups) and two quarts (8 cups) to measure the flour? The answer is no. Pints, quarts, and gallons are liquid measurements, not dry measurements. The flour will have to be measured using ounces or pounds.

Measurement Conversions

For the most part, all measurements in a commercial kitchen are done by weight. One cookbook lists that four cups of flour equals one pound. The two cups of flour for the cake that serves ten would weigh one-half pound. Therefore, for a cake that serves two hundred, the recipe would call for twenty times that amount, or ten pounds of flour.

Chefs are extremely well versed in making measurement conversions such as ounces to pounds and teaspoons to quarts; measurement conversion is a tool of the trade. In actuality, it would be extremely uncommon for a chef to bring a recipe from home in the hope of using it for a large group. Most recipes used in commercial kitchens are passed along from other commercial kitchens.

Interestingly, the type of measurement used in a recipe depends on where that recipe originated. Recipes from the United States are written using ounces and pounds, whereas European recipes are written using metric measurements. Dry quantities are measured by weight using grams, and liquids are measured by weight (grams) or volume (liters). In a home kitchen, measuring cups and spoons are used, whereas commercial kitchens tend to use larger metric and standard cups for liquids and a scoop and scale for the dry ingredients. The scoop may seem odd, but in those large kitchens dry ingredients are commonly stored in hefty bins, and the scoop is used to transfer materials to the scale. On the scale sits a removable bowl that is used, after holding the ingredients while they are measured, to transfer the measured quantity into the mixing bowl, or, more accurately, the mixing tub.

The use of metric measurement is becoming more common in commercial kitchens and actually minimizes the necessity for computational conversions. The metric system works so easily because it is based upon quantities of ten and mirrors our decimal number system. Often when a chef receives a recipe that uses standard measurements, he or she will take the time to convert the recipe to metric amounts. Rarely is the opposite true (unless the recipe is being shared with a nonchef).

Attempting to get the correct measurement is not the only obstacle in commercial cooking; the subject of chemistry must be considered when determining whether a recipe can be enlarged or reduced to a great extent. Some recipes can be doubled, but if they are to be increased by three times or more, the recipes will not work. It is not because the mathematics cannot be done but because of the chemical reaction that occurs among flour, baking soda or baking powder, and liquid.

Baking soda and baking powder are leavening agents that react with flour and liquid, but their ratio of use is not linear. Just because a recipe calls for one teaspoon of baking powder, two cups of flour, and one-half cup of liquid, that does not mean that five times that amount (five teaspoons of baking powder, ten cups of flour, and two and one-half cups of liquid) would bake properly. More than likely, only half the amount of baking powder would be necessary, but there is not an exact relationship between the two products. A chef would need to do experimentation to figure out how far a recipe can be taken up or down. Yet some recipes can be increased or decreased endlessly as long as their ingredients do not form specific chemical reactions.

Yeast is another leavening agent, but it is different from baking soda and baking powder. Yeast is actually a live organism. When it is mixed with warm liquid and flour, a reaction occurs in which the organism releases gas. These gases change the organic composition of the mixture and cause a volume increase.

Estimation in the Culinary World

Chefs are not only responsible for doing the actual preparation of food but also for planning menus and assisting in the purchase of groceries. Whether it is a hotel, restaurant, or cafeteria, the chef must estimate the number of people who will eat and plan to have enough food. Not only is it important to have enough food, but the chef must also attempt to minimize the amount

of wasted food. One possibility for waste occurs when a chef prepares too much food and the excess cannot be saved and used at a future time. Another type of waste occurs when too much food has been ordered and spoils before it can be used.

When preparing food for a cafeteria, the chef will take into account the number of people who will be eating and the size of a serving for each food item. A cushion amount of about 10 percent would then be added to ensure (or at least hope to ensure) that there is enough food. A cafeteria setting is more forgiving; when an item runs out, the pan can just be removed from the display window.

Determining the quantities necessary for preparation in a restaurant is more difficult, and much is based on previous business. The tricky part is becoming familiar with the number of food servings that can be prepared given a purchased quantity. For example, one pound of uncooked pasta yields three pounds of cooked pasta, beef tenderloin yields about 50 percent to 60 percent when cleaned and trimmed, and one case of romaine lettuce makes about seventy-five salads.

There is more to measurement in cooking than some might think. Not only do chefs deal with precise measurements when following a recipe, but they must also deal with vast quantities (of patrons and of food packaged in bulk). Chefs are also responsible for measurement when planning menus and purchasing food. For the most part, these skills are not learned from a textbook but rather from experience. SEE ALSO Restaurant Manager.

Elizabeth Sweeney

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Coordinate System, Polar

The polar coordinate system is an adaptation of the two-dimensional coordinate system invented in 1637 by French mathematician René Descartes (1596–1650). Several decades after Descartes published his two-dimensional coordinate system, Sir Isaac Newton (1640–1727) developed ten different coordinate systems. One of the ten systems was a polar coordinate system. Newton and others used the polar coordinate system to plot a complex curve known as a spiral. It was Swiss mathematician Jakob Bernoulli (1654–1705) who first used a polar coordinate system for a wider array of calculus problems and coined the terms "pole" and "polar axis" that are still used today in polar coordinate systems.

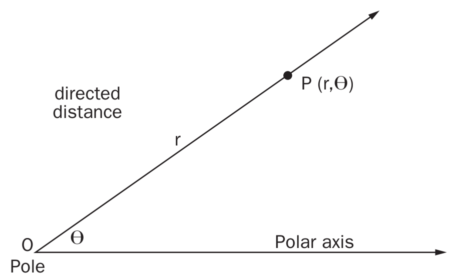
Understanding Polar Coordinates

As with the two-dimensional Cartesian coordinate system, one can describe the location of points in a polar coordinate system by means of coordinates. Both systems involve an origin point and axis lines. In the polar coordinate system a single axis, or a polar axis, extends indefinitely from the origin, known in the polar coordinate system as the pole. This polar axis is a fixed

line and the location of all points and figures is based on this fixed, polar axis. Every point in the polar coordinate system is described in terms of the **directed distance** the point is from the pole, and the angle of rotation the directed distance line makes with the polar axis.

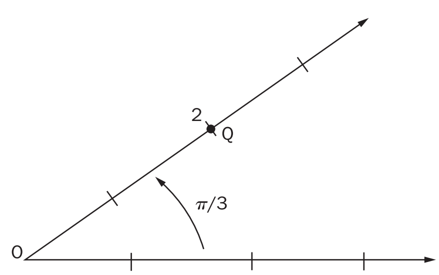
directed distance the distance from the pole to a point in the polar coordinate plane

In the diagram below, Point P is on a directed distance line that is at an angle θ from the polar axis. Point P is at a distance r from the pole along the directed distance line. The polar coordinates for Point P are represented as (r, θ). In a polar coordinate system, the angle of the directed distance line follows the distance from the pole that the point is located along the directed distance line.



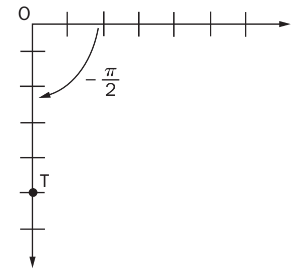
Although it is possible to locate a point by representing the angle of rotation with degrees, in a polar coordinate system the angle is usually represented by **radians**. Radians can conveniently express angle measures in terms of π. For example, an angle $\frac{\pi}{2}$ of radians is equal to an angle of 90° and an angle of π radians is equal to an angle of 180°. The rotation of any positive angle is always in a counterclockwise direction. Thus Point Q $(2, \frac{\pi}{3})$ is shown below.

radian an angle measure approximately equal to 57.3 degrees, it is the angle that subtends an arc of a circle equal to one radius



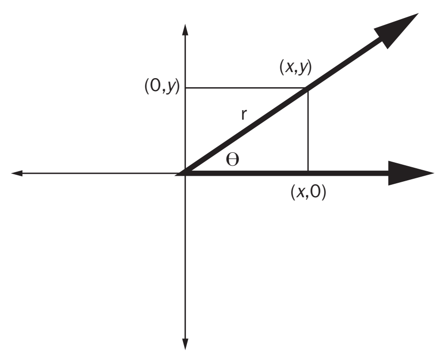
An angle measured as $\frac{\pi}{3}$ radians is equal to a 60° angle. Point Q is located on a directed distance line that makes a 60° angle with the polar axis, at a distance of 2 units from the pole (Point O).

The angle for the rotation of the directed distance may be generated in a clockwise direction by labeling the angle with a negative sign. In the following diagram, Point T $(5, -\frac{\pi}{2})$ is shown. Point T is located 5 units from the origin on a directed distance line that makes a 90° angle with the polar axis, but in a clockwise or negative rotation.



The polar coordinates for Point T may also be written as $(5, \frac{3\pi}{2})$ to indicate a positive or counterclockwise rotation for the directed distance line. Likewise, every point in a polar coordinate system may be represented by positive or negative rotations by the directed distance line. Generally, positive rotations are used to represent the location of a point in polar coordinate systems.

When the location of a point is represented in polar coordinates, it may be converted into two-dimensional or Cartesian coordinates. The angle of rotation for the directed distance line and the distance form the pole may be used to find the x, y coordinates on a Cartesian coordinate system as pictured below.



The point (x, y) in the Cartesian coordinate system is the same as (r, θ) in the polar coordinate system. Simple **trigonometry** procedures enable mathematicians to convert from one system to the other.

trigonometry the branch of mathematics that studies triangles and trigonometric functions

Using Polar Coordinates

Mathematicians find the polar coordinate system may be used more easily than the two-dimensional coordinate system to represent circles and other figures with curved lines. The polar coordinate system can also be more easily applied to certain real-life situations because it requires only a single axis line in order to represent the location of any point, as for example with navigation. Any location may be selected as the pole, and then the polar axis may be determined by sighting along any line from the pole. Once the pole and polar axis have been determined, any location may be represented by polar coordinates. In addition, the directed distance line can be expressed simply as an angle of rotation from the polar axis. Thus, an expression such

as $\frac{\pi}{4}$ can represent the heading of a ship at sea. SEE ALSO BERNOULLI FAMILY; CIRCLES, MEASUREMENT OF; COORDINATE SYSTEM, THREE-DIMENSIONAL; DESCARTES AND HIS COORDINATE SYSTEM; NAVIGATION; TRIGONOMETRY.

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Coordinate System, Three-Dimensional

The three-dimensional coordinate system is an extension of the two-dimensional coordinate system invented in 1637 by French mathematician René Descartes (1596–1650). Soon after Descartes wrote about his two-dimensional coordinate system, other mathematicians took Descartes's idea and expanded it from a two-dimensional plane to three-dimensional space. This new development greatly enlarged the uses of the coordinate system.

With the enhancement of the two-dimensional coordinate system to three dimensions, it was possible to locate any object in three-dimensional space. For example, with Descartes's two-dimensional coordinate system, you could describe the location of a coin on the floor of a room by referring to how many feet the coin was located from a front wall and a side wall. However, in a two-dimensional coordinate system, it is impossible to describe the location of an object that is off the **plane** of the floor, such as the location of a crystal on a chandelier that is hanging from the ceiling. In a three-dimensional coordinate system, it is possible to refer to the location of a chandelier crystal with reference to the two walls of the room and also the floor.

plane generally considered an undefinable term, a plane is a flat surface extending in all directions without end, and that has no thickness

perpendicular forming a right angle with a line or plane

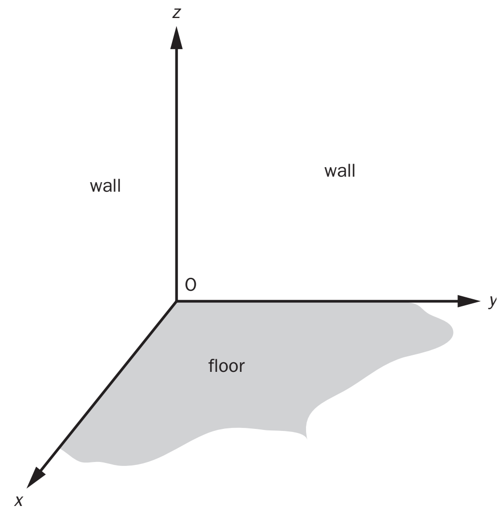
Understanding Three-Dimensional Coordinates

A two-dimensional coordinate system is formed by two **perpendicular** lines designated as an x-axis and a y-axis. The two lines meet at a point called the origin. With a three-dimensional coordinate system, a third axis line is added. The third axis line intersects the other two axis lines at the origin, forming right angles with each of them. The x-axis shows locations of points in space lying in front and back of the origin. The y-axis shows the location of points in space to the left and to the right of the origin. The z-axis shows points in space lying above and below the origin.

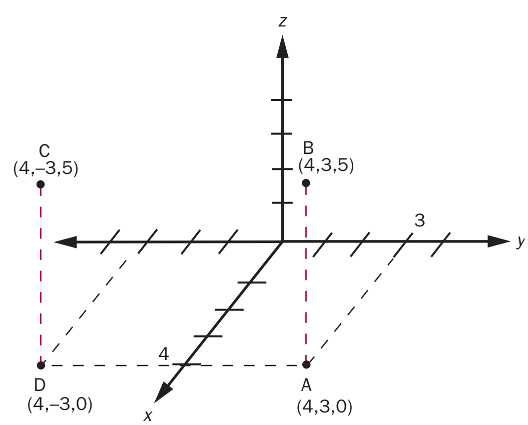
One way to visualize a three-dimensional coordinate system is to imagine yourself standing in a room near one of its corners, and facing the corner. The x-axis extends along the base of the wall coming towards you. The y-axis extends along the base of the other wall (either extending to your left or right depending on which side of the corner you are facing). The z-axis extends upward to the ceiling.

USING A THREE-DIMENSIONAL COORDINATE SYSTEM

Graphic artists, fashion designers, tool and die makers, and blueprint designers are examples of people who use a three-dimensional coordinate system to map three-dimensional objects—in this case, on a two-dimensional computer monitor or a drawing board. A three-dimensional coordinate system helps the artist draw the object in proper perspective. This system also insures that any object drawn by the artist can be properly interpreted by anyone familiar with three-dimensional coordinates.



In the sketch above and its three-dimensional coordinate system shown below, suppose you are standing at point A. The coordinates of the soles of your feet would be (4, 3, 0), meaning that you are standing four units in front of the origin along the *x*-axis; three units to the right of the origin along the *y*-axis; and are neither above nor below the origin with respect to the *z*-axis. The coordinates of the top of your head might be (4, 3, 5), meaning that your head is 5 units above the origin. If your identical twin were standing in the next room directly "across" from you (through the wall) and at mirrored distances from both walls, the coordinates of his feet and head would be (4, −3, 0) and (4, −3, 5).



This diagram gives a limited illustration of how objects can be located using a three-dimensional coordinate system, because it shows only the four quadrants lying "toward" the observer or "in front of" the origin relative to the *x*-axis. There are four complementary quadrants lying "away from" the observer or "behind" the origin relative to the *x*-axis. Hence, the three-dimensional coordinate system can be used to represent any point in space relative to the three axis lines.

The three-dimensional coordinate system may also be used to determine the distance between any two points of a three-dimensional object. In

the sketch above, the distance between Point B and the origin may be calculated by using the following formula:

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

In this formula x_1, y_1, z_1 represents the coordinates of Point B and x_2, y_2, z_2 represents the coordinates of the origin. Thus the distance between Point B and the origin can be calculated as shown below.

$$\text{distance} = \sqrt{[(4-0)^2 + (3-0)^2 + (5-0)^2]}$$

$$\text{distance} = \sqrt{4^2 + 3^2 + 5^2}$$

$$\text{distance} = \sqrt{16 + 9 + 25}$$

$$\text{distance} = \sqrt{50}$$

$$\text{distance} \approx 7$$

SEE ALSO Descartes and His Coordinate System; Computer Animation; Coordinate System, Polar; Dimensions.

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Coordinate System, Two-Dimensional *See Descartes and His Coordinate System.*

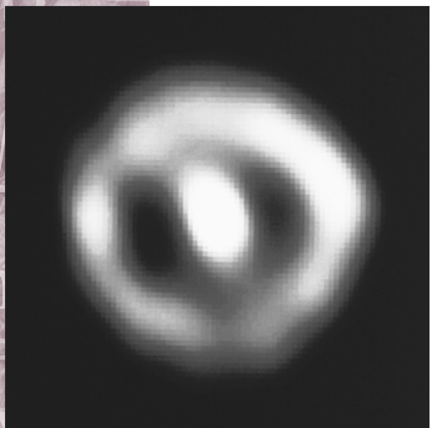
Cosine *See Trigonometry.*

Cosmos

Reading the chapter titles in a modern college astronomy textbook is much like reading the titles of science fiction stories. Astronomers regularly talk about subjects like black holes, neutron stars, pulsars, quasars, dark matter, novae, supernovae, and even more bizarre topics. Some of the ideas in astronomy push at the limits of what we know, or can know. Many ideas in astronomy are so strange that some astronomers have difficulty accepting them. Understanding these ideas in astronomy requires knowledge of all branches of mathematics, including some less well-known branches like tensor calculus and gauge theories.

Novae and Supernovae

When a star with a mass about the same as our Sun reaches the end of its life, its outer layers slough off, leaving behind a solid carbon core. This core, known as a "white dwarf," is very small, about the size of a small planet. Initially, the core is extremely hot with a temperature of over one million kelvin, although it does eventually cool off. It is also very dense, packing half the mass of the Sun into a sphere the size of Earth or smaller.



The star death of Nova Cygni 1992, seen here in an ultraviolet image with a ring-like shell of gas surrounding it, occurred in May 1993.

Sometimes, the white dwarf can be part of a binary star system. The brightest star in our night sky, Sirius, is an example of this kind of system. Sirius has a white dwarf companion called Sirius B. In a binary star system with a normal star and a white dwarf the companion to the white dwarf may also reach the end of its life cycle and become a "red giant." When this happens, the red giant can swell up so much that its outer envelope can be pulled onto the surface of the white dwarf. This happens because the red giant swells up past the **Roche limit**. Outside the red giant's Roche limit, the gravity from the white dwarf is stronger than the gravity from the red giant.

When matter spills over it can collect on the surface of the small, dense white dwarf. The new material is compressed by the white dwarf's intense gravity and is also heated from below. Compression and heating can raise the temperature of the new material until it passes 10,000,000 K, the temperature at which **nuclear fusion** begins. Normally, nuclear fusion is taking place deep in the interior of a star. In this case, however, nuclear fusion happens on the surface of the white dwarf. It can be likened to millions of hydrogen bombs going off all at once. This event is called a "nova" (from the Latin *nova stellaris*, "new star").

Supernovae are thousands of times more luminous than novae. One kind of supernova results from the death of a high-mass star, several times more massive than the Sun. Through nuclear fusion, these stars have converted hydrogen into helium, then helium into carbon, and continued to fuse nuclei into heavier and heavier elements until the core of the star is made of iron. However, the appearance of iron in the core terminates the fusion process. When iron fuses with another nucleus, it absorbs energy instead of releasing energy. This turns off energy production in the core and it begins to collapse causing more iron fusion. As more iron fuses the collapse goes faster and faster, approaching the speed of light. The upper layers of the star are no longer supported by heat and pressure from the core, and they also come crashing down, and the star **implodes**.

At the time of implosion, the core temperature of the star is several *billion* kelvin. The intense radiation from this enormously hot material has so much energy it begins to tear the atoms of iron and other elements apart, converting the core into protons, electrons, and neutrons. In less than a second, the star undoes all the effects of the last 10 million years! The core is now so dense that the protons and electrons are forced to combine and also become **neutrons**. The core of the star becomes a solid, rigid sphere of neutrons. When the outer layer of the star crashes down onto this rigid sphere, the whole thing bounces, sending all of the remaining layers of the star off into space in an enormous explosion called a supernova.✶ All that remains of the original star is the core of neutrons.

Another type of supernova has a different cause. It is similar to an ordinary nova, and also occurs in a double star system with one white dwarf. When the companion becomes a red giant, material can fall onto the surface of the white dwarf. In this case, the material does not explode but simply collects, increasing the mass of the white dwarf. However, if the resulting object exceeds 1.4 solar masses, it can collapse. The 1.4 solar mass limit is known as the **Chandrasekhar limit**, named after the Nobel prize winning

Roche limit an imaginary surface around a star in a binary system; outside the Roche limit, the gravitational attraction of the companion will pull matter away from a star

nuclear fusion mechanism of energy formation in a star; lighter nuclei are combined into heavier nuclei, releasing energy in the process

implode violently collapse; fall in

neutron an elementary particle with approximately the same mass as a proton and neutral charge

✶**At its peak, a supernova will emit more light than an entire galaxy of stars.**

Chandrasekhar limit the 1.4 solar mass limit imposed on a white dwarf by quantum mechanics; a white dwarf with greater than 1.4 solar masses will collapse to a neutron star

physicist Subramanyan Chandrasekhar, who proposed the idea. Above 1.4 solar masses, the white dwarf cannot support itself and it collapses as the carbon atoms begin to fuse. The core is so dense and rigid that all of the carbon fuses into heavier elements in a few moments. The star blows up and becomes as bright as a galaxy for a few days. Unlike an implosion-type supernova, the carbon detonation supernova leaves nothing behind.

Near the beginning of our universe, when ordinary matter condensed out a sea of radiation, the universe contained just two kinds of atoms, hydrogen and helium. However, the universe now contains many more kinds of atoms. The elements of the book you are holding, the elements that you are made of, the iron in your blood, the nitrogen in your DNA, the carbon stored in your tissues, the oxygen you are breathing, none of these existed when the material universe formed. So where did these elements come from? Amazingly, all of the elements in the periodic table, including those elements that make people, came from the explosions of stars. Except for the hydrogen, you are made of stardust!

Neutron Stars and Pulsars

When that implosion-type supernova blew up and scattered its outer layers into space, it left a remnant behind. The core of the star, where all of the protons and electrons were forced together into neutrons, is still there. It is called a "neutron star" although it is technically no longer a star. It is a stellar remnant. Neutron stars are extremely hot at first, but also extremely small. A neutron star is only a few kilometers across, the size of an asteroid in our solar system. Yet a neutron star has more mass than our sun. A teaspoon of matter from a neutron star would have the mass of a mountain! In addition to large mass and small size, they also have one other important characteristic: They spin very rapidly. You may have noticed an ice skater pulling her or his arms in and spinning faster as a result. A neutron star does the same thing. The star from which it was formed may have been rotating once a month or so, but by the time the core has collapsed to the size of an asteroid, it is spinning several times a minute or faster.

The neutron star also has a powerful magnetic field, which is captured from the star that exploded. It is because of the rapid spin and strong magnetic fields of neutron stars that we know they exist. In 1967, a graduate student named Jocelyn Bell detected radio waves coming in rapid, regular pulses. They were so regular that Bell and her advisor, Antony Hewish, first referred to them as LGMs (for "Little Green Men"!) because they thought at first that they might be artificial. It soon became apparent that they were far too powerful to be any sort of artificial beacon. These objects are now known as pulsars. Most emit radio waves and a few also emit pulses of light or radiation at even higher frequencies.

A few pulsars are associated with supernova remnants. When a star blows up, it scatters its outer layers back into space. These tatters are visible for a few hundred years after the explosion. The best known is the Crab nebula, which originated from a supernova known to have exploded in 1054 C.E. At the heart of the Crab nebula, right where a neutron star would be expected, is the Crab nebula pulsar. Astronomers now suggest that pulsars are spinning neutron stars. But what causes the pulses?

The axis of Earth's magnetic field does not line up with its axis of rotation. As the particles trapped in Earth's magnetic fields crash into the atmosphere above the north and south magnetic poles, auroras are created. From space, it is sometimes possible to see one or the other of these areas of aurora flash on and off as Earth rotates every 24 hours. Astronomers think that a similar thing is happening with some neutron stars. If the magnetic field of the spinning neutron star is at an angle to the axis of rotation, then two rotating beams of radiation might be emitted, one from each of the magnetic poles of the neutron star. Because of its resemblance to the rotating beacon in a lighthouse, this is called the "lighthouse" model of pulsars. As these beams sweep by Earth, we perceive a pulsar. So pulsars are evidence that neutron stars exist and neutron stars are the explanation for pulsars.

Black Holes

Sometimes even more massive stars collapse and blow up. If the remaining core has a mass greater than 3.0 solar masses, then no force is strong enough to stop its collapse. The core passes the neutron star phase and simply keeps on collapsing. It collapses right out of our universe, leaving behind nothing but its gravity! This bizarre end point of stellar evolution is called a "black hole." Black holes are just about the strangest objects in our universe. The principles of Newtonian mechanics do not apply in the space near black holes. To understand what is going on, astronomers must use a more modern theory of gravity known as **Einstein's General Theory of Relativity**. This theory deals with ideas like **curved space** and **time dilation**.

Einstein's General Theory of Relativity Einstein's generalization of relativity to include systems accelerated with respect to one another; a theory of gravity

curved space the notion suggested by Albert Einstein to explain the properties of space near a massive object, space acts as if it were curved in four dimension

time dilation the principle of general relativity which predicts that to an outside observer, clocks would appear to run more slowly in a powerful gravitational field

escape speed the minimum speed an object must attain so that it will not fall back to the surface of a planet

We can get a hint of what is going on around a black hole by thinking about two consequences of relativity: Nothing can travel faster than light, and gravity acts on everything, including light. Imagine going outside and throwing a baseball straight up. It will rise to a certain height and then begin to fall back. If you throw it harder, it will rise higher before falling back. Now imagine throwing the baseball so hard that it will rise infinitely high before falling back. The speed at which you would have to throw the baseball (or any other object) so that it rise to an infinite height is known as **escape speed**. For Earth, escape speed is about 11 km/s.

Now imagine squeezing Earth into a smaller, denser sphere one-fourth the size of its present radius. It would have the same mass but a smaller radius. If you were still standing on this smaller sphere, you would be closer to the center of its mass, so gravity would be higher and the escape speed would be greater too. It would be twice as large, about 22 km/s. If Earth were squeezed down to 1/1000 of its present size, its escape speed would be 630 km/s. Squeeze Earth down to a radius of one centimeter, and its escape speed would be 300,000 km/s. But 300,000 km/s is the speed of light, the fastest speed allowed by the laws of physics. So at a size of one centimeter, nothing could escape Earth's gravity. It would be a black hole.

Black holes are not cosmic vacuum cleaners. If the Sun, by some strange and impossible process, turned into a black hole, Earth and all the other planets would continue to travel along in their orbits as if nothing had happened. The mass would still be there and the gravity of the Sun out at the orbit of Earth would be unchanged. However, matter that does fall into a black hole does get compressed and heated to extremely high temperatures.

Our galaxy, known as the "Milky Way," contains roughly 100 billion stars.

This is how we know black holes exist. The matter spiraling into a one solar mass black hole would be heated to the point that it emits X-rays.

Dark Matter

Our galaxy is rotating. It is not a solid disk—each individual star or star cluster orbits the center of mass of the entire galaxy. Out at the edge of the visible galaxy, stars should be orbiting as if they were outside all of the matter of the galaxy. However, stars close to the edge of the visible disk of the galaxy are moving faster than can be accounted for by just the visible matter in our galaxy. The conclusion is that there is a large quantity of mass in the galaxy, as much as 90 percent of the mass of the galaxy, which cannot be seen. It does not emit or reflect any form of electromagnetic radiation, so it is called **cold dark matter**. We cannot see it, but we know it must be there because of the effects of its gravity on the part of the galaxy we can see.

cold dark matter hypothetical form of matter proposed to explain the 90 percent of mass in most galaxies that cannot be detected because it does not emit or reflect radiation

Astronomers do not know what the dark matter is. It is unlikely that dark matter is composed of black holes because the massive stars needed to form a black hole are not that common. The best candidate for at least some of the dark matter is a hypothetical exotic subatomic particle. Astronomers have dubbed these particles "Weakly Interacting Massive Particles" (WIMPs).

Quasars

Out near the limits of what we are able to observe with telescopes lie enormously energetic objects called "quasars." The term is a combination of the words "quasi stellar" objects. The name originated because when quasars were first detected, they looked like points of light, similar to stars, but they had **spectra** completely different from the spectra expected from stars. At first, these were thought to be nearby objects. Then they were discovered to have very large **red shifts**. Some astronomers thought that some sort of cosmic explosion might have occurred to fling these objects at speeds near the speed of light, but no evidence of such an explosion has ever been found.

spectrum the range of frequencies of light emitted or absorbed by an object

red shift motion-induced change in the frequency of light emitted by a source moving away from the observer

cosmological distance the distance a galaxy would have to have in order for its red shift to be due to Hubble expansion of the universe

Most astronomers now think that quasars are at **cosmological distances**. However, if quasars are this far away—200 million light years or more—they must emit prodigious amounts of radiation. Astronomers have proposed and rejected many different explanations for the enormous energy output of quasars. The only mechanism that seems to fit the data is matter falling into a super-massive black hole. A black hole with 10^8 or 10^9 solar masses can account for the energy output of even the most energetic quasar. A quasar emitting 10^{40} watts of power can be nicely explained by a 10^9-solar mass black hole consuming the equivalent of 10 stars per year in mass. This means that quasars are not a distinctly different class of objects, but are simply the extreme end of a spectrum of energy-emitting galaxies, including our own galaxy, all powered by massive black holes at the center. It also suggests that quasars and other active galaxies can evolve into normal galaxies as the available matter gradually falls into the black hole, leaving the remaining stars and other material orbiting at safe distances. SEE ALSO Light Speed; Universe, Geometry of.

Elliot Richmond

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Cryptology

Cryptology is the study of encoding and decoding messages and the study of the mathematical foundations of cryptographic messages. The processes involved in cryptology rely on the principles of mathematics and statistics, and encompass areas such as **probability theory**, **number theory**, **abstract algebra**, and **formula analysis**.

probability theory the branch of mathematics that deals with quantities having random distributions

number theory the study of the properties of the natural numbers, including prime numbers, the number theorem, and Fermat's Last Theorem

abstract algebra the branch of algebra dealing with groups, rings, fields, Galois sets, and number theory

formula analysis a method of analysis of the Boolean formulas used in computer programming

Cryptography is the art of creating a code for a secret message, and is also known as encryption, or encrypting a message. Cryptanalysis is the art of breaking, or decrypting, the message without the use of the appropriate key. In other words, code-breakers are individuals who intercept or retrieve information not intended for them.

Historic Overview

The word "cryptology" comes from the Greek word *kryptos*, which means hidden. Cryptology dates back at least 4,000 years to its earliest recorded use in Egyptian hieroglyphics. Historically, the primary motivation for secure communications has been military: namely to keep the enemy from learning of plans even if they captured messages.

For example, when Julius Caesar (100 B.C.E.–44 B.C.E.) sent messages, he used a method that replaced every A with a D, every B with an E, and so on. Only those with whom Caesar entrusted his "shift by three" method could decrypt his messages.

Throughout the centuries, cryptology has continued to play a major role in military applications. For example, the success of mathematicians and

cryptanalysts in breaking the seemingly unbreakable Nazi Enigma code greatly influenced the outcome of World War II. The German Enigma machine was a complex electromechanical encoding apparatus whose invention in 1918 for banking purposes quickly led to use by the German military. To solve the Enigma machine cipher, analysts had to determine which of the 15 billion billion (15×10^{18}) settings were used.

Three young Polish mathematicians first broke the Enigma code in 1933. In 1939 they passed along their knowledge to Great Britain and France, who—with American support—went on to break the ever-changing versions of the wartime Enigma. British mathematician Alan Turing further developed an idea originally proposed by the Poles: namely, an electromechanical machine (later known as The Bombe) that greatly reduced the odds, and therefore the time required, to break the daily-changing Enigma keys.

In today's world of e-commerce and global concerns, secure communications are an ever-increasing necessity. Diplomatic codes are now common. And the importance of information in everyday transactions has increased interest in both business and personal security.

DEVILISH SECRETS?

Because the science dealing with the transfer of secret messages hidden within codes is often associated with espionage, classified military information, secret meetings and activities, and other covert activities, cryptology has at times had a dark and mysterious reputation. Although early cryptographers were actually scientists, many common people thought they were engaged in the dark arts, and perhaps were even followers of the devil.

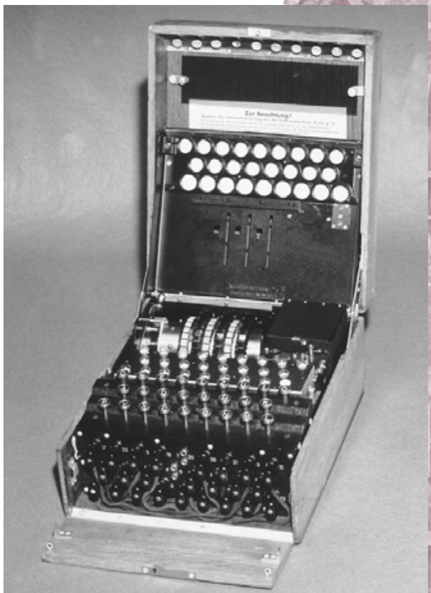
Encryption and Security

Most encryption involves a *key* and a set of steps or a procedure known as an **algorithm**. The key is an item of information used repeatedly in the algorithm while the message (called *plaintext*, even if it is sound or pictures) is being encrypted. The key customizes the algorithm. It is used with the algorithm for encoding and again later for decoding an entire message. If one knows the algorithm and the key, retrieving the plaintext from an encoded message is easy.

algorithm a rule or procedure used to solve a mathematical problem

Encryption systems fall into one of two main categories. In symmetric-key encryption, the key is the same for the encryption of the message as for the decryption: that is, both the encoder and the decoder know and use the same key. In asymmetric or "public-key" encryption, the key provided to the decoder is not the same as the encoder's key.

The transmission of symmetric-key encryptions is safe provided that nobody besides the sender and intended receiver obtains the key. However, the problem in security arises from the difficulty of securely transferring the key so that those receiving the encrypted message are able to decrypt it. Asymmetric keys minimize this problem because a public-key cryptosystem uses a pair of keys—a private key that remains with the encryptor, and a public key that is openly available to all users, thus eliminating the need to transfer a key in secrecy.



Messages encrypted by the German Enigma cipher machine (above) during World War II were first decoded by three Polish mathematicians. To honor these achievement, Britain's Prince Andrew presented a version of the Enigma to Poland's prime minister in September 2000.

Levels of Security. Systems that involve transmitting or storing the key with each message are very insecure. If an unauthorized person can recognize the key, then the next step is to recognize, guess at, or figure out the algorithm. Even without the key, the codebreaker can guess the algorithm, and then, by trying all the possible keys in succession, can conceivably recover the plaintext. For example, in Caesar's alphabetical cryptosystem discussed earlier, the cryptanalyst could simply try each of the 25 possible values of the key.

The security of transmissions can therefore be increased by increasing the number of possible keys and by increasing the amount of time it takes to try each key. If the same key is used for multiple messages, the cryptanalyst only has to figure out one key; but by varying the key from one message to another, the encryptor has essentially used a different procedure for encoding each one. With a complicated algorithm that may have a very large number of possible keys, even if the basic algorithm is known or guessed, the time and effort required to try all possible keys would take years, making decryption wholly impractical.

An example of a more secure algorithm is a book code. Both the sender and the receiver each have a copy of the same edition of some book. During encoding, each word in the plaintext is replaced with a code group that indicates where that same word appears in the book. Different occurrences of the same word in the plaintext may be represented by different code groups in the encoded message. With this method, the key is the book itself. Although a person who intercepts a message may guess that a book code is being used, the messages cannot be decoded unless the interceptor can determine what edition of what book is being used. The number of possible keys (books) is huge.

Before electronic computers became available, the most secure encryption method known was the *one-time pad.* The pad is a long list of different randomly chosen keys. Two (and only two) identical copies of the list of keys exist—one for the person encoding each message and another for whoever is decoding it. After being used for one message, a key is discarded and never used again. The next message will use the next key on the list. If the algorithm is even moderately complicated and the keys are long enough, cryptanalysis is practically impossible.

Cryptology and Computers

Electronic computing has revolutionized cryptology. Computers make it practical to use mathematical and logical algorithms that otherwise are much too complicated and time consuming. Encryption and decryption algorithms can be put into integrated circuit chips so that this technology can be economically applied anywhere. Yet governments are concerned that this will allow criminals to use modern communication and data storage methods without any fear of revealing their activities. Hence, consideration is being given to requiring that manufacturers include in encryption chips some facility for allowing authorized agents of the law to read messages encoded with them.

Computers have also brought cryptology into the home. As more and more everyday business transactions and personal communications are made at home by computer, personal privacy and transmission security have become everyone's concern. Present-day problems of security in transmission are compounded not only by the increased number of users but also by the users themselves—and a cryptosystem can be only as good as its users. Many attacks on private systems are inside jobs, and even honest users weaken the security of cryptosystems simply through careless use, such as failure to log off.

Because of the Internet, huge quantities of highly personal information are sent through a network of communications all over the world. Transmitted data includes credit card and bank account information and e-mail

correspondence, as well as sensitive company, military, and government information. Computer encryption systems provide the level of security that permits such large-scale transmission of data to be reasonably safe. Computer encryption programs use mathematical formulas to scramble data so that the code is extremely difficult to crack, thereby making cryptanalysis very time-consuming, expensive, and, in the end, often unsuccessful.

Toward an Unbreakable Code

The Data Encryption Standard (DES), developed by IBM and adopted in 1976 as the federal standard, remains today as America's official standard symmetric cryptosystem, although ongoing work on an advanced system may soon replace it. The DES is considered secure because it has a sufficiently large number of keys and encryption is done in eighteen steps, in each of which the bits are permuted and scrambled. Decryption basically involves running the entire eighteen-step process in reverse. An attacker would have to try so great a number of keys to crack the code as to make cryptanalysis infeasible.✶

✶When a message is encrypted according to the U.S. Data Encryption Standard, the coding relies on 72 quadrillion (10^{15}) keys.

Despite advances in cryptology, a known unbreakable code has yet to be discovered. From time to time, mathematicians and cryptographers have derived methods that they believed to be—or that actually were for a period of time—secure, but none has remained unbroken. However, as reported in newspapers across the country in March 2001, Harvard professor Michael Rabin has advanced a claim that he, along with his student Yangzhong Ding, has developed a mathematical proof that can be used to create a code unbreakable even by the most powerful computers.

In Rabin's method, a stream of random characters generated by a computer disappears after it is decoded by Rabin's mathematical proof, leaving nothing for a hacker to break and the method safe to repeat. Some professors have expressed reservations about the security of Rabin's method, and even his supporters remark that no code will remain unbreakable for very long.

On the technological horizon, however, looms the eventual advent of quantum and DNA computers, which will be capable of performing multiple tasks at a speed not possible for today's serial computers. Perhaps then cryptology will be able to boast of the discovery of a truly unbreakable code. SEE ALSO Analog and Digital; Communication Methods; Computers, Evolution of Electronic; Computers, Future of; Randomness; Turing, Alan.

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Cycling, Measurements of

The neighborhood kids riding their bicycles up and down the streets on a summer afternoon are not likely to be thinking about science as they feel the warm air against their faces or struggle to pedal up a hill. But in today's highly competitive sports environment, world-class racing bicyclists have to be part-time scientists, interested in the role mathematics, physics, and engineering play in improving the design of wheels, gears, frames, and other parts of a bicycle. They also need to understand the aerodynamics of cycling. Comprehension of these variables provides information that can help them shave valuable seconds off their time in a race and gain an edge over other competitors.

Wheels and Tires

Wheels have come a long way since the ancient Mesopotamians invented them around 3500 B.C.E. It was not until the Industrial Revolution that the wheel was made useful for "human-powered machines." The development of the **pneumatic tire** helped turn cycling into an immensely popular activity in the late nineteenth century.

pneumatic tire air-filled tire, usually rubber or synthetic

The earliest bicycles were "high-wheelers," or bikes with a huge wheel in front and a small wheel in back. These bikes did not have gears or even chains; pedals were connected directly to the bike's front wheel. Because the wheel was so large, a cyclist was able to travel as far as 140 inches with a single rotation of the wheel, making this bicycle remarkably fast on level ground. They were dangerous, though, especially on rough surfaces, because they were hard to pedal uphill, and the rider sat so high over the front wheel. It was not until the 1880s that the so-called safety bicycle was developed. With the rider suspended on a metal frame between two equally sized wheels. These bicycles were much safer and easier to pedal than the older varieties.

An important part of a bike's wheel is the spokes. They help reduce the weight of the bike (because the wheel is not solid) and make the wheel more efficient. Spoking can be done in two ways: radial or tangential. Radial spokes run directly from the hub of the wheel to the rim in a straight line. Tangential spokes, in contrast, connect the hub and the rim at a slight angle—at a **tangent** to the hub. Whereas the front wheel of a bike can be spoked radially, rear wheels have to be spoked tangentially; otherwise, they can not transmit torque (turning power) out from the hub to the rim efficiently. Tangentially spoked wheels are also stronger, and thus better able to withstand the forces created by steering, braking, and bumping up and down on an irregular road surface.

tangent a line that intersects a curve at one and only one point in a local region

Early tires were made from leather or solid rubber, so cycling was a pretty bumpy affair. Pneumatic, or air-filled, tires provide a more comfortable ride. Road bikes and touring bikes use thin tires that are inflated to 100 or 120 pounds per square inch. This high pressure keeps them from "flattening out" on the road, which can create friction and slow the rider down. Flattening out is often referred to as "rolling resistance" and describes how much energy is "lost to the road" as the wheel moves forward. Unlike road bikes, mountain bikes use fat tires, which flatten out on smooth surfaces but tend to "float" on top of rough surfaces.

Drives and Gears

Leonardo da Vinci developed the idea of a chain and cog in the fifteenth century, but it was not until the 1880s that the chain drive was commonplace on bikes. On a bicycle with a chain drive the rider is positioned between the two wheels, providing better balance and safety.

A chain drive without gears works well enough on flat surfaces, but headwinds and uphill climbs demand the use of gears. Without gears, one turn of the pedal equals one turn of the wheels. But gears allow the rider to change that ratio such that one revolution of the front wheels creates, say, two revolutions of the back wheel. Competitive bikers know exactly what the ratio is for each gear on their bike and chart the best ratio they need depending on conditions, for example whether they are going uphill or downhill.

Closely related to gearing is cadence, or the rate of pedaling. A competitive cyclist's goal is to find the most efficient cadence, or the cadence at which his or her body delivers the most energy to the bike. Road racers' cadences tend to range between 75 and 120 revolutions per minute; mountain bikers, on the other hand, tend to strive for a cadence of about 50 cycles per minute, though the wider range of conditions in mountain biking demands a wider range of cadences.

Frames

In deciding on the best material to use in a frame, bike makers measure three factors. The first is elasticity, or the ability of a material to return to its original shape when bent. The second is yield strength, or the amount of force needed to bend the material to a point where it can not return to its original shape. The third is ultimate strength, or the amount of force needed to break the material.

Recent years have seen a revolution in the types of materials used to construct bicycle frames. Early frames were made of steel tubing. Although steel is strong and has high ultimate strength, it made early bikes quite heavy, often over 80 pounds. Steel frames are still used today, but the tubes have thinner walls and are much lighter. A major advantage of steel is that it is inexpensive.

Cost, however, is not likely to be a major consideration to a competitive racer, who is likely to choose one of three other materials. One is aluminum, which is much lighter than steel but does not have as much yield strength, so the tubing has to be of a relatively large diameter. Another is titanium, which is extremely strong relative to its weight and has high elasticity—but at a cost up to 15 times that of steel. A third is carbon fiber, whose major advantage is that it does not have to be forged like tubing. It is more like a fabric that can be molded and tailored to provide maximum strength at stress points in the frame. It also does not have to be round. Carbon fiber can be formed into an oval or even teardrop shape, making the bike more aerodynamic. Frames made of either titanium or carbon fiber, because of their high elasticity and high ultimate strength but relatively low yield strength, have to be well designed to be stiff enough to resist pedaling forces.

Aerodynamics and minimizing wind resistance have become an integral part of cycling, a sport in which fractions of a second can make a difference in the outcome of a race.

Aerodynamics

Every cyclist is familiar with wind resistance (aerodynamic drag) which accounts for 70 to 90 percent of the resistance a cyclist feels while pedaling. Aerodynamic drag consists of two forces: direct friction and air pressure drag. Direct friction is created by the air's contact with the surface of the rider and the bicycle. Of greater importance, though, is air pressure drag. As the rider and the bike cut through the air, they create an area of high air pressure in front and an area of lower pressure behind. The two forces combine to literally pull the cyclist backward.

Cyclists calculate drag by factoring in velocity, wind speeds, the rider's weight, and the grade of the road. The result of their calculation is expressed in "newtons," defined as the unit of force needed to impart an acceleration of one meter per second to a mass of one kilogram. With this information the cyclist can then calculate the propulsive power needed to maintain velocity, usually expressed in watts. With this information the cyclist can calculate the number of calories per minute he or she has to burn to maintain speed. In a major race such as the Tour de France, cyclists burn up to 10,000 calories per day.

Competitive cyclists minimize drag by:

1. Selecting frame materials and shapes that have a good strength-to-weight ratio while improving aerodynamic efficiency;
2. Using disc wheels, which, while heavier than spoked wheels, reduce the number of drag-producing "air eddies" that spokes create;
3. Using drop bars, which reduce the size of the surface area their bodies present to the air;
4. Wearing tight, synthetic "skinsuits," which reduce direct friction; and
5. Drafting.

Drafting, an important technique in competitive road racing, is riding in the low-pressure area created by the air-pressure drag of a rider in the lead, which is sometimes just inches away. This low-pressure "eddy" can actually suck the rider forward. The lead cyclist even gains an advantage by having the low-pressure eddy filled by another rider. In a race, cyclists compete for position in packs called "pelotons," or in diagonal pace lines called "echelons." By riding in a group cyclists can save up to 40 percent of their energy compared to a rider cycling alone. Because a road's width can accommodate only so many riders, cyclists especially value the "gutter" position, the final place in the echelon line, where the benefits of drafting are most pronounced. SEE ALSO Athletics, Technology in.

Michael J. O'Neal

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Glossary

abscissa: the x-coordinate of a point in a Cartesian coordinate plane

absolute: standing alone, without reference to arbitrary standards of measurement

absolute dating: determining the date of an artifact by measuring some physical parameter independent of context

absolute value: the non-negative value of a number regardless of sign

absolute zero: the coldest possible temperature on any temperature scale; −273° Celsius

abstract: having only intrinsic form

abstract algebra: the branch of algebra dealing with groups, rings, fields, Galois sets, and number theory

acceleration: the rate of change of an object's velocity

accelerometer: a device that measures acceleration

acute: sharp, pointed; in geometry, an angle whose measure is less than 90 degrees

additive inverse: any two numbers that add to equal 1

advection: a local change in a property of a system

aerial photography: photographs of the ground taken from an airplane or balloon; used in mapping and surveying

aerodynamics: the study of what makes things fly; the engineering discipline specializing in aircraft design

aesthetic: having to do with beauty or artistry

aesthetic value: the value associated with beauty or attractiveness; distinct from monetary value

algebra: the branch of mathematics that deals with variables or unknowns representing the arithmetic numbers

algorithm: a rule or procedure used to solve a mathematical problem

algorithmic: pertaining to an algorithm

ambiguity: the quality of doubtfulness or uncertainty

analog encoding: encoding information using continuous values of some physical quantity

analogy: comparing two things similar in some respects and inferring they are also similar in other respects

analytical geometry: describes the study of geometric properties by using algebraic operations

anergy: spent energy transferred to the environment

angle of elevation: the angle formed by a line of sight above the horizontal

angle of rotation: the angle measured from an initial position a rotating object has moved through

anti-aliasing: introducing shades of gray or other intermediate shades around an image to make the edge appear to be smoother

applications: collections of general-purpose software such as word processors and database programs used on modern personal computers

arc: a continuous portion of a circle; the portion of a circle between two line segments originating at the center of the circle

areagraph: a fine-scale rectangular grid used for determining the area of irregular plots

artifact: something made by a human and left in an archaeological context

artificial intelligence: the field of research attempting the duplication of the human thought process with digital computers or similar devices; also includes expert systems research

ASCII: an acronym that stands for American Standard Code for Information Interchange; assigns a unique 8-bit binary number to every letter of the alphabet, the digits, and most keyboard symbols

assets: real, tangible property held by a business corporation including collectible debts to the corporation

asteroid: a small object or "minor planet" orbiting the Sun, usually in the space between Mars and Jupiter

astigmatism: a defect of a lens, such as within an eye, that prevents focusing on sharply defined objects

astrolabe: a device used to measure the angle between an astronomical object and the horizon

astronomical unit (AU): the average distance of Earth from the Sun; the semi-major axis of Earth's orbit

asymptote: the line that a curve approaches but never reaches

asymptotic: pertaining to an asymptote

atmosphere (unit): a unit of pressure equal to 14.7 lbs/in^2, which is the air pressure at mean sea level

atomic weight: the relative mass of an atom based on a scale in which a specific carbon atom (carbon-12) is assigned a mass value of 12

autogiro: a rotating wing aircraft with a powered propellor to provide thrust and an unpowered rotor for lift; also spelled "autogyro"

avatar: representation of user in virtual space (after the Hindu idea of an incarnation of a deity in human form)

average rate of change: how one variable changes as the other variable increases by a single unit

axiom: a statement regarded as self-evident; accepted without proof

axiomatic system: a system of logic based on certain axioms and definitions that are accepted as true without proof

axis: an imaginary line about which an object rotates

axon: fiber of a nerve cell that carries action potentials (electrochemical impulses)

azimuth: the angle, measured along the horizon, between north and the position of an object or direction of movement

azimuthal projections: a projection of a curved surface onto a flat plane

bandwidth: a range within a band of wavelengths or frequencies

base-10: a number system in which each place represents a power of 10 larger than the place to its right

base-2: a binary number system in which each place represents a power of 2 larger than the place to its right

base-20: a number system in which each place represents a power of 20 larger than the place to the right

base-60: a number system used by ancient Mesopotamian cultures for some calculations in which each place represents a power of 60 larger than the place to its right

baseline: the distance between two points used in parallax measurements or other triangulation techniques

Bernoulli's Equation: a first order, nonlinear differential equation with many applications in fluid dynamics

biased sampling: obtaining a nonrandom sample; choosing a sample to represent a particular viewpoint instead of the whole population

bidirectional frame: in compressed video, a frame between two other frames; the information is based on what changed from the previous frame as well as what will change in the next frame

bifurcation value: the numerical value near which small changes in the initial value of a variable can cause a function to take on widely different values or even completely different behaviors after several iterations

Big Bang: the singular event thought by most cosmologists to represent the beginning of our universe; at the moment of the big bang, all matter, energy, space, and time were concentrated into a single point

binary: existing in only two states, such as "off" or "on," "one" or "zero"

binary arithmetic: the arithmetic of binary numbers; base two arithmetic; internal arithmetic of electronic digital logic

binary number: a base-2 number; a number that uses only the binary digits 1 and 0

binary signal: a form of signal with only two states, such as two different values of voltage, or "on" and "off" states

binary system: a system of two stars that orbit their common center of mass; any system of two things

binomial: an expression with two terms

binomial coefficients: coefficients in the expansion of $(x + y^n$, where n is a positive integer

binomial distribution: the distribution of a binomial random variable

binomial theorem: a theorem giving the procedure by which a binomial expression may be raised to any power without using successive multiplications

bioengineering: the study of biological systems such as the human body using principles of engineering

biomechanics: the study of biological systems using engineering principles

bioturbation: disturbance of the strata in an archaeological site by biological factors such as rodent burrows, root action, or human activity

bit: a single binary digit, 1 or 0

bitmap: representing a graphic image in the memory of a computer by storing information about the color and shade of each individual picture element (or pixel)

Boolean algebra: a logic system developed by George Boole that deals with the theorems of undefined symbols and axioms concerning those symbols

Boolean operators: the set of operators used to perform operations on sets; includes the logical operators AND, OR, NOT

byte: a group of eight binary digits; represents a single character of text

cadaver: a corpse intended for medical research or training

caisson: a large cylinder or box that allows workers to perform construction tasks below the water surface, may be open at the top or sealed and pressurized

calculus: a method of dealing mathematically with variables that may be changing continuously with respect to each other

calibrate: act of systematically adjusting, checking, or standardizing the graduation of a measuring instrument

carrying capacity: in an ecosystem, the number of individuals of a species that can remain in a stable, sustainable relationship with the available resources

Cartesian coordinate system: a way of measuring the positions of points in a plane using two perpendicular lines as axes

Cartesian plane: a mathematical plane defined by the x and y axes or the ordinate and abscissa in a Cartesian coordinate system

cartographers: persons who make maps

catenary curve: the curve approximated by a free-hanging chain supported at each end; the curve generated by a point on a parabola rolling along a line

causal relations: responses to input that do not depend on values of the input at later times

celestial: relating to the stars, planets, and other heavenly bodies

celestial body: any natural object in space, defined as above Earth's atmosphere; the Moon, the Sun, the planets, asteroids, stars, galaxies, nebulae

central processor: the part of a computer that performs computations and controls and coordinates other parts of the computer

centrifugal: the outwardly directed force a spinning object exerts on its restraint; also the perceived force felt by persons in a rotating frame of reference

cesium: a chemical element, symbol Cs, atomic number 55

Chandrasekhar limit: the 1.4 solar mass limit imposed on a white dwarf by quantum mechanics; a white dwarf with greater than 1.4 solar masses will collapse to a neutron star

chaos theory: the qualitative study of unstable aperiodic behavior in deterministic nonlinear dynamical systems

chaotic attractor: a set of points such that all nearby trajectories converge to it

chert: material consisting of amorphous or cryptocrystalline silicon dioxide; fine-grained chert is indistinguishable from flint

chi-square test: a generalization of a test for significant differences between a binomial population and a multinomial population

chlorofluorocarbons: compounds similar to hydrocarbons in which one or more of the hydrogen atoms has been replaced by a chlorine or fluorine atom

chord: a straight line connecting the end points of an arc of a circle

chromakey: photographing an object shot against a known color, which can be replaced with an arbitrary background (like the weather maps on television newscasts)

chromosphere: the transparent layer of gas that resides above the photosphere in the atmosphere of the Sun

chronometer: an extremely precise timepiece

ciphered: coded; encrypyted

circumference: the distance around a circle

circumnavigation: the act of sailing completely around the globe

circumscribed: bounded, as by a circle

circumspheres: spheres that touch all the "outside" faces of a regular polyhedron

client: an individual, business, or agency for whom services are provided by another individual, business, or industry; a patron or customer

clones: computers assembled of generic components designed to use a standard operation system

codomain: for a given function *f*, the set of all possible values of the function; the range is a subset of the codomain

cold dark matter: hypothetical form of matter proposed to explain the 90 percent of mass in most galaxies that cannot be detected because it does not emit or reflect radiation

coma: the cloud of gas that first surrounds the nucleus of a comet as it begins to warm up

combinations: a group of elements from a set in which order is not important

combustion: chemical reaction combining fuel with oxygen accompanied by the release of light and heat

comet: a lump of frozen gas and dust that approaches the Sun in a highly elliptical orbit forming a coma and one or two tails

command: a particular instruction given to a computer, usually as part of a list of instructions comprising a program

commodities: anything having economic value, such as agricultural products or valuable metals

compendium: a summary of a larger work or collection of works

compiler: a computer program that translates symbolic instructions into machine code

complex plane: the mathematical abstraction on which complex numbers can be graphed; the *x*-axis is the real component and the *y*-axis is the imaginary component

composite number: an integer that is not prime

compression: reducing the size of a computer file by replacing long strings of identical bits with short instructions about the number of bits; the information is restored before the file is used

compression algorithm: the procedure used, such as comparing one frame in a movie to the next, to compress and reduce the size of electronic files

concave: hollowed out or curved inward

concentric: sets of circles or other geometric objects sharing the same center

conductive: having the ability to conduct or transmit

confidence interval: a range of values having a predetermined probability that the value of some measurement of a population lies within it

congruent: exactly the same everywhere; having exactly the same size and shape

conic: of or relating to a cone, that surface generated by a straight line, passing through a fixed point, and moving along the intersection with a fixed curve

conic sections: the curves generated by an imaginary plane slicing through an imaginary cone

continuous quantities: amounts composed of continuous and undistinguishable parts

converge: come together; to approach the same numerical value

convex: curved outward, bulging

coordinate geometry: the concept and use of a coordinate system with respect to the study of geometry

coordinate plane: an imaginary two-dimensional plane defined as the plane containing the x- and y-axes; all points on the plane have coordinates that can be expressed as x, y

coordinates: the set of n numbers that uniquely identifies the location of a point in n-dimensional space

corona: the upper, very rarefied atmosphere of the Sun that becomes visible around the darkened Sun during a total solar eclipse

corpus: Latin for "body"; used to describe a collection of artifacts

correlate: to establish a mutual or reciprocal relation between two things or sets of things

correlation: the process of establishing a mutual or reciprocal relation between two things or sets of things

cosine: if a unit circle is drawn with its center at the origin and a line segment is drawn from the origin at angle theta so that the line segment intersects the circle at (x, y), then x is the cosine of theta

cosmological distance: the distance a galaxy would have to have in order for its red shift to be due to Hubble expansion of the universe

cosmology: the study of the origin and evolution of the universe

cosmonaut: the term used by the Soviet Union and now used by the Russian Federation to refer to persons trained to go into space; synonomous with astronaut

cotton gin: a machine that separates the seeds, hulls, and other undesired material from cotton

cowcatcher: a plow-shaped device attached to the front of a train to quickly remove obstacles on railroad tracks

cryptography: the science of encrypting information for secure transmission

cubit: an ancient unit of length equal to the distance from the elbow to the tip of the middle finger; usually about 18 inches

culling: removing inferior plants or animals while keeping the best; also known as "thinning"

curved space: the notion suggested by Albert Einstein to explain the properties of space near a massive object, space acts as if it were curved in four dimensions

deduction: a conclusion arrived at through reasoning, especially a conclusion about some particular instance derived from general principles

deductive reasoning: a type of reasoning in which a conclusion necessarily follows from a set of axioms; reasoning from the general to the particular

degree: 1/360 of a circle or complete rotation

degree of significance: a determination, usually in advance, of the importance of measured differences in statistical variables

demographics: statistical data about people—including age, income, and gender—that are often used in marketing

dendrite: branched and short fiber of a neuron that carries information to the neuron

dependent variable: in the equation $y = f(x)$, if the function f assigns a single value of y to each value of x, then y is the output variable (or the dependent variable)

depreciate: to lessen in value

deregulation: the process of removing legal restrictions on the behavior of individuals or corporations

derivative: the derivative of a function is the limit of the ratio of the change in the function; the change is produced by a small variation in the variable as the change in the variable is allowed to approach zero; an inverse operation to calculating an integral

determinant: a square matrix with a single numerical value determined by a unique set of mathematical operations performed on the entries

determinate algebra: the study and analysis of equations that have one or a few well-defined solutions

deterministic: mathematical or other problems that have a single, well-defined solution

diameter: the chord formed by an arc of one-half of a circle

differential: a mathematical quantity representing a small change in one variable as used in a differential equation

differential calculus: the branch of mathematics primarily dealing with the solution of differential equations to find lengths, areas, and volumes of functions

differential equation: an equation that expresses the relationship between two variables that change in respect to each other, expressed in terms of the rate of change

digit: one of the symbols used in a number system to represent the multiplier of each place

digital: describes information technology that uses discrete values of a physical quantity to transmit information

digital encoding: encoding information by using discrete values of some physical quantity

digital logic: rules of logic as applied to systems that can exist in only discrete states (usually two)

dihedral: a geometric figure formed by two half-planes that are bounded by the same straight line

Diophantine equation: polynomial equations of several variables, with integer coefficients, whose solutions are to be integers

diopter: a measure of the power of a lens or a prism, equal to the reciprocal of its focal length in meters

directed distance: the distance from the pole to a point in the polar coordinate plane

discrete: composed of distinct elements

discrete quantities: amounts composed of separate and distinct parts

distributive property: property such that the result of an operation on the various parts collected into a whole is the same as the operation performed separately on the parts before collection into the whole

diverge: to go in different directions from the same starting point

dividend: the number to be divided; the numerator in a fraction

divisor: the number by which a dividend is divided; the denominator of a fraction

DNA fingerprinting: the process of isolating and amplifying segments of DNA in order to uniquely identify the source of the DNA

domain: the set of all values of a variable used in a function

double star: a binary star; two stars orbiting a common center of gravity

duodecimal: a numbering system based on 12

dynamometer: a device that measures mechanical or electrical power

eccentric: having a center of motion different from the geometric center of a circle

eclipse: occurrence when an object passes in front of another and blocks the view of the second object; most often used to refer to the phenomenon

that occurs when the Moon passes in front of the Sun or when the Moon passes through Earth's shadow

ecliptic: the plane of the Earth's orbit around the Sun

eigenvalue: if there exists a vector space such that a linear transformation onto itself produces a new vector equal to a scalar times the original vector, then that scalar is called an eigenfunction

eigenvector: if there exists a vector space such that a linear transformation onto itself produces a new vector equal to a scalar times the original vector, then that vector is called an eigenvector

Einstein's General Theory of Relativity: Albert Einstein's generalization of relativity to include systems accelerated with respect to one another; a theory of gravity

electromagnetic radiation: the form of energy, including light, that transfers information through space

elements: the members of a set

ellipse: one of the conic sections, it is defined as the locus of all points such that the sum of the distances from two points called the foci is constant

elliptical: a closed geometric curve where the sum of the distances of a point on the curve to two fixed points (foci) is constant

elliptical orbit: a planet, comet, or satellite follows a curved path known as an ellipse when it is in the gravitational field of the Sun or another object; the Sun or other object is at one focus of the ellipse

empirical law: a mathematical summary of experimental results

empiricism: the view that the experience of the senses is the single source of knowledge

encoding tree: a collection of dots with edges connecting them that have no looping paths

endangered species: a species with a population too small to be viable

epicenter: the point on Earth's surface directly above the site of an earthquake

epicycle: the curved path followed by planets in Ptolemey's model of the solar system; planets moved along a circle called the epicycle, whose center moved along a circular orbit around the sun

epicylic: having the property of moving along an epicycle

equatorial bulge: the increase in diameter or circumference of an object when measured around its equator usually due to rotation, all planets and the sun have equatorial bulges

equidistant: at the same distance

equilateral: having the property that all sides are equal; a square is an equilateral rectangle

equilateral triangle: a triangle whose sides and angles are equal

equilibrium: a state of balance between opposing forces

equinox points: two points on the celestial sphere at which the ecliptic intersects the celestial equator

escape speed: the minimum speed an object must attain so that it will not fall back to the surface of a planet

Euclidean geometry: the geometry of points, lines, angles, polygons, and curves confined to a plane

exergy: the measure of the ability of a system to produce work; maximum potential work output of a system

exosphere: the outermost layer of the atmosphere extending from the ionosphere upward

exponent: the symbol written above and to the right of an expression indicating the power to which the expression is to be raised

exponential: an expression in which the variable appears as an exponent

exponential power series: the series by which e to the x power may be approximated; $e^x = 1 + x + x^{2/2!} + x^{3/3!} + \ldots$

exponents: symbols written above and to the right of expressions indicating the power to which an expression is to be raised or the number of times the expression is to be multiplied by itself

externality: a factor that is not part of a system but still affects it

extrapolate: to extend beyond the observations; to infer values of a variable outside the range of the observations

farsightedness: describes the inability to see close objects clearly

fiber-optic: a long, thin strand of glass fiber; internal reflections in the fiber assure that light entering one end is transmitted to the other end with only small losses in intensity; used widely in transmitting digital information

fibrillation: a potentially fatal malfunction of heart muscle where the muscle rapidly and ineffectually twitches instead of pulsing regularly

fidelity: in information theory a measure of how close the information received is to the information sent

finite: having definite and definable limits; countable

fire: the reaction of a neuron when excited by the reception of a neurotransmitter

fission: the splitting of the nucleus of a heavy atom, which releases kinetic energy that is carried away by the fission fragments and two or three neutrons

fixed term: for a definite length of time determined in advance

fixed-wing aircraft: an aircraft that obtains lift from the flow of air over a nonmovable wing

floating-point operations: arithmetic operations on a number with a decimal point

fluctuate: to vary irregularly

flue: a pipe designed to remove exhaust gases from a fireplace, stove, or burner

fluid dynamics: the science of fluids in motion

focal length: the distance from the focal point (the principle point of focus) to the surface of a lens or concave mirror

focus: one of the two points that define an ellipse; in a planetary orbit, the Sun is at one focus and nothing is at the other focus

formula analysis: a method of analysis of the Boolean formulas used in computer programming

Fourier series: an infinite series consisting of cosine and sine functions of integral multiples of the variable each multiplied by a constant; if the series is finite, the expression is known as a Fourier polynomial

fractal: a type of geometric figure possessing the properties of self-similarity (any part resembles a larger or smaller part at any scale) and a measure that increases without bound as the unit of measure approaches zero

fractal forgery: creating a natural landscape by using fractals to simulate trees, mountains, clouds, or other features

fractal geometry: the study of the geometric figures produced by infinite iterations

futures exchange: a type of exchange where contracts are negotiated to deliver commodites at some fixed price at some time in the future

g: a common measure of acceleration; for example 1 *g* is the acceleration due to gravity at the Earth's surface, roughly 32 feet per second per second

game theory: a discipline that combines elements of mathematics, logic, social and behavioral sciences, and philosophy

gametes: mature male or female sexual reproductive cells

gaming: playing games or relating to the theory of game playing

gamma ray: a high-energy photon

general relativity: generalization of Albert Einstein's theory of relativity to include accelerated frames of reference; presents gravity as a curvature of four-dimensional space-time

generalized inverse: an extension of the concept of the inverse of a matrix to include matrices that are not square

generalizing: making a broad statement that includes many different special cases

genus: the taxonomic classification one step more general than species; the first name in the binomial nomenclature of all species

geoboard: a square board with pegs and holes for pegs used to create geometric figures

geocentric: Earth-centered

geodetic: of or relating to geodesy, which is the branch of applied mathematics dealing with the size and shape of the earth, including the precise location of points on its surface

geometer: a person who uses the principles of geometry to aid in making measurements

geometric: relating to the principles of geometry, a branch of mathematics related to the properties and relationships of points, lines, angles, surfaces, planes, and solids

geometric sequence: a sequence of numbers in which each number in the sequence is larger than the previous by some constant ratio

geometric series: a series in which each number is larger than the previous by some constant ratio; the sum of a geometric sequence

geometric solid: one of the solids whose faces are regular polygons

geometry: the branch of mathematics that deals with the properties and relationships of points, lines, angles, surfaces, planes, and solids

geostationary orbit: an Earth orbit made by an artificial satellite that has a period equal to the Earth's period of rotation on its axis (about 24 hours)

geysers: springs that occasionally spew streams of steam and hot water

glide reflection: a rigid motion of the plane that consists of a reflection followed by a translation parallel to the mirror axis

grade: the amount of increase in elevation per horizontal distance, usually expressed as a percent; the slope of a road

gradient: a unit used for measuring angles, in which the circle is divided into 400 equal units, called gradients

graphical user interface: a device designed to display information graphically on a screen; a modern computer interface system

Greenwich Mean Time: the time at Greenwich, England; used as the basis for universal time throughout the world

Gross Domestic Product: a measure in the change in the market value of goods, services, and structures produced in the economy

group theory: study of the properties of groups, the mathematical systems consisting of elements of a set and operations that can be performed on that set such that the results of the operations are always members of the same set

gyroscope: a device typically consisting of a spinning wheel or disk, whose spin-axis turns between two low-friction supports; it maintains its angular orientation with respect to inertial conditions when not subjected to external forces

Hagia Sophia: Instanbul's most famous landmark, built by the emperor Justinian I in 537 C.E. and converted to a mosque in 1453 C.E.

Hamming codes: a method of error correction in digital information

headwind: a wind blowing in the opposite direction as that of the course of a vehicle

Heisenberg Uncertainty Principle: the principle in physics that asserts it is impossible to know simultaneously and with complete accuracy the values of certain pairs of physical quantities such as position and momentum

heliocentric: Sun-centered

hemoglobin: the oxygen-bearing, iron-containing conjugated protein in vertebrate red blood cells

heuristics: a procedure that serves to guide investigation but that has not been proven

hominid: a member of family Hominidae; *Homo sapiens* are the only surviving species

Huffman encoding: a method of efficiently encoding digital information

hydrocarbon: a compound of carbon and hydrogen

hydrodynamics: the study of the behavior of moving fluids

hydrograph: a tabular or graphical display of stream flow or water runoff

hydroscope: a device designed to allow a person to see below the surface of water

hydrostatics: the study of the properties of fluids not in motion

hyperbola: a conic section; the locus of all points such that the absolute value of the difference in distance from two points called foci is a constant

hyperbolic: an open geometric curve where the difference of the distances of a point on the curve to two fixed points (foci) is constant

Hypertext Markup Language: the computer markup language used to create documents on the World Wide Web

hypertext: the text that contains hyperlinks, that is, links to other places in the same document or other documents or multimedia files

hypotenuse: the long side of a right triangle; the side opposite the right angle

hypothesis: a proposition that is assumed to be true for the purpose of proving other propositions

ice age: one of the broad spans of time when great sheets of ice covered the Northern parts of North America and Europe; the most recent ice age was about 16,000 years ago

identity: a mathematical statement much stronger than equality, which asserts that two expressions are the same for all values of the variables

implode: violently collapse; fall in

inclination: a slant or angle formed by a line or plane with the horizontal axis or plane

inclined: sloping, slanting, or leaning

incomplete interpretation: a statistical flaw

independent variable: in the equation $y = f(x)$, the input variable is x (or the independent variable)

indeterminate algebra: study and analysis of solution strategies for equations that do not have fixed or unique solutions

indeterminate equation: an equation in which more than one variable is unknown

index (number): a number that allows tracking of a quantity in economics by comparing it to a standard, the consumer price index is the best known example

inductive reasoning: drawing general conclusions based on specific instances or observations; for example, a theory might be based on the outcomes of several experiments

Industrial Revolution: beginning in Great Britain around 1730, a period in the eighteenth and nineteenth centuries when nations in Europe, Asia, and the Americas moved from agrarian-based to industry-based economies

inertia: tendency of a body that is at rest to remain at rest, or the tendency of a body that is in motion to remain in motion

inferences: the act or process of deriving a conclusion from given facts or premises

inferential statistics: analysis and interpretation of data in order to make predictions

infinite: having no limit; boundless, unlimited, endless; uncountable

infinitesimals: functions with values arbitrarily close to zero

infinity: the quality of unboundedness; a quantity beyond measure; an unbounded quantity

information database: an array of information related to a specific subject or group of subjects and arranged so that any individual bit of information can be easily found and recovered

information theory: the science that deals with how to separate information from noise in a signal or how to trace the flow of information through a complex system

infrastructure: the foundation or permanent installations necessary for a structure or system to operate

initial conditions: the values of variables at the beginning of an experiment or of a set at the beginning of a simulation; chaos theory reveals that small changes in initial conditions can produce widely divergent results

input: information provided to a computer or other computation system

inspheres: spheres that touch all the “inside” faces of a regular polyhedron; also called “enspheres”

integer: a positive whole number, its negative counterpart, or zero

integral: a mathematical operation similar to summation; the area between the curve of a function, the x-axis, and two bounds such as $x = a$ and $x = b$; an inverse operation to finding the derivative

integral calculus: the branch of mathematics dealing with the rate of change of functions with respect to their variables

integral number: integer; that is, a positive whole number, its negative counterpart, or zero

integral solutions: solutions to an equation or set of equations that are all integers

integrated circuit: a circuit with the transistors, resistors, and other circuit elements etched into the surface of a single chip of silicon

integration: solving a differential equation; determining the area under a curve between two boundaries

intensity: the brightness of radiation or energy contained in a wave

intergalactic: between galaxies; the space between the galaxies

interplanetary: between planets; the space between the planets

interpolation: filling in; estimating unknown values of a function between known values

intersection: a set containing all of the elements that are members of two other sets

interstellar: between stars; the space between stars

intraframe: the compression applied to still images, interframe compression compares one image to the next and only stores the elements that have changed

intrinsic: of itself; the essential nature of a thing; originating within the thing

inverse: opposite; the mathematical function that expresses the independent variable of another function in terms of the dependent variable

inverse operations: operations that undo each other, such as addition and subtraction

inverse square law: a given physical quality varies with the distance from the source inversely as the square of the distance

inverse tangent: the value of the argument of the tangent function that produces a given value of the function; the angle that produces a particular value of the tangent

invert: to turn upside down or to turn inside out; in mathematics, to rewrite as the inverse function

inverted: upside down; turned over

ionized: an atom that has lost one or more of its electrons and has become a charged particle

ionosphere: a layer in Earth's atmosphere above 80 kilometers characterized by the existence of ions and free electrons

irrational number: a real number that cannot be written as a fraction of the form a/b, where a and b are both integers and b is not zero; when expressed in decimal form, an irrational number is infinite and nonrepeating

isometry: equality of measure

isosceles triangle: a triangle with two sides and two angles equal

isotope: one of several species of an atom that has the same number of protons and the same chemical properties, but different numbers of neutrons

iteration: repetition; a repeated mathematical operation in which the output of one cycle becomes the input for the next cycle

iterative: relating to a computational procedure to produce a desired result by replication of a series of operations

iterator: the mathematical operation producing the result used in iteration

kinetic energy: the energy an object has as a consequence of its motion

kinetic theory of gases: the idea that all gases are composed of widely separated particles (atoms and molecules) that exert only small forces on each other and that are in constant motion

knot: nautical mile per hour

Lagrange points: two positions in which the motion of a body of negligible mass is stable under the gravitational influence of two much larger bodies (where one larger body is moving)

latitude: the number of degrees on Earth's surface north or south of the equator; the equator is latitude zero

law: a principle of science that is highly reliable, has great predictive power, and represents the mathematical summary of experimental results

law of cosines: for a triangle with angles A, B, C and sides a, b, c, $a^2 = b^2 + c^2 - 2bc \cos A$

law of sines: if a triangle has sides a, b, and c and opposite angles A, B, and C, then $\sin A/a = \sin B/b = \sin C/c$

laws of probability: set of principles that govern the use of probability in determining the truth or falsehood of a hypothesis

light-year: the distance light travels within a vaccuum in one year

limit: a mathematical concept in which numerical values get closer and closer to a given value

linear algebra: the study of vector spaces and linear transformations

linear equation: an equation in which all variables are raised to the first power

linear function: a function whose graph on the x-y plane is a straight line or line segment

litmus test: a test that uses a single indicator to prompt a decision

locus (pl: loci): in geometry, the set of all points, lines, or surfaces that satisfies a particular requirement

logarithm: the power to which a certain number called the base is to be raised to produce a particular number

logarithmic coordinates: the x and y coordinates of a point on a cartesian plane using logarithmic scales on the x- and y-axes.

logarithmic scale: a scale in which the distances that numbers are positioned, from a reference point, are proportional to their logarithms

logic circuits: circuits used to perform logical operations and containing one or more logic elements: devices that maintain a state based on previous input to determine current and future output

logistic difference equation: the equation $x_{(n+1)} = r \times x_{n(1-xn)}$ is used to study variability in animal populations

longitude: one of the imaginary great circles beginning at the poles and extending around Earth; the geographic position east or west of the prime meridian

machine code: the set of instructions used to direct the internal operation of a computer or other information-processing system

machine language: electronic code the computer can utilize

magnetic trap: a magnetic field configured in such a way that an ion or other charged particle can be held in place for an extended period of time

magnetosphere: an asymmetric region surrounding the Earth in which charged particles are trapped, their behavior being dominated by Earth's magnetic field

magnitude: size; the measure or extent of a mathematical or physical quantity

mainframes: large computers used by businesses and government agencies to process massive amounts of data; generally faster and more powerful than desktops but usually requiring specialized software

malfunctioning: not functioning correctly; performing badly

malleability: the ability or capability of being shaped or formed

margin of error: the difference between the estimated maximum and minimum values a given measurement could have

mathematical probability: the mathematical computation of probabilities of outcomes based on rules of logic

matrix: a rectangular array of data in rows and columns

mean: the arithmetic average of a set of data

median: the middle of a set of data when values are sorted from smallest to largest (or largest to smallest)

megabyte: term used to refer to one million bytes of memory storage, where each byte consists of eight bits; the actual value is 1,048,576 (2^{20})

memory: a device in a computer designed to temporarily or permanently store information in the form of binomial states of certain circuit elements

meridian: a great circle passing through Earth's poles and a particular location

metallurgy: the study of the properties of metals; the chemistry of metals and alloys

meteorologist: a person who studies the atmosphere in order to understand weather and climate

methanol: an alcohol consisting of a single carbon bonded to three hydrogen atoms and an O–H group

microcomputers: an older term used to designate small computers designed to sit on a desktop and to be used by one person; replaced by the term personal computer

microgravity: the apparent weightless condition of objects in free fall

microkelvin: one-millionth of a kelvin

minicomputers: a computer midway in size between a desktop computer and a main frame computer; most modern desktops are much more powerful than the older minicomputers and they have been phased out

minimum viable population: the smallest number of individuals of a species in a particular area that can survive and maintain genetic diversity

mission specialist: an individual trained by NASA to perform a specific task or set of tasks onboard a spacecraft, whose duties do not include piloting the spacecraft

mnemonic: a device or process that aids one's memory

mode: a kind of average or measure of central tendency equal to the number that occurs most often in a set of data

monomial: an expression with one term

Morse code: a binary code designed to allow text information to be transmitted by telegraph consisting of "dots" and "dashes"

mouse: a handheld pointing device used to manipulate an indicator on a screen

moving average: a method of averaging recent trends in relation to long term averages, it uses recent data (for example, the last 10 days) to calculate an average that changes but still smooths out daily variations

multimodal input/output (I/O): multimedia control and display that uses various senses and interaction styles

multiprocessing: a computer that has two or more central processers which have common access to main storage

nanometers: billionths of a meter

nearsightedness: describes the inability to see distant objects clearly

negative exponential: an exponential function of the form $y = e^{-x}$

net force: the final, or resultant, influence on a body that causes it to accelerate

neuron: a nerve cell

neurotransmitters: the substance released by a neuron that diffuses across the synapse

neutron: an elementary particle with approximately the same mass as a proton and neutral charge

Newtonian: a person who, like Isaac Newton, thinks the universe can be understood in terms of numbers and mathematical operations

nominal scales: a method for sorting objects into categories according to some distinguishing characteristic, then attaching a label to each category

non-Euclidean geometry: a branch of geometry defined by posing an alternate to Euclid's fifth postulate

nonlinear transformation: a transformation of a function that changes the shape of a curve or geometric figure

nonlinear transformations: transformations of functions that change the shape of a curve or geometric figure

nuclear fission: a reaction in which an atomic nucleus splits into fragments

nuclear fusion: mechanism of energy formation in a star; lighter nuclei are combined into heavier nuclei, releasing energy in the process

nucleotides: the basic chemical unit in a molecule of nucleic acid

nucleus: the dense, positive core of an atom that contains protons and neutrons

null hypothesis: the theory that there is no validity to the specific claim that two variations of the same thing can be distinguished by a specific procedure

number theory: the study of the properties of the natural numbers, including prime numbers, the number theorem, and Fermat's Last Theorem

numerical differentiation: approximating the mathematical process of differentiation using a digital computer

nutrient: a food substance or mineral required for the completion of the life cycle of an organism

oblate spheroid: a spheroid that bulges at the equator; the surface created by rotating an ellipse 360 degrees around its minor axis

omnidirectional: a device that transmits or receives energy in all directions

Öort cloud: a cloud of millions of comets and other material forming a spherical shell around the solar system far beyond the orbit of Neptune

orbital period: the period required for a planet or any other orbiting object to complete one complete orbit

orbital velocity: the speed and direction necessary for a body to circle a celestial body, such as Earth, in a stable manner

ordinate: the y-coordinate of a point on a Cartesian plane

organic: having to do with life, growing naturally, or dealing with the chemical compounds found in or produced by living organisms

oscillating: moving back and forth

outliers: extreme values in a data set

output: information received from a computer or other computation system based on the information it has received

overdubs: adding voice tracks to an existing film or tape

oxidant: a chemical reagent that combines with oxygen

oxidizer: the chemical that combines with oxygen or is made into an oxide

pace: an ancient measure of length equal to normal stride length

parabola: a conic section; the locus of all points such that the distance from a fixed point called the focus is equal to the perpendicular distance from a line

parabolic: an open geometric curve where the distance of a point on the curve to a fixed point (focus) and a fixed line (directrix) is the same

paradigm: an example, pattern, or way of thinking

parallax: the apparent motion of a nearby object when viewed against the background of more distant objects due to a change in the observer's position

parallel operations: separating the parts of a problem and working on different parts at the same time

parallel processing: using at least two different computers or working at least two different central processing units in the same computer at the same time or "in parallel" to solve problems or to perform calculation

parallelogram: a quadrilateral with opposite sides equal and opposite angles equal

parameter: an independent variable, such as time, that can be used to rewrite an expression as two separate functions

parity bits: extra bits inserted into digital signals that can be used to determine if the signal was accurately received

partial sum: with respect to infinite series, the sum of its first n terms for some n

pattern recognition: a process used by some artificial-intelligence systems to identify a variety of patterns, including visual patterns, information patterns buried in a noisy signal, and word patterns imbedded in text

payload specialist: an individual selected by NASA, another government agency, another government, or a private business, and trained by NASA to operate a specific piece of equipment onboard a spacecraft

payloads: the passengers, crew, instruments, or equipment carried by an aircraft, spacecraft, or rocket

perceptual noise shaping: a process of improving signal-to-noise ratio by looking for the patterns made by the signal, such as speech

perimeter: the distance around an area; in fractal geometry, some figures have a finite area but infinite perimeter

peripheral vision: outer area of the visual field

permutation: any arrangement, or ordering, of items in a set

perpendicular: forming a right angle with a line or plane

perspective: the point of view; a drawing constructed in such a way that an appearance of three dimensionality is achieved

perturbations: small displacements in an orbit

phonograph: a device used to recover the information recorded in analog form as waves or wiggles in a spiral grove on a flat disc of vinyl, rubber, or some other substance

photosphere: the very bright portion of the Sun visible to the unaided eye; the portion around the Sun that marks the boundary between the dense interior gases and the more diffuse

photosynthesis: the chemical process used by plants and some other organisms to harvest light energy by converting carbon dioxide and water to carbohydrates and oxygen

pixel: a single picture element on a video screen; one of the individual dots making up a picture on a video screen or digital image

place value: in a number system, the power of the base assigned to each place; in base-10, the ones place, the tens place, the hundreds place, and so on

plane: generally considered an undefinable term, a plane is a flat surface extending in all directions without end, and that has no thickness

plane geometry: the study of geometric figures, points, lines, and angles and their relationships when confined to a single plane

planetary: having to do with one of the planets

planisphere: a projection of the celestial sphere onto a plane with adjustable circles to demonstrate celestial phenomena

plates: the crustal segments on Earth's surface, which are constantly moving and rotating with respect to each other

plumb-bob: a heavy, conical-shaped weight, supported point-down on its axis by a strong cord, used to determine verticality in construction or surveying

pneumatic drill: a drill operated by compressed air

pneumatic tire: air-filled tire, usually rubber or synthetic

polar axis: the axis from which angles are measured in a polar coordinate system

pole: the origin of a polar coordinate system

poll: a survey designed to gather information about a subject

pollen analysis: microscopic examination of pollen grains to determine the genus and species of the plant producing the pollen; also known as palynology

polyconic projections: a type of map projection of a globe onto a plane that produces a distorted image but preserves correct distances along each meridian

polygon: a geometric figure bounded by line segments

polyhedron: a solid formed with all plane faces

polynomial: an expression with more than one term

polynomial function: a functional expression written in terms of a polynomial

position tracking: sensing the location and/or orientation of an object

power: the number of times a number is to be multiplied by itself in an expression

precalculus: the set of subjects and mathematical skills generally necessary to understand calculus

predicted frame: in compressed video, the next frame in a sequence of images; the information is based on what changed from the previous frame

prime: relating to, or being, a prime number (that is, a number that has no factors other than itself and 1)

Prime Meridian: the meridian that passes through Greenwich, England

prime number: a number that has no factors other than itself and 1

privatization: the process of converting a service traditionally offered by a government or public agency into a service provided by a private corporation or other private entity

proactive: taking action based on prediction of future situations

probability: the likelihood an event will occur when compared to other possible outcomes

probability density function: a function used to estimate the likelihood of spotting an organism while walking a transect

probability theory: the branch of mathematics that deals with quantities having random distributions

processor: an electronic device used to process a signal or to process a flow of information

profit margin: the difference between the total cost of a good or service and the actual selling cost of that good or service, usually expressed as a percentage

program: a set of instructions given to a computer that allows it to perform tasks; software

programming language processor: a program designed to recognize and process other programs

proliferation: growing rapidly

proportion: the mathematical relation between one part and another part, or between a part and the whole; the equality of two ratios

proportionately: divided or distributed according to a proportion; proportional

protractor: a device used for measuring angles, usually consisting of a half circle marked in degrees

pseudorandom numbers: numbers generated by a process that does not guarantee randomness; numbers produced by a computer using some highly complex function that simulates true randomness

Ptolemaic theory: the theory that asserted Earth was a spherical object at the center of the universe surrounded by other spheres carrying the various celestial objects

Pythagorean Theorem: a mathematical statement relating the sides of right triangles; the square of the hypotenuse is equal to the sums of the squares of the other two sides

Pythagorean triples: any set of three numbers obeying the Pythogorean relation such that the square of one is equal to the sum of the squares of the other two

quadrant: one-fourth of a circle; also a device used to measure angles above the horizon

quadratic: involving at least one term raised to the second power

quadratic equation: an equation in which the variable is raised to the second power in at least one term when the equation is written in its simplest form

quadratic form: the form of a function written so that the independent variable is raised to the second power

quantitative: of, relating to, or expressible in terms of quantity

quantum: a small packet of energy (matter and energy are equivalent)

quantum mechanics: the study of the interactions of matter with radiation on an atomic or smaller scale, whereby the granularity of energy and radiation becomes apparent

quantum theory: the study of the interactions of matter with radiation on an atomic or smaller scale, whereby the granularity of energy and radiation becomes apparent

quaternion: a form of complex number consisting of a real scalar and an imaginary vector component with three dimensions

quipus: knotted cords used by the Incas and other Andean cultures to encode numeric and other information

radian: an angle measure approximately equal to 57.3 degrees, it is the angle that subtends an arc of a circle equal to one radius

radicand: the quantity under the radical sign; the argument of the square root function

radius: the line segment originating at the center of a circle or sphere and terminating on the circle or sphere; also the measure of that line segment

radius vector: a line segment with both magnitude and direction that begins at the center of a circle or sphere and runs to a point on the circle or sphere

random: without order

random walks: a mathematical process in a plane of moving a random distance in a random direction then turning through a random angle and repeating the process indefinitely

range: the set of all values of a variable in a function mapped to the values in the domain of the independent variable; also called range set

rate (interest): the portion of the principal, usually expressed as a percentage, paid on a loan or investment during each time interval

ratio of similitude: the ratio of the corresponding sides of similar figures

rational number: a number that can be written in the form a/b, where a and b are intergers and b is not equal to zero

rations: the portion of feed that is given to a particular animal

ray: half line; line segment that originates at a point and extends without bound

real number: a number that has no imaginary part; a set composed of all the rational and irrational numbers

real number set: the combined set of all rational and irrational numbers, the set of numbers representing all points on the number line

realtime: occuring immediately, allowing interaction without significant delay

reapportionment: the process of redistributing the seats of the U. S. House of Representatives, based on each state's proportion of the national population

recalibration: process of resetting a measuring instrument so as to provide more accurate measurements

reciprocal: one of a pair of numbers that multiply to equal 1; a number's reciprocal is 1 divided by the number

red shift: motion-induced change in the frequency of light emitted by a source moving away from the observer

reflected: light or soundwaves returned from a surface

reflection: a rigid motion of the plane that fixes one line (the mirror axis) and moves every other point to its mirror image on the opposite side of the line

reflexive: directed back or turning back on itself

refraction: the change in direction of a wave as it passes from one medium to another

refrigerants: fluid circulating in a refrigerator that is successively compressed, cooled, allowed to expand, and warmed in the refrigeration cycle

regular hexagon: a hexagon whose sides are all equal and whose angles are all equal

relative: defined in terms of or in relation to other quantities

relative dating: determining the date of an archaeological artifact based on its position in the archaeological context relative to other artifacts

relativity: the assertion that measurements of certain physical quantities such as mass, length, and time depend on the relative motion of the object and observer

remediate: to provide a remedy; to heal or to correct a wrong or a deficiency

retrograde: apparent motion of a planet from east to west, the reverse of normal motion; for the outer planets, due to the more rapid motion of Earth as it overtakes an outer planet

revenue: the income produced by a source such as an investment or some other activity; the income produced by taxes and other sources and collected by a governmental unit

rhomboid: a parallelogram whose sides are equal

right angle: the angle formed by perpendicular lines; it measures 90 degrees

RNA: ribonucleic acid

robot arm: a sophisticated device that is standard equipment on space shuttles and on the International Space Station; used to deploy and retrieve satellites or perform other functions

Roche limit: an imaginary surface around a star in a binary system; outside the Roche limit, the gravitational attraction of the companion will pull matter away from a star

root: a number that when multiplied by itself a certain number of times forms a product equal to a specified number

rotary-wing design: an aircraft design that uses a rotating wing to produce lift; helicopter or autogiro (also spelled autogyro)

rotation: a rigid motion of the plane that fixes one point (the center of rotation) and moves every other point around a circle centered at that point

rotational: having to do with rotation

round: also to round off, the systematic process of reducing the number of decimal places for a given number

rounding: process of giving an approximate number

sample: a randomly selected subset of a larger population used to represent the larger population in statistical analysis

sampling: selecting a subset of a group or population in such a way that valid conclusions can be made about the whole set or population

scale (map): the numerical ratio between the dimensions of an object and the dimensions of the two or three dimensional representation of that object

scale drawing: a drawing in which all of the dimensions are reduced by some constant factor so that the proportions are preserved

scaling: the process of reducing or increasing a drawing or some physical process so that proper proportions are retained between the parts

schematic diagram: a diagram that uses symbols for elements and arranges these elements in a logical pattern rather than a practical physical arrangement

schematic diagrams: wiring diagrams that use symbols for circuit elements and arranges these elements in a logical pattern rather than a practical physical arrangement

search engine: software designed to search the Internet for occurences of a word, phrase, or picture, usually provided at no cost to the user as an advertising vehicle

secant: the ratio of the side adjacent to an acute angle in a right triangle to the side opposite; given a unit circle, the ratio of the x coordinate to the y coordinate of any point on the circle

seismic: subjected to, or caused by an earthquake or earth tremor

self-similarity: the term used to describe fractals where a part of the geometric figure resembles a larger or smaller part at any scale chosen

semantic: the study of how words acquire meaning and how those meanings change over time

semi-major axis: one-half of the long axis of an ellipse; also equal to the average distance of a planet or any satellite from the object it is orbiting

semiconductor: one of the elements with characteristics intermediate between the metals and nonmetals

set: a collection of objects defined by a rule such that it is possible to determine exactly which objects are members of the set

set dancing: a form of dance in which dancers are guided through a series of moves by a caller

set theory: the branch of mathematics that deals with the well-defined collections of objects known as sets

sextant: a device for measuring altitudes of celestial objects

signal processor: a device designed to convert information from one form to another so that it can be sent or received

significant difference: to distinguish greatly between two parameters

significant digits: the digits reported in a measure that accurately reflect the precision of the measurement

silicon: element number 14, it belongs in the category of elements known as metalloids or semiconductors

similar: in mathematics, having sides or parts in constant proportion; two items that resemble each other but are not identical

sine: if a unit circle is drawn with its center at the origin and a line segment is drawn from the origin at angle theta so that the line segment intersects the circle at (x, y), then y is the sine of theta

skepticism: a tendency towards doubt

skew: to cause lack of symmetry in the shape of a frequency distribution

slope: the ratio of the vertical change to the corresponding horizontal change

software: the set of instructions given to a computer that allows it to perform tasks

solar masses: dimensionless units in which mass, radius, luminosity, and other physical properties of stars can be expressed in terms of the Sun's characteristics

solar wind: a stream of particles and radiation constantly pouring out of the Sun at high velocities; partially responsible for the formation of the tails of comets

solid geometry: the geometry of solid figures, spheres, and polyhedrons; the geometry of points, lines, surfaces, and solids in three-dimensional space

spatial sound: audio channels endowed with directional and positional attributes (like azimuth, elevation, and range) and room effects (like echoes and reverberation)

spectra: the ranges of frequencies of light emitted or absorbed by objects

spectrum: the range of frequencies of light emitted or absorbed by an object

sphere: the locus of points in three-dimensional space that are all equidistant from a single point called the center

spin: to rotate on an axis or turn around

square: a quadrilateral with four equal sides and four right angles

square root: with respect to real or complex numbers s, the number t for which $t^2 = s$

stade: an ancient Greek measurement of length, one stade is approximately 559 feet (about 170 meters)

standard deviation: a measure of the average amount by which individual items of data might be expected to vary from the arithmetic mean of all data

static: without movement; stationary

statistical analysis: a set of methods for analyzing numerical data

statistics: the branch of mathematics that analyzes and interprets sets of numerical data

stellar: having to do with stars

sterographics: presenting slightly different views to left and right eyes, so that graphic scenes acquire depth

stochastic: random, or relating to a variable at each moment

Stonehenge: a large circle of standing stones on the Salisbury plain in England, thought by some to be an astronomical or calendrical marker

storm surge: the front of a hurricane, which bulges because of strong winds; can be the most damaging part of a hurricane

stratopause: the boundary in the atmosphere between the stratosphere and the mesosphere usually around 55 kilometers in altitude

stratosphere: the layer of Earth's atmosphere from 15 kilometers to about 50 kilometers, usually unaffected by weather and lacking clouds or moisture

sublimate: change of phase from a solid to a gas

sublunary: "below the moon"; term used by Aristotle and others to describe things that were nearer to Earth than the Moon and so not necessarily heavenly in origin or composition

subtend: to extend past and mark off a chord or arc

sunspot activity: one of the powerful magnetic storms on the surface of the Sun, which causes it to appear to have dark spots; sunspot activity varies on an 11-year cycle

superconduction: the flow of electric current without resistance in certain metals and alloys while at temperatures near absolute zero

superposition: the placing of one thing on top of another

suspension bridge: a bridge held up by a system of cables or cables and rods in tension; usually having two or more tall towers with heavy cables anchored at the ends and strung between the towers and lighter vertical cables extending downward to support the roadway

symmetric: to have balanced proportions; in bilateral symmetry, opposite sides are mirror images of each other

symmetry: a correspondence or equivalence between or among constituents of a system

synapse: the narrow gap between the terminal of one neuron and the dendrites of the next

tactile: relating to the sense of touch

tailwind: a wind blowing in the same direction of that of the course of a vehicle

tangent: a line that intersects a curve at one and only one point in a local region

tectonic plates: large segments of Earth's crust that move in relation to one another

telecommuting: working from home or another offsite location

tenable: defensible, reasonable

terrestrial refraction: the apparent raising or lowering of a distant object on Earth's surface due to variations in atmospheric temperature

tessellation: a mosaic of tiles or other objects composed of identical repeated elements with no gaps

tesseract: a four-dimensional cube, formed by connecting all of the vertices of two three-dimensional cubes separated by the length of one side in four-dimensional space

theodolite: a surveying instrument designed to measure both horizontal and vertical angles

theorem: a statement in mathematics that can be demonstrated to be true given that certain assumptions and definitions (called axioms) are accepted as true

threatened species: a species whose population is viable but diminishing or has limited habitat

time dilation: the principle of general relativity which predicts that to an outside observer, clocks would appear to run more slowly in a powerful gravitational field

topology: the study of those properties of geometric figures that do not change under such nonlinear transformations as stretching or bending

topspin: spin placed on a baseball, tennis ball, bowling ball, or other object so that the axis of rotation is horizontal and perpendicular to the line of flight and the top of the object is rotating in the same direction as the motion of the object

trajectory: the path followed by a projectile; in chaotic systems, the trajectory is ordered and unpredictable

transcendental: a real number that cannot be the root of a polynomial with rational coefficients

transect: to divide by cutting transversly

transfinite: surpassing the finite

transformation: changing one mathematical expression into another by translation, mapping, or rotation according to some mathematical rule

transistor: an electronic device consisting of two different kinds of semiconductor material, which can be used as a switch or amplifier

transit: a surveyor's instrument with a rotating telescope that is used to measure angles and elevations

transitive: having the mathematical property that if the first expression in a series is equal to the second and the second is equal to the third, then the first is equal to the third

translate: to move from one place to another without rotation

translation: a rigid motion of the plane that moves each point in the same direction and by the same distance

tree: a collection of dots with edges connecting them that have no looping paths

triangulation: the process of determining the distance to an object by measuring the length of the base and two angles of a triangle

trigonometric ratio: a ratio formed from the lengths of the sides of right triangles

trigonometry: the branch of mathematics that studies triangles and trigonometric functions

tropopause: the boundry in Earth's atmosphere between the troposphere and the stratosphere at an altitude of 14 to 15 kilometers

troposphere: the lowest layer of Earth's atmosphere extending from the surface up to about 15 kilometers; the layer where most weather phenomena occur

ultra-violet radiation: electromagnetic radiation with wavelength shorter than visible light, in the range of 1 nanometer to about 400 nanometer

unbiased sample: a random sample selected from a larger population in such a way that each member of the larger population has an equal chance of being in the sample

underspin: spin placed on a baseball, tennis ball, bowling ball, or other object so that the axis of rotation is horizontal and perpendicular to the line of flight and the top of the object is rotating in the opposite direction from the motion of the object

Unicode: a newer system than ASCII for assigning binary numbers to keyboard symbols that includes most other alphabets; uses 16-bit symbol sets

union: a set containing all of the members of two other sets

upper bound: the maximum value of a function

vaccuum: theoretically, a space in which there is no matter

variable: a symbol, such as letters, that may assume any one of a set of values known as the domain

variable star: a star whose brightness noticeably varies over time

vector: a quantity which has both magnitude and direction

velocity: distance traveled per unit of time in a specific direction

verify: confirm; establish the truth of a statement or proposition

vernal equinox: the moment when the Sun crosses the celestial equator marking the first day of spring; occurs around March 22 for the northern hemisphere and September 21 for the southern hemisphere

vertex: a point of a graph; a node; the point on a triangle or polygon where two sides come together; the point at which a conic section intersects its axis of symmetry

viable: capable of living, growing, and developing

wavelengths: the distance in a periodic wave between two points of corresponding phase in consecutive cycles

whole numbers: the positive integers and zero

World Wide Web: the part of the Internet allowing users to examine graphic "web" pages

yield (interest): the actual amount of interest earned, which may be different than the rate

zenith: the point on the celestial sphere vertically above a given position

zenith angle: from an observer's viewpoint, the angle between the line of sight to a celestial body (such as the Sun) and the line from the observer to the zenith point

zero pair: one positive integer and one negative integer

ziggurat: a tower built in ancient Babylonia with a pyramidal shape and stepped sides

Topic Outline

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mathematics

mathematics

VOLUME 2
Da-Lo

Barry Max Brandenberger, Jr., Editor in Chief

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Dance, Folk

D

Both mathematics and dance are languages that use symbols to convey ideas and expressions. Mathematics uses written symbols to represent abstractions so that users can arrive at a greater understanding of a problem without **ambiguity**. Dancers use **abstract** symbols to represent thoughts, feelings, emotions, and ideas, and these symbols may be interpreted in multiple ways. Both disciplines rely to a large extent on pattern recognition.

ambiguity the quality of doubtfulness or uncertainty

abstract having only intrinsic form

Many forms of dance, such as classical ballet, involve complex patterns and take years of practice to master. Yet other forms of dance use everyday movements with more simplistic patterns. For example, folk dances have evolved from common movements of work and play.

Although folk dances require concentration and focus, their use of everyday movement invites observers to participate. Similarly, mathematics can be studied at the basic level of arithmetic, which is used to make simple transactions and to understand how things work. More advanced mathematics, such as **calculus**, **chaos theory**, or **abstract algebra** require years to master.

calculus a method of dealing mathematically with variables that may be changing continuously with respect to each other

chaos theory the qualitative study of unstable aperiodic behavior in deterministic nonlinear dynamical systems

abstract algebra the branch of algebra dealing with groups, rings, fields, Galois sets, and number theory

Discreteness in Mathematics and Dance

Many dances are based on a simple method of counting and **discrete** sequences, which enables participants to recognize and learn a variety of dances. The word "discrete" also has a common, similar usage in mathematics. Discrete mathematics involves counting separate elements, such as the number of arrangements of letters on a license place, or the number of ways that a presidential candidate can visit all fifty states. Solutions in discrete mathematics can be only whole units. Discrete math is therefore one of the most accessible areas of modern mathematics since many of the questions are easy for anyone to understand.

discrete composed of distinct elements

Contradancing. Contradancing is a popular form of folk dance in the United States that illustrates the mathematics of dance. Its origins go back to colonial days, and its roots can be traced to English country dances.

Contradancing, which shares elements of traditional square dancing, is a form of **set dancing** in which a dancer's position relative to another dancer traces patterns on the dance floor. As in most dancing, timing is crucial, as is the ability to rapidly carry out called instructions.

set dancing a form of dance in which dancers are guided through a series of moves by a caller

Counting is crucial to timing in dance, which can be very structured. Traditional folk dancers—like this troupe from Morocco—often form lines, circles, squares, and other geometric figures while keeping rhythm.

Music for contradancing is highly structured. Everything occurs in multiples of four. In one common format, the band plays a tune for sixteen beats, repeats the tune, then plays a new tune for sixteen beats and repeats. An eight-beat section is known as a call, during which each block of four dancers executes a called-out instruction.

When contradancers line up in their groups of four to produce a long column "down" the dance floor (extending away from the band), each square block of two couples can be thought of as a mathematical **matrix** with the dimension 2×2. Each dancer, or element of the matrix, is in a specific position within the array. The called instructions correspond to rearrangements of the elements (dancers). After sixty-four beats, for example, the first and second rows of the matrix may be interchanged. Of course, this could be done in one step, but the fun of dancing comes from performing the various **permutations** by which groups of four can reach the end result.

matrix a rectangular array of data in rows and columns

permutation any arrangement of ordering, of items in a set

There are many called instructions in contradancing, ranging in complexity from simply circling once around to the left or right within each group of four to sequences of moves that involve exchanging partners or stepping one-quarter, one-half, or three-quarters of the way around the ring. With each call, the matrix representing four dancers changes. In the final configuration, the two rows of the original 2×2 matrix may be interchanged, or they may be the same as when the dance started.

Chaos Theory and Dance

Computer scientists have applied the basics of chaos theory to generate variations on dance movement sequences. Special symbols represent human body postures, and positions for each of the body's main joints are encoded by defining an **axis** and **angle of rotation** given in the form of a mathematical expression called a **quaternion**. A motion sequence is then mapped onto a **chaotic attractor**. Following a new **trajectory** around the attractor produces a variation of the original motion sequence. To smooth out abrupt transitions introduced by the chaotic mapping, the researchers have developed schemes that capture and enforce particular dance styles. SEE ALSO Chaos, Mathematics of.

Marilyn K. Simon

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What Is Contra Dance? <http://www.sbcds.org/contradance/whatis/>.

THE COMMUNICATION OF MATHEMATICS AND DANCE

According to Keith Devlin in *The Language of Mathematics,* mathematics seeks to communicate a sense of what humans experience. The simplicity, precision, purity, and elegance of mathematical expressions and patterns give mathematics an aesthetic value. The mathematical connections to dance similarly give dancers a creative, aesthetic, and interpretive means of expressing the human experience.

Data Analyst

A data analyst does more than simply analyze information. Data are collected for a variety of reasons—to learn about something new, to find relationships and generate statistics, or to create **information databases**. Likewise, there are numerous fields of study that collect data, such as finance, medicine, sales and marketing, and engineering, to name a few. All this data needs to be correlated into useful and relevant information.

When a data analyst faces a printout of numbers or facts, he or she must make sense of it all. First, a data analyst will determine where the data came from, if anything has corrupted the collection, and if more data is needed. As data is sorted, the analyst needs to find relationships among the data, select samples that are indicative of the whole, convert data from one form to another, and even predict results. In short, the analyst helps make the data useful.

Data analysts may also summarize the data in a report and communicate this information to colleagues or the public. Sometimes data analysts maintain routine records in a database or archive data for future use and analysis. Those data analysts with more advanced computer and engineering training may be called upon to design programs or models that collect data, calibrate instruments that run tests, or troubleshoot systems that are not functioning properly.

axis an imaginary line about which an object rotates

angle of rotation the angle, measured from an initial position to a final position, that a rotating object has moved through

quaternion a form of complex number consisting of a real scalar and an imaginary vector component with three dimensions

chaotic attractor a set of points such that all nearby trajectories converge to it

trajectory the path followed by a projectile; in chaotic systems, the trajectory is ordered and unpredictable

information database an array of information related to a specific subject or groups of subjects and arranged so that any individual bit of information can be easily found and recovered

Some data analysts with engineering expertise may work with electronic instrumentation that measures and processes data.

Mathematics and computer programming are essential skills for data analysts. The most important math skills are strong knowledge of **statistics** and statistical analysis, since a data analyst will often be asked whether a set of data is statistically significant. Data analysts must also have the computer skills necessary to operate a wide variety of databases. SEE ALSO Data Collection and Interpretation.

Lorraine Savage

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Data Collection and Interpretation

Data interpretation is part of daily life for most people. Interpretation is the process of making sense of numerical data that has been collected, analyzed, and presented. People interpret data when they turn on the television and hear the news anchor reporting on a poll, when they read advertisements claiming that one product is better than another, or when they choose grocery store items that claim they are more effective than other leading brands.

A common method of assessing numerical data is known as **statistical analysis**, and the activity of analyzing and interpreting data in order to make predictions is known as **inferential statistics**. Informed consumers recognize the importance of judging the reasonableness of data interpretations and predictions by considering sources of bias such as sampling procedures or misleading questions, **margins of error**, **confidence intervals**, and incomplete interpretations.

statistics the branch of mathematics that analyzes and interprets sets of numerical data

statistical analysis a set of methods for analyzing numerical data

inferential statistics analysis and interpretation of data in order to make predictions

margin of error the difference between estimated maximum and minimum values a given measurement could have

confidence interval a range of values having a predetermined probability that the value of some measurement of a population lies within it

reapportionment the process of redistributing the seats of the U.S. House of Representatives, based on each state's proportion of the national population

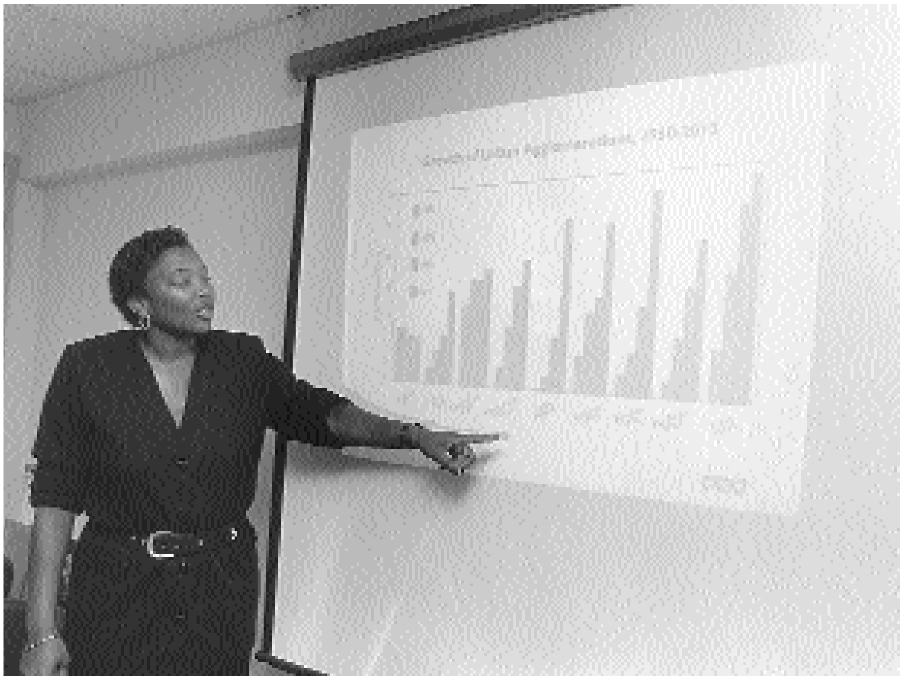
Why Is Accurate Data Collection Important?

The repercussions of inaccurate or improperly interpreted data are wide-ranging. For example, every 10 years a major census is done in the United States. The results are used to help determine the number of congressional seats that are assigned to each district; where new roads will be built; where new schools and libraries are needed; where new nursing homes, hospitals, and day care centers will be located; where new parks and recreational centers will be built; and the sizes of police and fire departments.

In the past 30 years there has been a major shift in the U.S. population. People have migrated from the northern states toward the southern states, and the result has been a major shift in congressional representation. With a net change of nearly 30 percent (a 17 percent drop in the Northeast and Midwest coupled with a 12 percent gain in the South), the South has gone from a position of less influence to one of greater influence in Congress as a result of population-based **reapportionment**. This is just one of many possible examples that reveal how data gathering and interpretation related to population can have a marked affect on the whole country.

Gathering Reliable Data

The process of data interpretation begins by gathering data. Because it is often difficult, or even impossible, to look at all the data (for example, to



In data presentations, the audience must think critically about both the data collection methods and the interpretation being offered by the presenter.

poll every high school student in the United States), data are generally obtained from a smaller unit, a subset of the population known as a **sample**. Then data from the sample are used to predict (or infer) what the characteristics of the population as a whole may be. For example, a telephone survey of one thousand car owners in the United States might be conducted to predict the popularity of various cars among all U.S. car owners. The one thousand U.S. car owners who are surveyed are the sample and all car owners in the United States are the population.

sample a randomly selected subset of a larger population used to represent the larger population in statistical analysis

But there both an art and science to collecting high-quality data. Several key elements must be considered: bias, sample size, question design, margin of error, and interpretation.

Avoiding Bias. In order for data interpretation to be reliable, a number of factors must be in place. First and perhaps foremost, an **unbiased sample** must be used. In other words, every person (or item) in the population should have an equal chance of being in the sample.

unbiased sample a random sample selected from a larger population in such a way that each member of the larger population has an equal chance of being in the sample

For example, what if only Ford owners were surveyed in the telephone survey? The survey would be quite likely to show that Fords were more popular. A biased sample is likely to **skew** the data, thus making data interpretation unreliable. If we want to know what sorts of cars are preferred by U.S. car owners, we need to be sure that our sample of car owners is representative of the entire car owner population.

skew to cause lack of symmetry in the shape of a frequency distribution

One way of ensuring an unbiased sample is to choose randomly from the population. However, it is often difficult to design a study that will produce a truly unbiased sample. For example, suppose a surveyor decides to choose car owners at **random** to participate in a phone interview about car preferences. This may sound like a good plan, but car owners who do not have telephones or whose telephone numbers are unavailable will not have a chance to participate in the survey. Maybe car owners with unlisted telephone numbers have

random without order

very different car preferences than the broader population, but we will never know if they are not included in the sample.

Biased sampling continues to challenge census takers. In 1990, nearly 35 percent of the households that were mailed census forms did not mail them back. If a form is not returned, the Census Bureau must send someone to the person's house. Even with census takers visiting homes door to door, the Census Bureau was still unable to contact one out of every five of the families who did not return their census form.

Although this may not sound like a lot, consider that in 1990 there were approximately 250 million people in the United States. If a household contains an average of four people, that means that there were 62.5 million forms mailed out. Multiplying that figure by 35 percent (the number of households that did not return the forms) gives the staggering figure of 21.875 million forms that were not returned. Of the 21.875 million households that did not return forms, census takers were unable to track down 20 percent, or 4.375 million households.

Why is this biased sampling? It is believed that of the more than 4 million households not counted, the overwhelming majority was from poorer sections of large cities. This implies that certain parts of the country may be over-represented in Congress and are the recipients of more federal funds than may be deserved.

BIAS IN NEWS CALL-IN POLLS

Many news programs have call-in polls. The results of the poll are usually shown later in the program. This type of data collection is very unreliable because the information is coming from a biased sample.

People who watch or listen to the news make up only a small percentage of the population. Of that group, only an even smaller percentage will call to offer their opinion. And of those who call, more are likely to disagree with the question because people with strong feelings against an issue are more likely to respond.

biased sampling process of obtaining a nonrandom sample; choosing a sample to represent a particular viewpoint instead of the whole population

Achieving a Large Enough Sample. A second important factor in data collection is whether the chosen sample is large enough. Are one thousand car owners a sufficient number of car owners from which to infer the opinion of all car owners? In order to answer this question, the margin of error needs to be calculated.

The margin of error is a statistic that represents a range in which the surveyor feels confident that the population as a whole will fall. A sufficient sample size needs to have a small margin of error, usually around 5 percent. To determine the margin of error (m), divide one by the square root of the sample size (s): $m = 1 / \sqrt{s}$. Therefore, the sample of one thousand car owners gives us a margin of error of about 3 percent, an allowable margin of error.

Asking the Proper Questions. Informed citizens who are assessing survey results must consider the type of questions that are asked when a survey is conducted. Were the questions leading? Were they easy or difficult to understand? For example, suppose a study carried out by a local ice cream manufacturer states that 75 percent of Americans prefer ice cream. It seems self-evident that an ice cream company would not report a study that showed Americans do not like ice cream. So perhaps the question in the study was leading: for example, "Do you prefer ice cream or spinach?" It is therefore important to find out exactly what questions were asked and of whom.

Giving a Proper Interpretation. Data are often interpreted with a bias, and the results can therefore be misleading or incomplete. For example, a bath soap company claims that its soap is 99 percent pure. This statement is misleading because the soap manufacturer does not explain what "pure" is. When reading an unclarified percentage such as in the previous exam-

ple, one needs to ask such questions. An example of another incomplete or misleading interpretation is that the average child watches approximately 5 hours of television a day. The reader should question what an "average child" is.

Considering Margin of Error. Margin of error is important to consider when statistics are reported. For example, we might read that the high school dropout rate declined from 18 percent to 16 percent with a margin of error of 3 percent. Because the 2-percentage point decline is smaller than the margin of error (3 percent), the new dropout rate may fall between 13 percent to 19 percent. We cannot be entirely sure that the high school dropout rate actually declined at all.

Confidence intervals, a term usually employed by statisticians, and related to margins of error, is reported by a percentage and is constructed to relay how confident one can be that the sample is representative of the population. The producers of this survey may only be 95 percent confident that their sample is representative of the population. If this is the case then there is a 5 percent chance that this sample data does not typify or carry over to the population of the United States. The margin of error represents the range of this 95-percent confidence interval (the range that represents plus or minus two **standard deviations** from the **mean**).

Understanding the nature of particular data sets is critical for anyone who deals with numbers and measurements.

standard deviation a measure of the average amount by which individual items of data might be expected to vary from the arithmetic mean of all data

mean the arithmetic average of a set of data

Understanding and Interpreting Data

Figuring out what data means is just as important as collecting it. Even if the data collection process is sound, data can be misinterpreted. When interpreting data, the data user must not only attempt to discern the differences between causality and coincidence, but also must consider all possible factors that may have led to a result.

After considering the design of a survey, consumers should look at the reported data interpretation. Suppose a report states that 52 percent of all Americans prefer Chevrolet to other car manufacturers. The surveyors want you to think that more than half of all Americans prefer Chevrolet, but is this really the case? Perhaps not all those surveyed were Americans. Also, the 52 percent comes from the sample, so it is important to ask if the sample was large enough, unbiased, and randomly chosen. One also needs to be aware of margins of error and confidence intervals. If the margin of error for this survey is 5 percent than this means that the percentage of car owners in the United States who prefer Chevrolet could actually be between 47 and 57 percent (5 percent higher or lower than the 52 percent).

Similar questions are important to consider when we try to understand **polls**. During the 2000 presidential race, the evening news and newspapers were often filled with poll reports. For example, one poll stated 51 percent of Americans preferred George W. Bush, 46 percent preferred Al Gore, and 3 percent were undecided, with a margin of error of plus or minus 5 percent.

poll a survey designed to gather information about a subject

The news anchor then went on to report that *most* Americans prefer George W. Bush. However, given the data outlined above, this conclusion is questionable. Because the difference between George W. Bush and Al Gore is the same as the margin of error, it is impossible to know which

AN UNNECESSARY SCARE

A January 1991 report by the American Cancer Society proclaimed the odds of a woman getting breast cancer had risen to one in nine. For obvious reasons, this scared women all over the country. The research was sound but was based on a lifetime of over 110 years. In other words, if a woman lives to be 110 years old, there is an 11 percent chance she will get breast cancer. The odds for a woman under 50 are closer to one in a thousand, or 0.1 percent. There was nothing wrong with the sampling—but the interpretation and presentation of the data were incomplete.

candidate was actually preferred. In addition, if we do not know any of the circumstances behind the poll, we should be skeptical about its findings.

As another example, consider census data that shows a radical increase in the number of people living in Florida and Arizona along with a decrease in the number of people living in New York. One could easily (and falsely) conclude that the data "proves" that people are finding New York to be a less desirable place to live and therefore are moving away.

But this hasty conclusion could be missing the big picture. What if the data also reveals that the average age of New Yorkers has dropped since 1990? Further interpretation of the data may reveal that when New Yorkers grow older, they move to warmer climates to retire. This illustrates why data must be thoroughly interpreted before any conclusions can be drawn.

A Data Checklist. When reading any survey, listening to an advertisement, or hearing about poll results, informed consumers should ask questions about the soundness of the data interpretation. A recap of key points follows.

1. Was the sample unbiased (representative of the whole population)?
2. Was the sample large enough for the purpose of the survey (margin of error of the sample)?
3. What type of questions did the surveyor ask? Were they simple and unambiguous? Were they leading (constructed in such a way to get the desired response)?
4. Can the conclusions drawn be justified based on the information gathered?
5. How was the survey done (mail, phone, interview)? Does the survey report mention margins of error or confidence intervals, and, if so, are these such that the conclusions drawn are warranted?

By using these checkpoints and learning to think critically about data collection and interpretation, individuals can become more savvy consumers of information. SEE ALSO Census; Central Tendency, Measures of; Graphs; Mass Media, Mathematics and the; Predictions; Randomness; Statistical Analysis.

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Dating Techniques

Movies and television have presented a romantic vision of archaeology as adventure in far-away and exotic locations. A more realistic picture might show researchers digging in smelly mud for hours under the hot sun while battling relentless mosquitoes. This type of archaeological research produces hundreds of small plastic bags containing pottery shards, animal bones, bits of worked stone, and other fragments. These findings must be classified, which requires more hours of tedious work in a stuffy tent. At its best, archaeology involves a studious examination of the past with the goal of learning important information about the culture and customs of ancient (or not so ancient) peoples. Much archaeology in the early twenty-first century investigates the recent past, a sub-branch called "historical archaeology."

What Is Archaeology?

Archaeology is the study of the material remains of past human cultures. It is distinguished from other forms of inquiry by its method of study, excavation. (Most archaeologists call this "digging.") Excavation is not simply digging until something interesting is found. That sort of unscientific digging destroys the archaeological information. Archaeological excavation requires the removal of material layer by layer to expose artifacts in place. The removed material is carefully sifted to find small **artifacts**, tiny animal bones, and other remains. Archaeologists even examine the soil in various layers for microscopic material, such as pollen. Excavations, in combination with surveys, may yield maps of a ruin or collections of artifacts.

artifact something made by a human and left in an archaeological context

Time is important to archaeologists. There is rarely enough time to complete the work, but of even greater interest is the time that has passed since the artifact was created. An important part of archaeology is the examination of how cultures change over time. It is therefore essential that the archaeologist is able to establish the age of the artifacts or other material remains and arrange them in a chronological sequence. The archaeologist must be able to distinguish between objects that were made at the same time and objects that were made at different times. When objects that were made at different times are excavated, the archaeologist must be able to arrange them in a sequence from the oldest to the most recent.

Relative Dating and Absolute Dating

Before scientific dating techniques such as dendrochronology and radiocarbon dating were introduced to archaeology, the discipline was dominated by extensive discussions of the chronological sequence of events. Most of those questions have now been settled and archaeologists have moved on to other issues. Scientific dating techniques have had a huge impact on archaeology.

Archaeologists use many different techniques to determine the age of an object. Usually, several different techniques are applied to the same object. **Relative dating** arranges artifacts in a chronological sequence from oldest to most recent without reference to the actual date. For example, by studying the decorations used on pottery, the types of materials used in the pottery, and the types and shapes of pots, it is often possible to arrange them

relative dating determining the date of an archaeological artifact based on its position in the archaeological context relative to other artifacts

absolute dating determining the date of an artifact by measuring some physical parameter independent of context

superposition the placing of one thing on top of another

into a sequence without knowing the actual date. In **absolute dating**, the age of an object is determined by some chemical or physical process without reference to a chronology.

Relative Dating Methods. The most common and widely used relative dating technique is ***stratigraphy***. The principle of **superposition** (borrowed from geology) states that higher layers must be deposited on top of lower layers. Thus, higher layers are more recent than lower layers. This only applies to undisturbed deposits. Rodent burrows, root action, and human activity can mix layers in a process known as bioturbation. However, the archaeologist can detect bioturbation and allow for its effects.

Discrete layers of occupation can often be determined. For example, Hisarlik, which is a hill in Turkey, is thought by some archaeologists to be the site of the ancient city of Troy. However, Hisarlik was occupied by many different cultures at various times both before and after the time of Troy, and each culture built on top of the ruins of the previous culture, often after violent conquest. Consequently, the layers in this famous archaeological site represent many different cultures. An early excavator of Hisarlik, Heinrich Schleimann, inadvertently dug through the Troy layer into an earlier occupation and mistakenly assigned the gold artifacts he found there to Troy. Other sites have been continuously occupied by the same culture for a long time and the different layers represent gradual changes. In both cases, stratigraphy will apply.

correlate to establish a mutual or reciprocal relation between two things or sets of things

A chronology based on stratigraphy often can be **correlated** to layers in other nearby sites. For example, a particular type or pattern of pottery may occur in only one layer in an excavation. If the same pottery type is found in another excavation nearby, it is safe to assume that the layers are the same age. Archaeologists rarely make these determinations on the basis of a single example. Usually, a set of related artifacts is used to determine the age of a layer.

Seriation simply means ordering. This technique was developed by the inventor of modern archaeology, Sir William Matthew Flinders Petrie. Seriation is based on the assumption that cultural characteristics change over time. For example, consider how automobiles have changed in the last 50 years (a relatively short time in archaeology). Automobile manufacturers frequently introduce new styles about every year, so archaeologists thousands of years from now will have no difficulty identifying the precise date of a layer if the layer contains automobile parts.

Cultural characteristics tend to show a particular pattern over time. The characteristic is introduced into the culture (for example, using a certain type of projectile point for hunting or wearing low-riding jeans), becomes progressively more popular, then gradually wanes in popularity. The method of seriation uses this distinctive pattern to arrange archaeological materials into a sequence. However, seriation only works when variations in a cultural characteristic are due to rapid and significant change over time. It also works best when a characteristic is widely shared among many different members of a group. Even then, it can only be applied to a small geographic area, because there is also geographic variation in cultural characteristics. For example, 50 years ago American automobiles changed every year while the Volkswagen Beetle hardly changed at all from year to year.

Cross dating is also based on stratigraphy. It uses the principle that different archaeological sites will show a similar collection of artifacts in layers of the same age. Sir Flinders Petrie used this method to establish the time sequence of artifacts in Egyptian cemeteries by identifying which burials contained Greek pottery vessels. These same Greek pottery styles could be associated with monuments in Greece whose construction dates were fairly well known. Since absolute dating techniques have become common, the use of cross dating has decreased significantly.

Pollen grains also appear in archaeological layers. They are abundant and they survive very well in archaeological contexts. As climates change over time, the plants that grow in a region change as well. People who examine pollen grains (the study of which is known as **pollen analysis**) can usually determine the **genus**, and often the exact species producing a certain pollen type. Archaeologists can then use this information to determine the relative ages of some sites and layers within sites. However, climates do not change rapidly, so this type of analysis is best for archaeological sites dating back to the last ice age.

pollen analysis microscopic examination of pollen grains to determine the genus and species of the plant producing the pollen; also known as palynology

genus the taxonomic classification one step more general than species; the first name in the binomial nomenclature of all species

Absolute Dating Methods. Absolute dating methods produce an actual date, usually accurate to within a few years. This date is established independent of stratigraphy and chronology. If a date for a certain layer in an excavation can be established using an absolute dating method, other artifacts in the same layer can safely be assigned the same age.

Dendrochronology, also known as tree-ring dating, is the earliest form of absolute dating. This method was first developed by the American astronomer Andrew Ellicott Douglas at the University of Arizona in the early 1900s. Douglas was trying to develop a **correlation** between climate variations and **sunspot activity**, but archaeologists quickly recognized its usefulness as a dating tool. The technique was first applied in the American Southwest and later extended to other parts of the world.

correlation the process of establishing a mutual or reciprocal relation between two things or sets of things

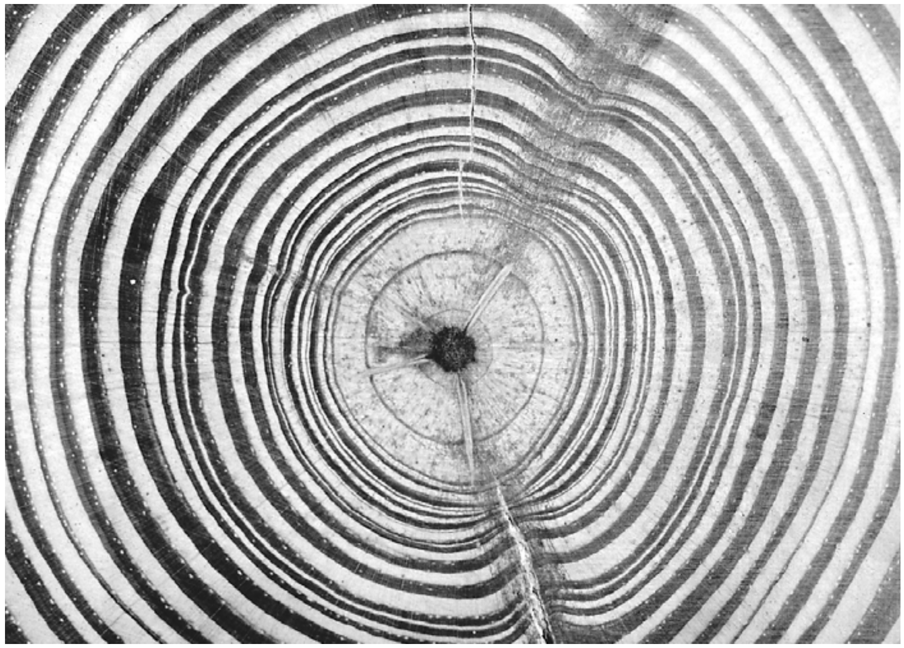
sunspot activity one of the powerful magnetic storms on the surface of the Sun, which causes it to appear to have dark spots; sunspot activity varies on an 11-year cycle

Tree-ring dating is relatively simple. Trees add a new layer of cambium (the layer right under the bark) every year. The thickness of the layer depends on local weather and climate. In years with plenty of rain, the layer will be thick and healthy. Over the lifetime of the tree, these rings accumulate, and the rings form a record of regional variation in climate that may extend back hundreds of years. Since all of the trees in a region experience the same climate variations, they will have similar growth patterns and similar tree ring patterns.

One tree usually does not cover a period sufficiently long to be archaeologically useful. However, patterns of tree ring growth have been built up by "overlapping" ring sequences from different trees so that the tree ring record extends back several thousand years in many parts of the world. The process starts with examination of the growth ring patterns of samples from living trees. Then older trees are added to the sequence by overlapping the inner rings of a younger sample with the outer rings of an older sample. Older trees are recovered from old buildings, archaeological sites, peat bogs, and swamps. Eventually, a regional master chronology is constructed.

When dendrochronology can be used, it provides the most accurate dates of any technique. In the American Southwest, the accuracy and precision of dendrochronology has enabled the development of one of the most

Widely spaced tree rings indicate periods of rapid growth, whereas narrowly spaced rings indicate slow growth.



accurate prehistoric cultural chronologies anywhere in the world. Often events can be dated to within a decade. This precision has allowed archaeologists working in the American Southwest to reconstruct patterns of village growth and subsequent abandonment with a fineness of detail unmatched in most of the world.

Radiometric dating methods are more recent than dendrochronology. However, dendrochronology provides an important calibration technique for radiocarbon dating techniques. All radiometric-dating techniques are based on the well-established principle from physics that large samples of radioactive **isotopes** decay at precisely known rates. The rate of decay of a radioactive isotope is usually given by its half-life. The decay of any individual nucleus is completely random. The half-life is a measure of the probability that a given atom will decay in a certain time. The shorter the half-life, the more likely the atom will decay. This probability does not increase with time. If an atom has not decayed, the probability that it will decay in the future remains exactly the same. This means that no matter how many atoms are in a sample, approximately one-half will decay in one half-life. The remaining atoms have exactly the same decay probability, so in another half-life, one half of the remaining atoms will decay. The amount of time required for one-half of a radioactive sample to decay can be precisely determined. The particular radioisotope used to determine the age of an object depends on the type of object and its age.

isotope one of several species of an atom that has the same number of protons and the same chemical properties, but different numbers of neutrons

Radiocarbon is the most common and best known of radiometric dating techniques, but it is also possibly the most misunderstood. It was developed at the University of Chicago in 1949 by a group of American scientists led by Willard F. Libby. Radiocarbon dating has had an enormous impact on archaeology. In the last 50 years, radiocarbon dating has provided the basis for a worldwide cultural chronology. Recognizing the importance of this technique, the Nobel Prize committee awarded the Prize in Chemistry to Libby in 1960.

The physics behind radiocarbon dating is straightforward. Earth's atmosphere is constantly bombarded with cosmic rays from outer space. Cosmic-ray neutrons collide with atoms of nitrogen in the upper atmosphere, converting them to atoms of radioactive carbon-14. The carbon-14 atom quickly combines with an oxygen molecule to form carbon dioxide. This radioactive carbon dioxide spreads throughout Earth's atmosphere, where it is taken up by plants along with normal carbon-12. As long as the plant is alive, the relative amount (ratio) of carbon-14 to carbon-12 remains constant at about one carbon-14 atom for every one trillion carbon-12 atoms. Some animals eat plants and other animals eat the plant-eaters. As long as they are alive, all living organisms have the same ratio of carbon-14 to carbon-12 as in the atmosphere because the radioactive carbon is continually replenished, either through **photosynthesis** or through the food animals eat.

photosynthesis the chemical process used by plants and some other organisms to harvest light energy by converting carbon dioxide and water to carbohydrates and oxygen

However, when the plant or animal dies, the intake of carbon-14 stops and the ratio of carbon-14 to carbon-12 immediately starts to decrease. The half-life of carbon-14 is 5,730 years. After 5,730 years, about one-half of the carbon-14 atoms will have decayed. After another 5,730 years, one-half of the remaining atoms will have decayed. So after 11,460 years, only one-fourth will remain. After 17,190 years, one-eighth of the original carbon-14 will remain. After 22,920 years, one-sixteenth will remain.

Radiocarbon dating has become the standard technique for determining the age of organic remains (those remains that contain carbon). There are many factors that must be taken into account when determining the age of an object. The best objects are bits of charcoal that have been preserved in completely dry environments. The worst candidates are bits of wood that have been saturated with sea water, since sea water contains dissolved atmospheric carbon dioxide that may throw off the results. Radiocarbon dating can be used for small bits of clothing or other fabric, bits of bone, baskets, or anything that contains **organic** material.

organic having to do with life, growing naturally, or dealing with the chemical compounds found in or produced by living organisms

There are well over 100 labs worldwide that do radiocarbon dating. In the early twenty-first century, the dating of objects up to about 10 half-lives, or up to about 50,000 years old, is possible. However, objects less than 300 years old cannot be reliably dated because of the widespread burning of fossil fuels, which began in the nineteenth century, and the production of carbon-14 from atmospheric testing of nuclear weapons in the 1950s and 1960s. Another problem with radiocarbon dating is that the production of carbon-14 in the atmosphere has not been constant, due to variation in solar activity. For example, in the 1700s, solar activity dropped (a phenomenon called the "Maunder Minimum"), so carbon-14 production also decreased during this period. To achieve the highest level of accuracy, carbon-14 dates must be calibrated by comparison to dates obtained from dendrochronology.

Calibration of Radiocarbon Dates. Samples of Bristlecone pine, a tree with a very long life span, have been dated using both dendrochronology and radiocarbon dating. The results do not agree, but the differences are consistent. That is, the radiocarbon dates were always wrong by the same number of years. Consequently, tree-ring chronologies have been used to calibrate radiocarbon dates to around 12,000 years ago.

When radiocarbon dating was first put into use, it was decided that dates would always be reported as B.P., where B.P. stood for "before present" and

"present" was defined as 1950. That way, dates reported in magazine articles and books do not have to be adjusted as the years pass. So if a lab determines that an object has a radiocarbon age of 1,050 years in 2000, its age will be given as 1000 B.P. Calibrated dates are given using the actual date, such as 950 C.E.

Potassium-Argon Dating. If an object is too old to be dated by radiocarbon dating, or if it contains no organic material, other methods must be used. One of these is potassium-argon dating. All naturally occurring rocks contain potassium. Some of the potassium in rocks is the radioactive isotope potassium-40. Potassium-40 gradually decays to the stable isotope argon-40, which is a gas. When the rock is melted, as in a volcano, any argon gas trapped in the rock escapes. When the rock cools, the argon will begin to build up. So this method can be used to measure the age of any volcanic rock, from 100,000 years up to around 5 billion years old.

This method is not widely used in archaeology, since most archaeological deposits are not associated with volcanic activity. However, Louis and Mary Leakey successfully used the method to determine the ages of fossils in Olduvai Gorge in Tanzania by examining rocks from lava flows above and below the fossils. They were able to establish an absolute chronology for humans and human ancestors extending back two million years. At Laetolli, in Tanzania, volcanic ash containing early **hominid** footprints was dated by this method at 3.5 million years.

hominid a member of family Hominidae; *Homo sapiens* are the only surviving species

fission the splitting of the nucleus of a heavy atom, which releases kinetic energy that is carried away by the fission fragments and two or three neutrons

Other Methods. Uranium-238 is present in most rocks. This isotope of uranium spontaneously undergoes **fission**. The fission fragments have a lot of energy, and they plow through the rock, leaving a track that can be made visible by treating the rock. So by counting fission tracks, the age of the rock can be determined. Like potassium-argon dating, this can only be used to determine the age of the rock, not the age of the artifact itself.

Thermoluminescence is a recently developed technique that uses the property of some crystals to "store" light. Sometimes an electron will be knocked out of its position in a crystal and will "stick" somewhere else in the crystal. These displaced electrons will accumulate over time. If the sample is heated, the electrons will fall back to their normal positions, emitting a small flash of light. By measuring the light emitted, the time that has passed since the artifact was heated can be determined. This method should prove to be especially useful in determining the age of ceramics, rocks that have been used to build fire rings, and samples of **chert** and flint that have been deliberately heated to make them easier to flake into a projectile point.

chert material consisting of amorphous or cryptocrystalline silicon dioxide; fine-grained chert is indistinguishable from flint

Conclusion

Science continues to develop new methods to determine the age of objects. As our knowledge of past chronologies improves, archaeologists will be better able to understand how cultures change over time, and how different cultures interact with each other. As a result, this knowledge will enable us to achieve a progressively better understanding of our own culture. SEE ALSO Time, Measurement of.

Elliot Richmond

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Decimals

The number system most commonly used today is based on the Hindu-Arabic number system, which was developed in what is now India around 300 B.C.E. The Arab mathematicians adopted this system and brought it to Spain, where it slowly spread to the rest of Europe.

The present-day number system, which is called the decimal* or **base-10** number system, is an elegant and efficient way to express numbers. The rules for performing arithmetic calculations are simple and straightforward.

***The prefix "deci" in the word "decimal" means ten.**

base-10 a number system in which each place represents a power of 10 larger than the place to its right

digit one of the symbols used in a number system to represent the multipliers of each place

place value in a number system, the power of the base assigned to each place; in base-10, the ones place, the tens place, the hundreds place, and so on

power the number of times a number is to be multiplied by itself in an expression

Basic Properties

The decimal number system is based on two fundamental properties. First, numbers are constructed from ten **digits**, or numerals—0, 1, 2, 3, 4, 5, 6, 7, 8, and 9—that are arranged in a sequence. Second, the position of a digit in the sequence determines its value, called the place value. Because each digit, by its **place value**, represents a multiple of a **power** of 10, the system is called base-10.

Numbers Greater and Less Than One. The expansion of numbers greater than 1 consists of sums of groups of tens. Each digit in the expansion represents how many groups of tens are present. The groups are 1s, 10s, 100s, and so on. The groups are arranged in order, with the right-most representing 10^0, or groups of 1s. The next digit to the left in the group stands for 10^1, or groups of 10s, and so on. For example, the expansion of the whole number 3,254 in the decimal system is expressed as:

$$3{,}254 = (3 \times 10^3) + (2 \times 10^2) + (5 \times 10^1) + (4 \times 10^0)$$
$$= 3000 + 200 + 50 + 4.$$

A number less than 1 is represented by a sequence of numbers to the right of a decimal point. The digits to the right of a decimal point are multiplied by 10 raised to a negative power, starting from negative 1. In moving to the right, the place value decreases in value, each being one-tenth as small as the previous place value. Thus, each successive digit to the right of the decimal point denotes the number of tenths, hundredths, thousandths, and so forth. For example, the expansion of 0.3574 is expressed as:

$$0.3574 = (3 \times 10^{-1}) + (5 \times 10^{-2}) + (7 \times 10^{-3}) + (4 \times 10^{-4})$$
$$= 0.3 + 0.05 + 0.007 + 0.0004.$$

Decimal Fractions. The example above illustrates standard decimal notation for a number less than 1. But what is commonly referred to as a decimal is actually a decimal fraction. A decimal fraction is a number written in decimal notation that does not have any digits except 0 to the left of the decimal point. For example, .257 is a decimal fraction, which can also be

written as 0.257. It is preferable to use 0 to the left of the decimal point when there are no other digits to the left of the decimal point.

Decimal fractions can be converted into a fraction using a power of 10. For example, $0.486 = \frac{486}{1000}$, and $0.35 = \frac{35}{100}$. The fraction may be converted to an equivalent fraction by writing it in its simplest or reduced form. The fraction $\frac{35}{100}$ equals $\frac{7}{20}$, for example.

Decimal Expansion of Rational and Irrational Numbers

rational number a number that can be written as the ratio of two integers in the form $\frac{a}{b}$ where a and b are integers and b is not 0

Is it possible to convert all fractions into decimals and all decimals into fractions? All fractions, $\frac{a}{b}$, are **rational numbers**, and all rational numbers are fractions, provided b is not 0. So the question is: Can every rational number be converted into a decimal?

Taking the rational number, or fraction $\frac{a}{b}$, and dividing the numerator by the denominator will always produce a decimal. For example, $\frac{1}{4} = 0.25$, $\frac{1}{10} = 0.1$, and $\frac{2}{5} = 0.4$. In each of these examples, the numbers to the right of the decimal point end, or terminate.

However, the conversion of $\frac{1}{3} = 0.333...$, $\frac{2}{9} = 0.222...$, and $\frac{3}{11} = 0.2727...$, results in decimals that are nonterminating. In the first example, the calculation continuously produces 3, which repeats indefinitely. In the second example, 2 repeats indefinitely. In the third example, the pair of digits 2 and 7 repeats indefinitely.

When a digit or group of digits repeats indefinitely, a bar is placed above the digit or digits. Thus, $\frac{1}{3} = 0.\overline{3}$, $\frac{2}{9} = 0.\overline{2}$, and $\frac{3}{11} = 0.\overline{27}$. These are examples of repeating but nonterminating decimals.

It may not always be apparent when decimals start repeating. One interesting example is $\frac{3}{17}$. The division needs to be carried out to sixteen digits before the repeating digits appear:

$$\frac{3}{17} = 0.\overline{1764705882352941}.$$

Nonterminating and nonrepeating decimal numbers exist, and some may have predictable patterns, such as 0.313113111311113. . .. However, these numbers are not considered rational, but instead are known as irrational. An **irrational number** cannot be represented as a decimal with terminating or repeating digits, but instead yields a nonterminating, nonrepeating sequence.

irrational number a real number that cannot be written as fractions of the form $\frac{a}{b}$, where a and b are both integers and b is not 0

A familiar example of an irrational number is $\sqrt{2}$, which in decimal form is expressed as 1.414213. . ., where the digits after the decimal point neither end nor repeat. Another example of an irrational number is π, which in decimal form is expressed as 3.141592. . . .

Daily Applications of Decimals

In daily use, decimal numbers are used for counting and measuring. Consider a shopping trip to a sporting good store, where a person buys a pair of running shoes, a warm-up jacket, and a pair of socks. The total cost is calculated by aligning at the right all three numbers on top of each other.

\$39.95	running shoes
\$29.95	warm-up jacket
\$ 9.49	socks
\$79.39	total

DECIMALS AND FRACTIONS

Every fraction can be represented by a decimal that either terminates or eventually repeats. Also, every decimal that either terminates or eventually repeats can be represented as a fraction.

As another example, consider a car's gas tank that has a capacity of 18.5 gallons. If the gas tank is full after pumping 7.2 gallons, how much gasoline was already in the tank? The answer is 18.5 minus 7.2, which equals 11.3 gallons. SEE ALSO Fractions; Numbers, Irrational; Numbers, Rational; Powers and Exponents.

Rafiq Ladhani

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Denominator *See Fractions.*

Derivative *See Calculus.*

Descartes and His Coordinate System

Every time you graph an equation on a Cartesian coordinate system, you are using the work of René Descartes. Descartes, a French mathematician and philosopher, was born in La Haye, France (now named in his honor) on March 31, 1596. His parents taught him at home until he was 8 years old, when he entered the Jesuit college of La Flèche. There he continued his studies until he graduated at age 18.

René Descartes built the foundation for modern philosophical method with a simple catchphrase: "I think, therefore I am." He is also regarded as the founder of analytic geometry.

skepticism a tendency towards doubt

Descartes was an outstanding student at La Flèche, especially in mathematics. Because of his delicate health, his teachers allowed him to stay in bed until late morning. Despite missing most of his morning classes, Descartes was able to keep up with his studies. He would continue the habit of staying late in bed for his entire adult life.

After graduating from La Flèche, Descartes traveled to Paris and eventually enrolled at the University of Poitiers. He graduated with a law degree in 1616 and then enlisted in a military school. In 1619, he joined the Bavarian army and spent the next nine years as a soldier, touring throughout much of Europe in between military campaigns. Descartes eventually settled in Holland, where he spent most of the rest of his life. There Descartes gave up a military career and decided on a life of mathematics and philosophy.

Descartes attempted to provide a philosophical foundation for the new mechanistic physics that was developing from the work of Copernicus and Galileo. He divided all things into two categories—mind and matter—and developed a dualistic philosophical system in which, although mind is subject to the will and does not follow physical laws, all matter must obey the same mechanistic laws.

GEOMETRY AND THE FLY

Some mathematics historians claim it may be that Descartes's inspiration for the coordinate system was due to his lifelong habit of staying late in bed. According to some accounts, one morning Descartes noticed a fly walking across the ceiling of his bedroom. As he watched the fly, Descartes began to think of how the fly's path could be described without actually tracing its path. His further reflections about describing a path by means of mathematics led to *La Géometrie* and Descartes's invention of coordinate geometry.

The philosophical system that Descartes developed, known as Cartesian philosophy, was based on **skepticism** and asserted that all reliable

knowledge must be built up by the use of reason through logical analysis. Cartesian philosophy was influential in the ultimate success of the Scientific Revolution and provides the foundation upon which most subsequent philosophical thought is grounded.

Descartes published various treatises about philosophy and mathematics. In 1637 Descartes published his masterwork, *Discourse on the Method of Reasoning Well and Seeking Truth in the Sciences.* In *Discourse*, Descartes sought to explain everything in terms of matter and motion. *Discourse* contained three appendices, one on optics, one on meteorology, and one titled *La Géometrie* (The Geometry). In *La Géometrie*, Descartes described what is now known as the system of Cartesian Coordinates, or coordinate geometry. In Descartes's system of coordinates, **geometry** and **algebra** were united for the first time to create what is known as **analytic geometry**.

The Cartesian Coordinate System

Cartesian coordinates are used to locate a point in space by giving its relative distance from **perpendicular** intersecting lines. In coordinate geometry, all points, lines, and figures are drawn in a **coordinate plane**. By reference to the two coordinate axes, any point, line, or figure may be precisely located.

In Descartes's system, the first coordinate value (x-coordinate) describes where along the horizontal axis (the x-axis) the point is located. The second coordinate value (y-coordinate) locates the point in terms of the vertical axis (the y-axis). A point with coordinates (4, −2) is located four units to the right of the intersection point of the two axes (point O, or the origin) and then two units below the vertical position of the origin. In example (a) of the figure, point D is at the coordinate location (4, −2). The coordinates for point A are (3, 2); for point B, (2, −4); and for point C, (−2, −5).

The coordinate system also makes it possible to exactly duplicate geometric figures. For example, the triangle shown in (b) has coordinates A (3, 2), B (4, 5), and C (−2, 4) that make it possible to duplicate the triangle without reference to any drawing.

The triangle may be reproduced by using the coordinates to locate the position of the three **vertex** points. The vertex points may then be connected with segments to replicate triangle ABC. More complex figures may likewise be described and duplicated with coordinates.

A straight line may also be represented on a coordinate grid. In the case of a straight line, every point on the line has coordinate values that must

geometry the branch of mathematics that deals with the properties and relationships of points, lines, angles, surfaces, planes, and solids

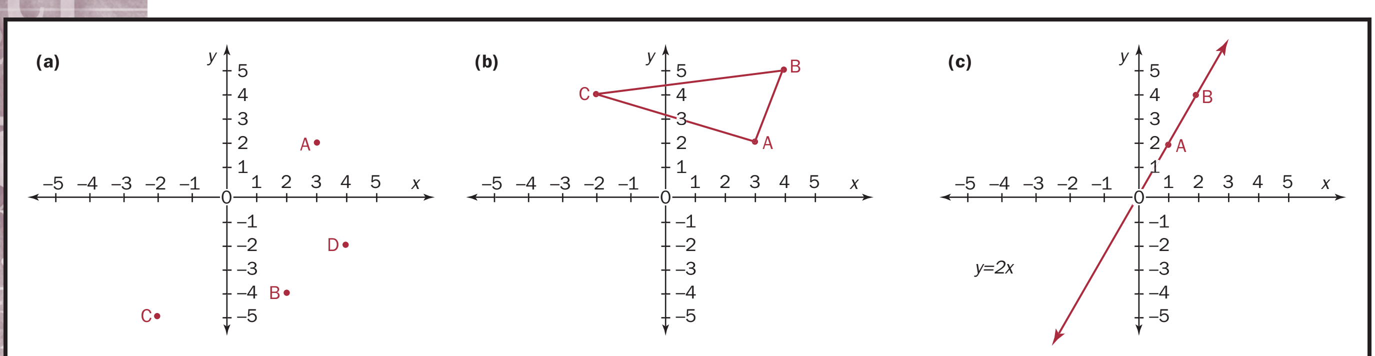
algebra the branch of mathematics that deals with variables or unknowns representing the arithmetic numbers

analytic geometry the study of geometric properties by using algebraic operations

perpendicular forming a right angle with a line or plane

coordinate plane an imaginary two-dimensional plane defined as the plane containing the x- and y-axes; all points on the plane have coordinates that can be expressed as (x, y)

vertex the point on a triangle or polygon where two sides come together



The Cartesian coordinate system unites geometry and algebra, and is a universal system for unambiguous location of points. Applications range from computer animation to global positioning systems.

satisfy a specific equation. The line in (c) may be expressed as $y = 2x$. The coordinates of every point on the line will satisfy the equation $y = 2x$, as for example, point A (1, 2) and point B (2, 4). More complex equations are used to represent circles, ellipses, and curved lines.

Other Contributions

La Géometrie made Descartes famous throughout Europe. He continued to publish his philosophy, detailing how to acquire accurate knowledge. His philosophy is sometimes summed up in his statement, "I think, therefore I am."

Descartes also made a number of other contributions to mathematics. He discovered the Law of Angular Deficiency for all **polyhedrons** and was the first to offer a quantifiable explanation of rainbows. In *La Géometrie*, Descartes introduced a familiar mathematics symbol, a raised number to indicate an **exponent**. The expression $4 \times 4 \times 4 \times 4 \times 4$ may be written as 4^5 using Descartes's notation. He also instituted using x, y, and z for unknowns in an equation.

In 1649, Descartes accepted an invitation from Queen Christina to travel to Sweden to be the royal tutor. Unfortunately for Descartes, the queen expected to be tutored while she did her exercises at 5:00 A.M. in an unheated library. Descartes had been used to a lifetime of sleeping late, and the new routine was much too rigorous for him. After only a few weeks of this regimen, Descartes contracted pneumonia and died on February 11, 1650. SEE ALSO COMPUTER ANIMATION; COORDINATE SYSTEM, POLAR; COORDINATE SYSTEM, THREE-DIMENSIONAL; GLOBAL POSITIONING SYSTEM; NAVIGATION.

Arthur V. Johnson II and J. William Moncrief

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WHO USES COORDINATES?

The system of coordinates that Descartes invented is used in many modern applications. For example, on any map the location of a country or a city is usually given as a set of coordinates. The location of a ship at sea is determined by longitude and latitude, which is an application of the coordinate system to the curved surface of Earth. Computer graphic artists create figures and computer animation by referencing coordinates on the screen.

polyhedron a solid formed with all plane faces

exponent the symbol written above and to the right of an expression indicating the power to which the expression is to be raised

Differential Equation *See Calculus.*

Dimensional Relationships

Usually, when mathematicians compare the size of two-dimensional objects, they compare their areas. For example, how many times larger is a larger square than a smaller one? One way to answer this question is to determine the lengths of the sides of the squares, and use this information to find the respective areas.

Use the formula for the area of a square, $A = S^2$, where A represents area and S represents the side length of the square. Suppose two squares have side lengths of 2 and 6, respectively. Hence, the respective areas are 4 and 36. Thus the area of the larger square is nine times that of the smaller square. Therefore, a square whose side length is three times that of a second square will have an area nine times as great.

Use the notation S_1 to denote the side of the smaller square and S_2 to denote the side of the larger square. With this notation, $S_2 = 3S_1$. The area of the larger square then becomes $(3S_1)^2 = 3S_1 \times 3S_1 = 9S_1^2$. This can be generalized further by letting one side of the square be k times the side of another, also known as the ratio of similitude (k) between the figures. Then $(kS_1)^2 = kS_1 \times kS_1 = k^2S_1^2$. From this, it is evident that if the side lengths of one square are k times the side lengths of another, the area of the first is k^2 that of the other.

This principle is true for any two-dimensional object. Suppose two circles have radii that are in the ratio of 2:1. Letting $R_2 = 2R_1$, the area of the larger circle can be represented by $A = \pi(2R_1)^2 = 4\pi R_1^2$.

As another example, suppose the sides and altitude of the larger triangle are twice those of a smaller triangle. Thus the area of the larger triangle can be written as $A = \frac{1}{2}(2b_1)(2h_1) = 2b_1h_1 = 4(\frac{1}{2}b_1h_1)$.

For three-dimensional objects, volumes of similar figures relate to each other in a manner akin to areas of two-dimensional figures. A cube, for example, with a side length twice that of another cube, will have a volume $2^3 = 8$ times as great. A sphere with a radius five times that of a smaller sphere will have a volume $5^3 = 125$ times as great.

ratio of similitude the ratio of the corresponding sides of similar figures

If k represents the **ratio of similitude** of two similar objects, then the areas of the two objects will be in the ratio of k^2, and the volumes of the two objects will be in the ratio of k^3. SEE ALSO Dimensions.

Albert Goetz

Dimensions

A dimension is a measurement of space. In a three-dimensional world, we usually think of three different directions as we measure the space in which we exist—length, width, and height. To indicate a certain location in space, we would provide three different coordinates on three different axes (x, y, and z).

The understanding of different dimensions is crucial in understanding much of mathematics. The ability to visualize and a flexibility in adjusting to n-dimensional worlds is a skill worth pursuing.

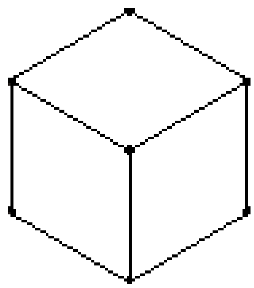
Portraying Three Dimensions

Even though most everyday experience is in three dimensions, most high school mathematics takes place in a two-dimensional world in which there is only length and width. Part of this is because traditional learning is largely communicated through books or on a chalkboard. Book pages or the planes of chalkboards have primarily two dimensions. Even if there is a picture in

a book of a three-dimensional object, the object is "flattened" so that its likeness can be communicated through a two-dimensional medium. This can cause differences between what we see in a three-dimensional object and how the object is portrayed in two dimensions.

For example, consider a cube. In real life, a **vertex** of a cube is formed when three 90° angles from three different square faces intersect at a point. However, in a two-dimensional representation of a cube, the right angles may measure other than 90° on the page. For example, in the picture of the cube below, each angle around the center vertex measures close to 120°. The image of the three-dimensional object must be altered this way in order for it to give us the perception that it is really three-dimensional.

vertex the point on a triangle or polygon where two sides come together



Portraying three dimensions in two-dimensional representations can also play tricks with our sense of perception. Many optical illusions are created by taking an impossible figure in three dimensions and drawing a two-dimensional picture of it.

Artist M. C. Escher was a master of communicating impossible ideas through two-dimensional drawings. In many of his engravings, the scene is impossible to construct in three dimensions, but by working with angles and making a few simple alterations, Escher fools our normally reliable sense of perception.

Imagining Dimensions

Points, lines, and planes are theoretical concepts typically modeled with three-dimensional objects in which we learn to ignore some of their dimensions. For example, earlier it was stated that a piece of paper or a plane is primarily two-dimensional. Although a piece of paper clearly has a thickness (height), albeit small, we think of it as being two-dimensional. So when considering objects with different dimensions, we must be able to visualize and think abstractly.

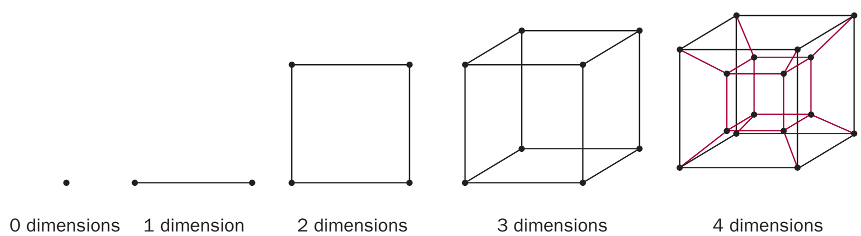
When we think of a plane, we start with a sheet of paper, which has length, width, and a very small height. Then we imagine that the height slowly disappears until all that remains is a length and a width. Similarly, when we draw lines on a chalkboard, we know that the chalkdust has length, a small width, and even an infinitesimal thickness. However, a mathematician imagines the line as having only one dimension: length. Finally when we consider a point in space, we must imagine that the point is merely a position or a location in space: that is, it has no size. To imagine a point, begin with an image of a fixed atom in space that is slowly

melting or disappearing until all that remains is its location. That location is a true mathematical image of a point.

A college professor gave the following way to think about dimensions. Start with a figure of zero dimensions: that is, a point. Set two of these items next to each other and connect them with a line segment. You now have a new entity of one dimension called a line segment. Again set two of these line segments next to each other and connect them with line segments. You now have a two-dimensional entity called a square. Connect two squares, and you have a three-dimensional entity called a cube. Connect the vertices of two cubes and you have a four-dimensional entity sometimes known as a hypercube or **tesseract**.✶

tesseract a four-dimensional cube, formed by connecting all of the vertices of two three-dimensional cubes separated by the length of one side in four-dimensional space

✶**A tesseract has 16 vertices, 32 edges, 24 squares, and 8 cubes.**



If you are having trouble visualizing a tesseract (as shown above), keep in mind that you are looking at a two-dimensional picture of something that is four-dimensional! Even if you build a three-dimensional physical model of a tesseract, you will still need to imagine the missing dimension that would bring us from three dimensions into four.

Abbott's *Flatland*

Imagining worlds of different dimensions is the premise of Edwin A. Abbott's book, *Flatland.* He describes an entire civilization that lives in a world with only two dimensions. All of its inhabitants are either lines, or polygons such as triangles, squares, and pentagons. These individuals cannot perceive of anything but length and width. Imagine, for example, living in a blackboard and only being able to move from side to side or up and down. Depth would be an unknown quantity.

Abbott's book describes a day when a sphere visits a family in Flatland. Because Flatlanders are unable to perceive a third dimension of depth, they are only able to perceive first a point (as the sphere first entered their world), then a small circle which increases in area until it reaches the very middle of the sphere, and then a circle of decreasing area until it becomes a point again. Then it disappears. Imagine trying to communicate to these Flatlanders anything about the third dimension when they had never experienced anything in that realm.

Use a similar analogy when thinking about what four dimensions might be like. Even if a creature from the fourth dimension visited us here in three-dimensional "Spaceland," and tried to describe the fourth dimension to us, we would not be equipped to understand it fully. SEE ALSO Coordinates, Three-Dimensional; Escher, M. C.; Mathematics, Impossible.

Jane Keiser Krumpe

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Distance, Measurement of

In the twenty-first century, societies need to make a wide range of measurements, from economic indicators and population trends to the times of sporting events and various standardized tests. Despite all of the varied measurement that is performed, modern societies still focus attention on the same two subjects, time and distance. In this entry, the focus will be on the measurement of distance.

Early Attempts to Measure Distance

Early civilizations used various crude instruments to measure distance, ranging from a simple **pace** to measuring rods and marked cords. The accuracy that was achieved with such basic devices can be astonishing. For example, the Great Pyramid of Egypt, built about 2400 B.C.E., has four base edges that are each within 0.01 percent of 751 feet in length.

pace an ancient measure of length equal to normal stride length

Most societies developed random units of measure that became standardized over time. The Egyptians used the *cubit,* which varied in length from 15 inches to 19 inches, depending on the ruling dynasty. To measure longer distances, some societies employed individuals to pace off the distance. In Egypt, these individuals were called "rope stretchers." They tied a specific length of rope to their two ankles and then paced off the distance so that each and every stride stretched the rope to its full length. The Egyptians used other crude measuring devices, such as marked cords and wooden rods. Each of the four faces of the Great Pyramid is positioned exactly on one of the four points of the compass. A *groma* was used to establish right angles for such measuring purposes. The groma was a wooden cross that pivoted and had plumb lines hanging from each arm to ensure proper positioning.

Distance measurement remained static for centuries after the Egyptians, with few advancements. About 500 B.C.E., Greek mathematician Thales demonstrated how to use the geometric principles of similarity to measure distances indirectly. For example, Thales used shadows to measure the height of the Great Pyramid. Still, the data needed for such indirect measurements had to be found using cords and measuring rods.

Two hundred years later, Greek mathematician Eratosthenes determined the circumference of Earth to within a few hundred miles by using geometric principles and data from rope stretchers. About 200 B.C.E., the astrolabe was invented to measure the **angle of elevation** of various stars. These data were used to find the time of day, which in turn was used by sailors for navigation. Later developments led to its replacement by the **sextant**. In the 1600s, the theodolite was invented, which is a device that measures angles of

angle of elevation the angle formed by a line of sight above the horizontal

sextant a device for measuring altitudes of celestial objects

theodolite a surveying instrument designed to measure both horizontal and vertical angles

elevation and horizontal angles simultaneously. The **theodolite** is essentially a telescope that pivots around horizontal and vertical axes. Modern theodolites can measure angles to an accuracy of 1/3600 of a degree.

In 1620, Edmund Gunther (1581–1626) invented a chain to survey land. It was 66 feet long and not subject to humidity, fraying, or other irregularities that might affect a rope or cord. The chain was the precursor to the modern steel tape measure. At about the same time, the telescope was invented. This made it possible to see great distances across the heavens and to see stars that had previously been beyond human sight. The distances between **celestial bodies** remained unknown, but advances in telescopes gave rise to the ability to measure distances across the heavens. During the same time period, Anthony Leeuwenhoek invented the first modern microscope, opening up a new world of discovery at the opposite end of the measurement spectrum.

celestial body any natural object in space, defined as above Earth's atmosphere; the Moon, the Sun, the planets, asteroids, stars, galaxies, nebula

Modern Advances in Distance Measurement

The early twentieth century saw great advancements in distance measurements. First, **aerial photography** made accurate measurements of distances across difficult terrain easily possible. This was followed by satellite photography, making it feasible to survey great tracts of land easily. In the twenty-first century, modern theodolites employ electromagnetic distance measurement processes. These instruments measure the time a laser or ultra high radiation (either microwave for longer distances or infrared radiation for shorter distances) needs to pass over a specific distance. Modern surveying instruments send out a laser pulse to a target on location. The target reflects the laser beam back to the surveyor. The time required to travel out and back is measured by a computer, which then calculates the distance. At the other end of measurable distances, powerful electronic microscopes enable scientists to see increasingly smaller objects.

aerial photography photographs of the ground taken from an airplane or balloon; used in mapping and surveying

Modern precision instruments have even affected standard measurement units like the meter. Originally the standard meter was a length of platinum that was exactly one meter long. However, a metallic object can grow or shrink according to atmospheric conditions. The effects of dust and handling can also affect such an object, although only at a microscopic level. To avoid such effects, a meter is now defined as the distance that light in a vacuum travels in 1/229,792,458 of a second.

Modern precision has made it possible to measure extremely large and small distances. The longest distance measurement is for galaxies that are 14 billion light years from Earth. (Light travels approximately 186,000 miles per second. A light year is how far light travels in a year.) This distance is referred to as 10^{25} meters. At the other end of the scale, scientists use scanning electron microscopes to see molecular particles that are 10^{-16} meters long, or 0.1 Fermi in length. (One Fermi is 10^{-15} meter.) How much more precision is still possible to achieve is difficult to predict. Most scientists agree that there are objects in the universe that are even farther away from Earth. Newer, more powerful telescopes will likely be able to detect them and measure their distance from Earth. Scientists also envision a time when objects as small as an individual atom or even subatomic particles will be "visible" and thus can be measured. SEE ALSO Astronomy, Measurements in;

Measurement, English System of; Measurement, Metric System of; Mile, Nautical and Statute; Navigation; Telescope; Time, Measurement of.

Arthur V. Johnson II

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Division by Zero

The number 0 has unique properties, including when a number is multiplied or divided by 0. Multiplying a number by 0 equals 0. For example, $256 \times 0 = 0$. Dividing a number by 0, however, is undefined.

Why is dividing a number by 0 undefined? Suppose dividing 5 by 0 produces a number x:

$$\frac{5}{0} = x.$$

From $\frac{5}{0} = x$ it follows that $0 \times x$ must be 5. But the product of 0 and any number is always 0. Therefore, there is no number x that works, and division by 0 is undefined.

A False Proof

If division by 0 were allowed, it could be proved—falsely—that $1 = 2$. Suppose $x = y$. Using valid properties of equations, the above equation is rewritten

$x^2 = xy$ (after multiplying both sides by x)

$x^2 - y^2 = xy - y^2$ (after subtracting y^2 from both sides)

$(x - y)(x + y) = y(x - y)$ (after factoring both sides)

$(x + y) = y$ (after dividing both sides by $(x - y)$)

$2y = y$ ($x = y$, based on the original supposition)

$2 = 1$ (after dividing both sides by y)

This absurd result ($2 = 1$) comes from division by 0. If $x = y$, dividing by $(x - y)$ is essentially dividing by 0 because $x - y = 0$.

Approaching Limits

It is interesting to note that dividing a number such as 5 by a series of increasingly small numbers (0.1, 0.01, 0.001, and so on) produces increasingly large numbers (50, 500, 5000, and so on). This division sequence can be written as $\frac{5}{x}$ where x approaches but never equals 0. In mathematical language, as x approaches 0, $\frac{5}{x}$ increases without limit or that $\frac{5}{x}$ approaches **infinity**. see also Consistency; Infinity; Limit.

Frederick Landwehr

infinity a quantity beyond measure; an unbounded quantity

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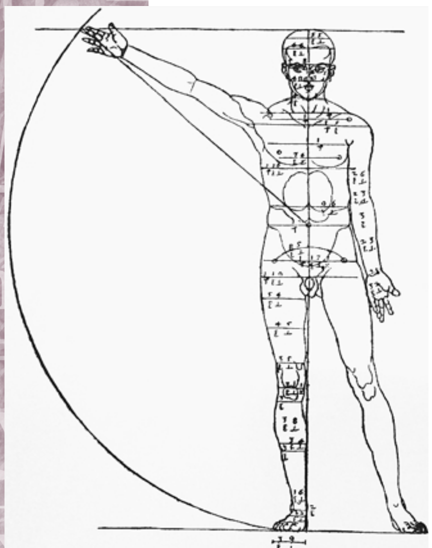
Dürer, Albrecht

German Painter, Printmaker, and Engraver
1471–1528



Albrecht Dürer's engraving "Proportion of Man" is both a mathematical and artistic representation of the human body.

Albrecht Dürer was born in 1471 in Nuremberg, Germany, and died there in 1528. He is regarded as one of the leading artists of the Renaissance. His use of mathematical methods in artistic composition influenced subsequent development of art.

Dürer first worked with his father, who was an accomplished goldsmith, then broadened his artistic training by assisting artist Michel Wohlgemuth. Developing his expertise quickly, Dürer was soon able to go out on his own as a painter and printmaker. He became widely known, traveling throughout Europe while studying and producing works of art, and was a particular favorite of Emperor Maximilian I.

Dürer attempted to portray nature realistically in his works, paying close attention to the appearance of animals, plants, and the human body and trying to reproduce them accurately. He was even known to have dissected **cadavers** to better understand the human body. Dürer's realistic paintings of plants influenced botanists to use drawings that more closely resembled the plants portrayed.

cadaver the term used to refer to a corpse intended for medical research or training

perspective the point of view; a drawing constructed in such a way that an appearance of three dimensionality is achieved

proportion the mathematical relation between one part and another part, or between a part and the whole; the equality of two ratios

To improve his paintings and etchings, Dürer sought a mathematical formulation for the ideal human body and for beauty in general. He studied the problems of space, **perspective**, and **proportion** and constructed his forms on the canvas, using arithmetic and geometric techniques. The results of his studies were published posthumously in 1528 as *The Four Books on Human Proportions*, a work that has had a significant effect on succeeding generations of artists. SEE ALSO Human Body.

J. William Moncrief

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Earthquakes, Measuring

Earthquakes happen every day. Thousands occur across the world every week. Most are not felt, but throughout history there have been many earthquakes that have been very strong and caused extensive destruction. Mathematics skills are invaluable to earthquake scientists, and with them they can sometimes predict when and where earthquakes will happen, and make measurements of earthquakes when they do occur.

Earthquakes happen when the **tectonic plates** of the Earth shift. As this movement happens, pressure builds on the plates and faults. Eventually this pressure is released through an earthquake. During an earthquake seismic waves radiate out from a central point in all directions. There are four basic types of seismic waves, two that travel through the Earth, and two that are felt at the surface. These waves are recorded on a seismograph, which is an instrument made up of sensitive detectors that produce a permanent recording.

tectonic plates large segments of Earth's crust that move in relation to one another

Size of Earthquakes

To determine the magnitude or size of an earthquake, scientists use the Richter scale. Developed by Charles Richter in 1934, this scale is based on a logarithmic increase of magnitude. With a scale such as this, for every whole number increase on the scale, the amplitude of the earthquake goes up ten times. For example, an earthquake with a reading of 5.0 on the Richter scale has a magnitude 10^5 (or 100,000) times as great as an earthquake with a magnitude of zero.

There are more than 1,000 earthquakes a day with recorded magnitudes of two or greater. An earthquake measuring between 6.1 and 6.9 can cause destruction in an area up to 100 kilometers in diameter. The San Francisco earthquake of 1906 measured 7.8 on the Richter scale.

Large sections of a major expressway collapsed during a 1995 earthquake in Kobe, Japan, where thousands of people perished.

Another way to measure an earthquake is to use the Mercalli Intensity Scale. This scale measures the intensity or energy of an earthquake. Though each earthquake has a fixed magnitude, the effects of it are different depending on location. This measurement is based on criteria such as structural damage and the observations of people who felt the earthquake. From these types of observations, the intensity of an earthquake can be estimated.

The Mercalli scale is not considered as scientific as the Richter scale, since it is based on factors that can change depending on where the infor-

mation is derived. After an earthquake, for example, witnesses may exaggerate or not agree on what they saw. In addition damage does not always accurately measure the strength of an earthquake.

Depth and Location of Earthquakes

Earthquakes occur between the Earth's surface and about 700 kilometers below the surface. The way to determine the depth of an earthquake is to look at wave characteristics on a seismogram, which is a graph of seismic waves. For example, an indication of a large earthquake with a deep focus would be surface waves with small amplitudes, and uncomplicated deep waves.

The point on Earth's surface directly above the origin of an earthquake is called its **epicenter**. To find out where an earthquake is located, scientists must examine its waves. The simplest method is to look at the different arrival times of wave types at multiple seismograph stations. Scientists then use standard travel-time tables and travel-time curves to find the distance to the earthquake from each station. Arcs are then drawn, with the distance from each station to the earthquake used as a radius. The point where all the arcs intersect is the epicenter of the earthquake.

epicenter the point on Earth's surface directly above the site of an earthquake

Calculating Earthquake Odds

Earthquakes are naturally recurring events, and scientists continue to develop better methods to predict when and where earthquakes might happen.✶ Earthquake probabilities are based on balancing the continual motions of the Earth's plates with the slip on faults, which occurs primarily during earthquakes. Scientists must also look at the history of a given fault to know when it last ruptured, potential quake sizes, and the rate at which the plate is moving. By combining geology, physics, and statistics, scientists continue to become more accurate in their earthquake predictions.

✶When predicting earthquakes, some scientists study the behavior of animals, which can become very erratic before a quake strikes.

SEE ALSO Logarithms; Probability, Theoretical.

Brook E. Hall

MAJOR QUAKE LIKELY TO STRIKE BETWEEN 2000 AND 2030

Scientists believe that there is an 80-percent probability of at least one earthquake with a Richter magnitude between 6.0 and 6.6 striking the San Francisco Bay region before 2030. By keeping aware of earthquake threats, people can make informed decisions on how to prepare for future quakes.

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Economic Indicators

Individuals and families have checking accounts, savings accounts, credit cards, and bills, so people make budgets to determine how much money they are making and spending. Businesses do the same thing. Countries, such as the United States, also use a budget to keep close track of their finances. The study of money and where it is going and where it came from

is called economics. The economy is the system through which money circulates. Economists are the people who study the movement of money through the economy. By looking at economic indicators (which are features of the economy that are represented in numbers) economists make predictions about the potential strengths and weaknesses in the economy. Economic indicators give economists valuable insights into a country's financial standing.

Interest rates on credit cards vary from lender to lender and from person to person, depending on one's credit history. Smart consumers shop around to find the best rates available.

Leading Indicators

Leading economic indicators are those that have the ability to forecast the probable future economy. An example of a leading indicator is the length of the workweek; that is, the average number of hours employees work in a week. As business increases, employers generally increase the number of hours that current employees work instead of immediately hiring new employees. A longer workweek tells economists that businesses are doing well. If business is increasing, the economy is doing well. If, on the other hand, the economy is doing poorly, employers will shorten their employees' workweek before laying off employees, a measure which is, for most employers, used only as a last effort to save money. Since the workweek lengthens or shortens before any effects on the economy are seen, a change in the length of the workweek in either direction can be used as a forecasting tool for economists.

Lagging Indicators

Other economic indicators result from changes in various features of the economy. An example of a lagging indicator is the unemployment rate, which is the number that represents the percentage of the labor force that is not employed. The labor force is defined as all people over 16 years of age who are able to work. However, it does not include stay-at-home mothers or fathers, students, people who cannot work because of their health, or people who are not looking for jobs. If many people are unemployed, the economy is doing poorly. If few people are unemployed, the economy is doing well. Since change in the unemployment rate tells how many people either lost or gained jobs, it simply describes what has already happened; it does not predict what might happen.

Numbers in the Economy

Most of the numbers used to calculate economic indicators come from the U.S. Census Bureau, which is the division of the U.S. Department of Commerce that keeps statistics, such as the unemployment rate, the workweek length, the number of new houses being built, the number of building permits being given out, the number of new jobs being created, as well as many other figures. All of these statistics are used to provide economists with the data they need to study the economy. Economists are most concerned with changes in these statistics, which raises questions such as "Are there more new jobs being created this year than last year?" or "Are people working fewer hours this month than last month?"

The numbers for economic indicators are often put into indexes. Indexes are combinations of data from different economic indicators. The index numbers provide a broader view of the economy. One of the most frequently used numbers is the index of leading economic indicators (LEI).

This index measures changes in several leading economic indicators. An increase in the LEI for 3 or more months signals that the economy is improving; a decrease in the LEI for 3 or more months suggests a possible recession. A recession is a temporary decrease in business and therefore a downturn in the economy.

Gross Domestic Product

One of the best indicators of the economy is the gross domestic product (GDP). This dollar figure is the value of all goods and services produced within a nation in a calendar year. The gross national product (GNP) is also a dollar figure, but differs in that it is the value of all goods and services produced by a nation's citizens within a year. For example, the profits from an American-owned business operating in Germany would be included in the GNP, but not in the GDP because the business is not within the United States. The gross domestic product (GDP) is more commonly used as an economic indicator.

Goods that are included in the GDP must be newly manufactured items, and they must be finished products. In making an automobile, first a manufacturer makes the steel and sells it to the auto manufacturer. At this stage the steel is not counted in the GDP because it is not a finished product. The steel is counted in the GDP only as part of the value of the finished automobile.

The goods and services counted in the GDP consist not only of the things that people buy: They can be things that people produce that get consumed without ever being sold. One example is a farmer who milks cows. If the farmer then drinks some of the milk he or she produces, it is still counted in the GDP, even though it was never sold. Economists will guess at its market value (its worth) and add it to the total GDP. If milk sells for $2.50 per gallon at the time and it is estimated that the farmer's family drank 100 gallons of milk during the year, $250 would be added to the GDP. In this sense, the GDP is not a precise measurement.

Information for calculating the GDP is collected every quarter (i.e., every 3 months). The GDP in each quarter is multiplied by 4 to calculate an "annual GDP." Economists often adjust the GDP even further. Since seasonal changes affect the economy, economists often make adjustments for the seasons of the year. In the summer, for example, tourism increases and more money is spent. Instead of saying that the GDP is higher in the summer, economists adjust it so it is standard throughout the year. They do this by looking at the change in the GDP over several summers. For example, if the average change every summer is an increase of 3 percent, but one summer it increased 5 percent, it will only be said to have increased 2 percent above "normal."

fluctuate to vary irregularly

Because prices **fluctuate** from year to year, another adjustment must be made to the GDP in order to allow economists to compare GDPs from different years. Most products, such as cars, homes, and clothing cost more now than they did 20 years ago. Therefore a dollar today purchases a different amount of product than it did in the past. To account for this, economists calculate each year's GDP in constant dollars. Constant dollars measure the value of products in a given year based on the prices of products in some base year. For example, in the early-1990s, economists used 1983 as a base year to convert to constant dollars. When economists refer

to constant dollars, the base year is often stated. By doing this, the change in the GDP from year to year is due to the change in the amount of goods and services being produced.

Inflation

Economists use the inflation rate to help them adjust the GDP to constant dollars. Inflation is the rate, expressed as a percentage, at which prices of goods and services are increasing each year. It may be reported on the news that "in March prices increased 1 percent, an annual inflation rate of 12 percent." The annual rate is calculated by multiplying the rate for the month times 12. But the annual rate of inflation for that year will only be 12 percent if prices continue to increase 1 percent per month for the rest of the year. Inflation rates may also be given as "the year's inflation." This does not mean an annual rate but a rate since January of that year. In the preceding example, the months included would be January, February, and March. It is important to understand the concept of inflation as well as the means by which it is reported.

The following example converts the GDP in 1980 to constant 1972 dollars. First, think of the base year (1972) prices as 1.0. By 1980, prices in the United States had increased to 1.8, an 80 percent increase since 1972. The GDP (in 1980 dollars) in 1980 was $2.62 trillion dollars. To change this to 1972 dollars, divide it by 1.8. This gives the constant dollar GDP of $1.45 trillion. This number is significantly lower than the current dollar GDP of 1980. However, it allows economists to compare the economy from year to year more accurately.

Consumer Price Index

Inflation is determined by the Consumer Price Index, which is a measurement of price increase. Surveyors from the U.S. Labor Department collect information from households around the country about what goods are being purchased and consumed. Beyond knowing how much a household is spending, for example, on groceries or entertainment, analysts determine how much is being spent on specific items, such as eggs, milk, and movie rentals. All of the information collected from the surveyors is then averaged. The result provides the Labor Department with a representative budget for an average household.

The surveyors then price all of the items that are in that representative budget every month. They can compare the prices each month to find out how much they have changed. The rate at which they change is inflation. It is important to recalculate the consumer price index periodically because of the changing habits of consumers.

These economic indicators are only a few of the many that economists study every day. However, these indicators are the ones that are most often reported to the public and the ones that most directly affect consumers. For example, inflation rates, which may affect the interest rate a consumer must pay on a car loan or mortgage, are used to adjust interest rates. As inflation rates rise, so do interest rates, thus affecting consumers' purchasing power. A basic understanding of economics and economic indicators is essential in sound financial management. SEE ALSO Agriculture; Stock Market.

Kelly J. Martinson

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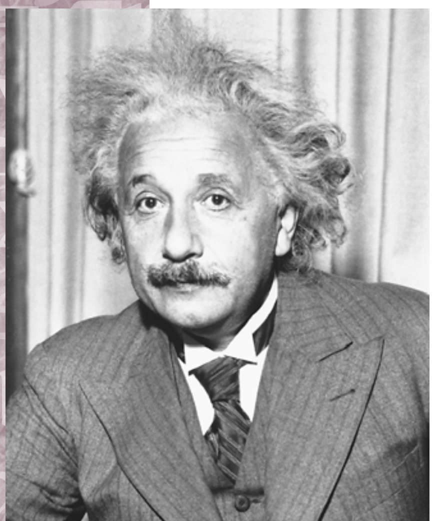

Although he attended school in both Germany and Switzerland, Albert Einstein was disinterested in the formalities of education, which led him to teach himself calculus, higher mathematics, and physics. He was awarded the Nobel Prize for Physics in 1921.

American Physicist and Mathematician
1879–1955

Albert Einstein is perhaps the best-known scientist who ever lived. His contributions include the special and general theories of **relativity**, the assertion of the equivalence of mass and energy, and the **quantum** explanation of the behavior of electromagnetic radiation, including light. Einstein was born in Ulm, Germany, in 1879 and died in Princeton, New Jersey, in 1955.

Einstein showed little academic ability before entering the Federal Polytechnic Academy in Zurich, Switzerland, in 1896, where he studied both mathematics and physics. After graduating in 1900, he briefly taught school and then took a position in the patent office. During this time, he wrote articles on theoretical physics in his spare time.

Einstein's ability to apply advanced mathematics in the solution of complex physical problems led to the publication of a group of momentous papers in 1905. A doctorate from the University of Zurich and world fame soon followed.

The subjects of the 1905 publications included special relativity, the equivalence of matter and energy, and the quantum nature of radiation. These revolutionary publications, in combination with the general theory of relativity, which he published in 1915, and the development of quantum mechanics, to which he made significant contributions, transformed science and again demonstrated the indispensability of mathematics in the scientific endeavor.

The atomic age, the space age, and the electronic age owe much to Einstein's contributions to physics, changing human civilization more dramatically in the twentieth century than in previous centuries combined.

J. William Moncrief

relativity the assertion that measurements of certain physical quantities such as mass, length, and time depend on the relative motion of the object and observer

quantum describes a small packet of energy (matter and energy are equivalent)

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Electronics Repair Technician

Modern life is powered by electricity, but electrical equipment sometimes breaks down. Electronics repair technicians can troubleshoot a piece of malfunctioning equipment and repair it. They can also test and maintain equipment to keep it from breaking down at all.

"Electronics" is a broad category, and electronics repair technicians find jobs in a number of settings and can pursue any range of specializations, including heating and air conditioning, home appliances, computers, and telecommunications, among others. Some electronics repair technicians even specialize in musical instruments such as electric guitars. While hands-on experience is important, most electronics repair technicians find that entry and advancement in the job market are easier if they have a strong math and science background in high school from classes such as physics, algebra, trigonometry, and calculus. It is also helpful to have a two-year degree in electronics technology from a community college or vocational-technical college. Such a program emphasizes not only electronics, but also applied mathematics and geometry. Although it is not mandatory, many pass an exam to be become certified.

Electronics repair technicians must be able to use mathematics to troubleshoot a wide range of problems.

Good electronics repair technicians have a firm grasp of the mathematics and physics of electricity. They have to measure and understand electrical charges, currents and amps, voltage, and resistance. They must be familiar with Ohm's law, which is the relationship between current, voltage, and resistance. An electronic repair technician should understand the structure and operation of electrical components, including resistors, capacitors, diodes, transistors, integrated circuits, and switches. In addition, they need to read and understand **schematic diagrams**, which visually present "outlines" of circuits, showing how electrical components connect. SEE ALSO BOOLE, GEORGE.

schematic diagram a diagram that uses symbols for circuit elements and arranges these elements in a logical pattern rather than a practical physical arrangement

Michael J. O'Neal

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Encryption *See Cryptology.*

End of the World, Predictions of

Keeping track of time has been an integral activity of every civilization, no matter how primitive. There are ample indications that even civilizations in pre-historic times made concerted attempts to measure time. Many civilizations, including the Mayan, and individuals, such as Stifel in sixteenth-century Germany, used measurements of time to predict the end of the world.

Keeping Track of Time

For early civilizations, keeping track of time involved two distinct pursuits: food and religion. The cycles for planting and harvesting could be anticipated and planned for with a calendar. Primitive societies could also plan for bird and animal migrations in advance of their arrival. In both cases, even a primitive calendar could help communities grow crops and hunt game in a better, more organized manner.

The religious aspect of keeping time was tied to the worship of the heavens. Ancient peoples viewed the motions of heavenly bodies and related phenomena, such as an eclipse of the moon, as acts of the gods that were important to predict. Archeologists think the standing stones of **Stonehenge** in England were a massive observatory that ancient peoples used to track the rising and setting of the sun on the summer and winter solstice. It also appears that they used Stonehenge to keep track of lunar eclipses. The ability to keep time and predict heavenly phenomena strengthened the hold that the religious class had over its followers.

Stonehenge a large circle of standing stones on the Salisbury plain in England, thought by some to be an astronomical or calendrical marker

Today's calendar is derived, in part, from the calendar used by ancient Sumerians and Babylonians. In approximately 3000 to 4000 B.C.E., these Middle Eastern peoples invented a written alphabet and a fairly accurate calendar. The Babylonians used a lunar calendar, which consisted of twelve lunar months of 29 or 30 days each, depending on the motion of the moon. The total number of days in their year was only 354 days. Religious feast days were added to some of the months to bring the total to 360 days.

This total of 360 days fit in well with the Babylonian mathematics of the times. Their number system was based on the number 60, in contrast to the modern-day number system, which is a decimal, or **base-10**, system. The 360-day year also accommodated Babylonian astrology.

base-10 a number system in which each place represents a power of 10 larger than the place to its right

In Babylonian astrology, the year was divided into twelve parts, each devoted to a god who was personified by the movement of various constellations in the heavens. The 360-day year was still 5 days shorter than the actual length of a year, so the Babylonians added a festival of 5 days at the end of the year to bring the total to 365 days.

Many cultures that followed the Babylonians adopted their 365-day year. This was still incorrect because a year is now known to be 365.2422 days. As a result, calendars descended from the Babylonian calendar required frequent adjustments, and the calendar for many cultures was constantly being revised to make up for lost or gained days.

The present calendar went through several refinements from its beginning during the Roman Empire. It was not until 1582 that a commission under Pope Gregory IX developed our present-day calendar, with an accuracy that will not need any revision for at least 3,000 years.

Around the year 500 C.E. the Mayas of Central America were using a calendar that is still more accurate than our calendar of today. The Mayan calendar✶ was used exclusively by the priestly classes and consisted of a ritual cycle of 260 individually named days and a yearly calendar of 365 days. The ritual cycle lasted 20 months, each month containing 13 days ($20 \times 13 = 260$). The yearly calendar, or *tun*, contained 18 months of 20 days each ($18 \times 20 = 360$), with a final month of 5 days added at the end of the year.

✶The last month in the Mayan calendar was a special month of bad luck and danger.

The Mayas also adjusted their calendar periodically, according to solar eclipses and observations of Venus. The two Mayan calendar cycles ran at the same time, and a named day would fall on the same day of the year every 18,980 days or every 52 years. This 52-year cycle is called a calendar round.

The calendar round was not used to indicate the specific year during which an event took place. Instead, the event was placed in a "long count."

A long count consists of 20 *tuns* making a *katun*, in turn 20 *katuns* make a *baktun*, and finally 13 *baktuns* make a "Great Cycle" of 1,872,000 days, or about 5,130 years.

According to Mayan tradition, the present creation is the fifth such creation of the gods. The gods were dissatisfied with their first four attempts to create mankind and destroyed each of the first four creations after a period of time called a long count. The fifth creation took place in 3133 B.C.E., and the present long count started at that time. This long count will expire on December 24, 2011. At that time, the gods will declare themselves either pleased with mankind or will destroy the world and begin again, starting the clock for a new long count.

Predicting the End of the World

Since the time of the ancient Babylonians, many individuals and cultures have looked to the heavens and to astrology to predict the future. Many mathematicians earned their living by casting horoscopes, including Jerome Cardano (1501–1576) and Johann Kepler (1571–1630).

During the sixteenth century, several mathematicians applied their knowledge of mathematics to questionable Bible scholarship and dubious Biblical interpretation to predict the exact date for the end of the world. After a careful study of the Bible and mathematics, German mathematician Michael Stifel (1486–1567) predicted a date for the end of the world.

Stifel was an educated man, who graduated with a degree in mathematics from the University of Wittenberg. He was a monk at a Catholic monastery for a time after graduation but eventually joined Protestant leader Martin Luther, even staying at Luther's house in Wittenberg. Soon after joining with Luther, Stifel began to apply his mathematics to the Bible. After some careful study, Stifel determined that a hidden message in the Bible revealed that the world would end on October 3, 1533.

When the fateful day arrived, Stifel gathered his small group of believers from the town of Lochau to the top of a hill to await the end of the world and their deliverance into heaven. As the morning wore on and nothing happened, some of the believers began to get worried. Soon most of the believers were anxiously questioning Stifel. At about midday, Stifel excused himself and hurried to town where he was put into a jail cell for protection against his now angry believers, many of whom sold houses and farms in expectation of the world's end. Eventually Stifel was able to leave town safely. In time, he gained Luther's forgiveness in return for a promise to never again apply mathematics to Biblical matters.

A more famous mathematician from the same time period also predicted a date for the world's end. Scottish mathematician John Napier (1550–1617) was a nobleman who spent nearly all his life in Scotland. Napier was an able mathematician, who was also active in religious controversies of the times. In fact, he devoted more time to his religious studies than to his mathematics. He was a zealous supporter of the Protestant cause.

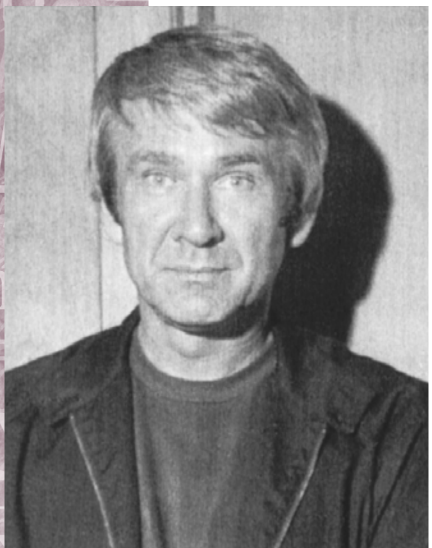
After 27 years of writing, Napier published *The Plaine Discovery of the Whole Revelation of St. John*. In it, he identified the Pope as the Antichrist. In *Discovery*, he also predicted that the end of the world would happen between 1688 and 1700. In this prediction, Napier had the good fortune to select a

day far beyond his life span. When his prediction failed to come true, he avoided the humiliation that had come to Stifel some 75 years earlier.

More recently, William Miller predicted the world would end on October 22, 1844. Thousands of believers were disappointed when the prophesied end did not come, but many shifted their belief to a future, unspecified time when the end would come.

While most predictions of the world's end today come from religious groups, there are scientists who are making such predictions about the world's end. Their predictions are based on our present knowledge of stars, such as the Sun. Our Sun has a finite life and eventually will burn out. When might such an event occur? The best estimates are that the core of the Sun will collapse in approximately 5 billion years. The resulting increase in temperature of the Sun's core will ultimately end all life on Earth. However, like Napier, the scientists making these estimates will not live to see whether their predictions come true. SEE ALSO Calendar, Numbers in the; Time, Measurement of.

Arthur V. Johnson II



Marshall Applewhite, leader of the Heaven's Gate Cult, believed that a spaceship would follow the Hale-Bopp comet of 1997, and that aliens aboard would take him and his followers to a better place. In order to be taken by the aliens, Applewhite and thirty-nine cult members committed suicide.

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Endangered Species, Measuring

The U.S. Endangered Species Act (ESA) is one of the most important and controversial legislative acts in recent years. This law requires the use of specific measures to protect certain species of plants or animals that are listed as **threatened species** or **endangered species**. Before a species is listed as threatened or endangered, biologists must determine if a **viable** population of the plant or animal in question exists in the wild. This usually means determining the number of existing individuals, the sex of each, the number within breeding age, the breeding success rate, mortality rates, birth rates, whether sufficient genetic diversity exists, and many other factors. To answer these questions, the number of plants or animals must be counted.

threatened species a species whose population is viable but diminishing or has limited habitat

endangered species a species with a population too small to be viable

viable capable of living, growing, and developing

transect to divide by cutting transversly

Study Methods Used to Estimate Population Size

Study methods include observation and photography, live trapping, and **transect** sampling. All of the methods result in an estimate of the number of individuals in the population. This number is then compared with what is considered a minimum viable population, which is the smallest number of individuals of a species in a particular area that can survive and maintain genetic diversity.

Observation and Photography. These are the simplest study methods. Observation and photography involve going into an area known to contain individuals from the population in question and simply counting how many can be found. This method works well for species that are not very mobile. For example, there is a species of snail that inhabits a small area on one mountain slope. This area is so small that one person can count the number of individuals present. In other cases, the number of individuals is so small that the entire population can be counted. Whooping cranes exist in such small numbers that the entire population is known from yearly censuses made on both the cranes's wintering grounds and their breeding grounds. Whooping cranes can be counted from observation towers, or they can be photographed from low flying airplanes or helicopters.

Observation can involve senses other than vision. Estimates of bird populations are often made by listening to singing males. The songs are distinctive enough that the species can be identified from their song. Knowledge of their breeding success, the number of offspring per pair, and mortality rates can then be used to determine if the population is viable.

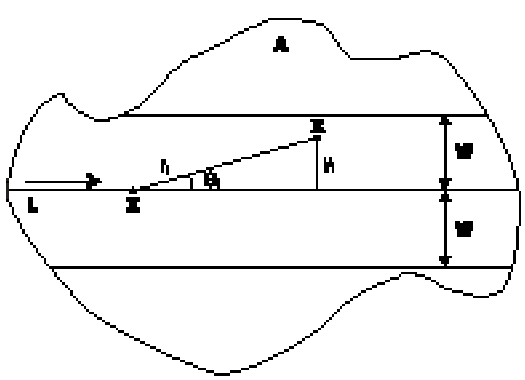
When one thinks of endangered or threatened species, one often thinks first of animals such as pandas or eagles. However, plants—like this species of fringed orchid—may also fall under the protection of the Endangered Species Act.

Live Trapping. Sometimes population estimates are made by placing traps in the area being studied. The number of animals captured is then related back to the total number of animals in the area. Live trapping generally does not harm the animals, and it has the advantage of allowing the researcher to gather other information about the species, such as age, sex, and health. Individuals can also be marked so that their movements can be followed. Birds trapped in this way are commonly banded. Bird banding has given biologists and wildlife managers extensive information about bird migration patterns.

Transect Sampling. Transect sampling is a standard statistical technique for determining the population in an area. The researcher or surveyor walks along a straight line (called a transect) through the area of interest and counts every individual that can be seen. Alternatively, the researcher may observe other evidence of the presence of an animal (such as droppings). Under ideal conditions, the number of individuals observed within the transect area has the same proportion to the total number of individuals within the total area.

Suppose a surveyor walks across the area A along the transect L. At some time, the surveyor will be at point Z, and may see an individual at X. The width of the strip that can be seen by the surveyor is 2W, and the distance of the animal or plant from the surveyor is r_i. The angle from the transect line to the observed animal or plant is θ_i. The **perpendicular** distance from the transect line to the observed individual is y_i. Note that $y_i = r_i \sin \theta_i$.

perpendicular forming a right angle with a line or plane



parameter an independent variable, such as time, that can be used to rewrite an expression as two separate functions

probability density function a function used to estimate the likelihood of spotting an organism while walking a transect

Fourier series an infinite series consisting of cosine and sine functions of integral multiples of the variable each multiplied by a constant; if the series is finite, the expression is known as a Fourier polynomial

exponential power series the series by which e to the x power may be approximated; $e^x = 1 + x + x^{2/2!} + x^{3/3!} + \ldots$

negative exponential an exponential function of the form $y = e^{-x}$

The mathematical model for estimating the size of a population from transect data depends on two assumptions: (1) not all individuals will be detected and (2) the probability of detecting an individual decreases as its perpendicular distance from the transect line increases. These two assumptions are generally expressed as a detection function, $g(y)$, which represents the probability that an individual will be observed at a distance x from the transect line. The function $g(y)$ decreases as y increases. Generally, estimating detection functions requires calculus. However, the result is an *effective* width, a, of the transect area that is different from the actual width. Once a is known, the population density is given by $D = \frac{n}{2La}$ where n is the number of individuals observed, and L is the length of the transect. The basic problem in estimating population density is estimating the **parameter** a. The parameter a can be estimated accurately by choosing an appropriate **probability density function** $f(y) = \frac{g(y)}{a}$. Division by a makes the probability density function equal to one, which is what is expected of probability functions. Candidate choices for $f(y)$ include **Fourier series**, **exponential power series**, and **negative exponentials**.

The critical assumption permitting estimation from distance data is that all objects located directly on the line (distance = 0) are certain to be detected, so $g(0) = 1$. If $g(0) = 1$, then $f(0) = \frac{1}{a}$. The equation for estimating population density can now be rewritten in terms of $f(0)$: $D = \frac{nf(0)}{2L}$. The estimator function $f(0)$ can usually be determined by trial and error or by one of several different mathematical models available. Wildlife managers commonly use computer programs that have the various mathematical models and estimator functions included.

Determining if a Species Has a Minimum Viable Population

Once the size of the population has been estimated, researchers must then decide if the population is healthy and can survive on its own, or if it is too small to be viable and requires protection. There are various methods of estimating viability of a population.

One method often used by biologists to estimate the viability of a population of vertebrates is the 50/500 rule. The minimum number of individuals in a breeding population required to prevent an unacceptable level of inbreeding is 50. The number of breeding individuals required for the long-term genetic variability necessary for a healthy population is 500. This rule is established by assuming that a mature male and a mature female are randomly drawn and randomly paired. However, in many populations, one male may dominate a large group of females, excluding other males from the genetic pool. In this and similar cases, a larger population is needed for viability.

Since not all individuals in a population are active breeders, the census population must be at least twice as large as the breeding population. Many researchers use a census population of 1,000 to 10,000 individuals as the minimum necessary for long-term genetic viability.

Other Factors Affecting Minimum Viable Population. When rules such as the 50/500 method are not applicable or when greater precision is de-

sired, one of several analytic approaches to calculating the Minumum Viable Population may be used. One analytic factor to be considered is the sex ratio. If the percentage of males is 50 percent, then the sex ratio is not skewed. However, in many populations the fraction of males can be larger or smaller than 50 percent. In these groups, larger census populations are necessary.

Another factor affecting the size of the census population is the average number of offspring. If a certain species typically produces many offspring, the census population can increase quite rapidly. Relative birth and death rates can also be used to predict population increase or decrease.

Simulations. None of the preceding approaches has been completely satisfactory in estimating populations and as a result, simulations have been suggested as an alternative. Simulations are computer programs that attempt to model population dynamics. Some are applicable to many different populations, while others have been developed for specific situations, such as a program to estimate the population of the grizzly bears of Yellowstone National Park.

The Endangered Species Act

The ultimate goal of counting specific species may be to determine if they should be listed as threatened or endangered. The first endangered species legislation was enacted in 1966. This act established a list of animals that were threatened with extinction. However, federal agencies were limited in their ability to protect these species. The most significant part of this first act was the establishment of the National Wildlife Refuge System, which was designed to protect the habitats of endangered species.

In 1969 the U.S. Congress passed the Endangered Species Conservation Act. This act included invertebrates as well as vertebrates. It also restricted interstate commerce in illegally taken animals. In 1973 the Endangered Species Act (ESA) was passed. The ESA has been called the most comprehensive legislation for the preservation of endangered species ever enacted by any nation. Since 1973, the ESA has been reauthorized and amended twice.

Under the ESA, two departments have the sole authority to list species: The National Marine Fisheries Service is authorized to list marine mammals and fish, and the U.S. Fish and Wildlife Service lists all other species. This division of responsibility between the U.S. Department of Commerce and the U.S. Department of the Interior has caused some difficulties in the consistent application of rules and criteria for listing species. The act was amended in 1982 to state that listing should be based solely on biological criteria.

The ESA identifies two different population conditions related to the viability of species. "Threatened" species are those whose populations are still viable, but may be declining or have a limited habitat, and therefore be in danger of becoming threatened. "Endangered" species have populations that are too small to be considered viable and thus are in danger of becoming extinct. SEE ALSO Statistical Analysis.

Elliot Richmond

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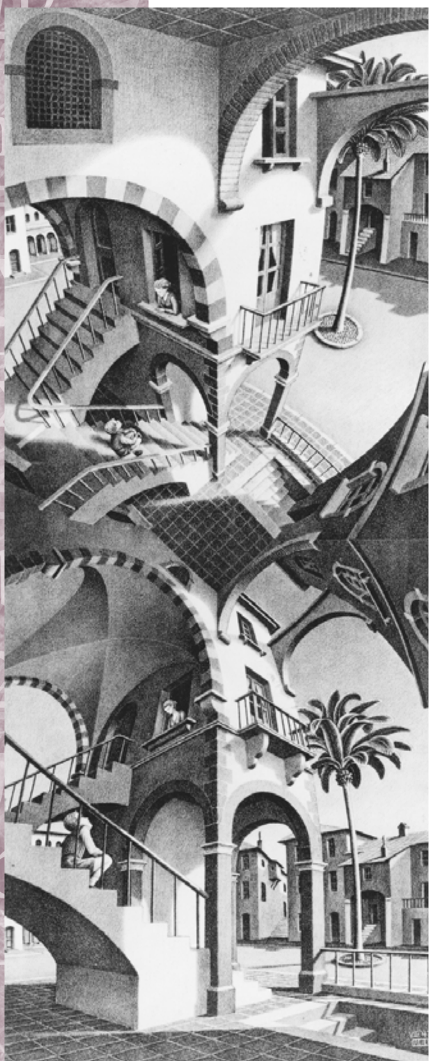

M. C. Escher's lithographs and woodcuts of geometric distortions and mathematical impossibilities made him famous throughout the world.

Escher, M. C.

Dutch Artist
1898–1972

Maurits Cornelis Escher was born in Leeuwarden, Holland, in 1898. He enrolled in the School for Architecture and Decorative Arts in Haarlem because his father, a civil engineer, wanted him to become an architect. Escher, however, left school in 1922 to pursue his interest in art. He married in 1924 and moved to Rome where he lived until 1934. Growing political tension in Europe caused him to move his family first to Switzerland, then to Belgium, and finally back to the Netherlands in 1941. He remained there until his death in 1972.

Mathematics in Art

Escher's art is of particular interest to mathematicians because, although he received no mathematical training beyond his early years, he used a variety of mathematical principles in unique and fascinating ways. Escher's artwork encompasses two broad areas: the geometry of space, and the so-called "logic" of space.

On a visit to the Alhambra in Spain, Escher was inspired by the colorful geometrical patterns of tiles. He began to explore the various ways of filling two-dimensional space with symmetrically repeated arrangements of images known as tessellations. In the process, he discovered the same principles that had been developed previously, unknown to Escher, within the branch of mathematics known as group theory. When mathematicians and scientists became aware of his work, they helped popularize his art, and he soon gained an international reputation.

Subsequent interactions with mathematicians introduced Escher to other mathematical concepts that he explored in his art. Among the results are his so-called impossible constructions that appear reasonable but prove to be impossible to construct in three-dimensional space. He also employed **non-Euclidean geometry**, representations of infinite space, and various aspects of **topology**.

non-Euclidean geometry a branch of geometry defined by posing an alternate to Euclid's fifth postulate

topology the study of those properties of geometric figures that do not change under such nonlinear transformations as stretching or bending

Although Escher completed his final graphic work in 1969, the popularity of his images continues today. Several Internet sites are dedicated to providing information about Escher and selling renditions of his art. SEE

ALSO DIMENSIONS; EUCLID AND HIS CONTRIBUTIONS; MATHEMATICS, IMPOSSIBLE; TESSELLATIONS; TOPOLOGY.

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Estimation

Adding, multiplying, and performing similar mathematical operations in one's head can be difficult tasks, even for the most skilled mathematics students. By estimating, however, basic operations are easier to calculate mentally. This can make daily calculation tasks, from figuring tips to monthly budgets, quickly attainable and understandable.

How to Estimate

Although the core of estimation is **rounding**, *place value* (for example, rounding to the nearest hundreds) makes estimating flexible and useful. For instance, calculating $2.4 + 13.7 - 10.8 + 8 - 124.2 - 32$ to equal -142.9 in one's head may be a daunting task. But if the equation is estimated by 10s, that is, if each number is rounded to the nearest 10, the problem becomes $0 + 10 - 10 + 10 - 120 - 30$, and it is easier to calculate its value at -140. Estimating to the 1s makes the equation $2 + 14 - 11 + 8 - 124 - 32 = -143$, which is more accurate but more difficult to calculate mentally. Note that the smaller the place value used, the closer the estimation is to the actual sum.

rounding process of giving an approximate number

Estimation by Tens

Multiplication and division can be estimated with any place value, but estimating by 10s is usually the quickest method. For example, the product $8 \times 1{,}294 = 10{,}352$ can be estimated by 10s as $10 \times 1{,}290 = 12{,}900$, which is calculated with little effort. Division is similar in that estimating by 10s allows for the quickest calculation, even with decimals. For instance, $1{,}232.322 \div 12.2 = 101.01$ is quicker to estimate by 10s as $1{,}230.0 \div 10.0 = 123.0$.

Regardless of the ease of estimating by 10s, there is a greater degree of inaccuracy as compared to estimating by 1s. However, this estimation method need not be abandoned in order to gain accuracy; instead, it can be used to obtain estimations that are more accurate, as the following example illustrates.

Suppose a couple on a date enjoys a dinner that costs \$24.32. The customary tip is 15 percent, but the couple does not have a calculator, tip table, or pencil to help figure the amount that should be added to the bill. Using the estimating-by-10s method, they figure that 15 percent of \$10 is \$1.50; if the bill is around \$20, then the tip doubles to \$3. However, a \$3 tip is not enough because they have not included tip for the \$4.32 remaining on the bill. Yet if \$1.50 is the tip for \$10, then \$0.75 would be an appropriate tip for \$5, which is near enough to \$4.32. A total estimated tip of \$3.75 is

close (in fact, an overestimation) to 15 percent of $24.32, which is $3.65 (rounded to the nearest cent).

Conservative Estimation

As seen in several of the examples, estimations tend to be more (an overestimation) or less (an underestimation) than the actual calculation. Whether this is important depends upon the situation. For example, overestimating the distance for a proposed trip may be a good idea, especially in figuring how much gas money will be needed.

This property of rounding and estimation is the foundation of conservative estimation found in financial planning. When constructing a monthly budget, financial planners will purposely underestimate income and overestimate expenses, usually by hundreds. Although an accurate budget seems ideal, this estimating technique creates a "cushion" for unexpected changes, such as a higher water bill or fewer hours worked. Furthermore, financial planners will round down (regardless of rounding rules) for underestimation and round up for overestimation.

The following table represents a sample budget for an individual. The first column includes amounts expected to pay; the second is a conservative estimate of the next month's budget; the third is a list of the actual amounts incurred; and the fourth is the difference between actual and budgeted amounts. Note that negative numbers, or amounts that take away from income, are written in parentheses.

The table shows that the individual earned less than expected and in some cases spent more than expected. Nevertheless, because the budget is conservative, there is a surplus (money left over) at the end of the month.

ESTIMATING A MONTHLY BUDGET

	Expected Amount	Budget	Actual Amount	Difference
Income	$3,040	$3,000	$2,995	($5)
Tax	(578)	(600)	(579)	21
Rent	(575)	(600)	(575)	25
Utilities	(40)	(100)	(62)	38
Food	(175)	(200)	(254)	(54)
Insurance	(175)	(200)	(175)	25
Medical	(45)	(100)	(97)	3
Car Payments	(245)	(300)	(245)	55
Gas	(85)	(100)	(133)	(33)
Student Loans	(325)	(400)	(325)	75
Savings	(300)	(300)	(300)	0
Fun Money	(49)	(100)	(175)	(75)
Surplus (Deficit)	$28	$0	$75	$75

Estimation by Average

Counting the number of words on a page can be a tedious task. Therefore, writers often estimate the total by averaging the number of words on the first few lines and then multiplying that average by the number of lines on the page.

Another application of estimation by average is the classic game of guessing how many jellybeans are in a jar. The trick is to average the number of beans on the top and bottom layers and then to multiply that average by the number of layers in the jar. Because it is customary to declare a winner

who guessed the closest but not over the actual count, it is best to estimate conservatively.

Estimation is a powerful skill that can be applied to tasks from proofing arithmetic to winning a counting game. However, the use of estimation is not always appropriate to the task. For example, estimating distance and direction of space debris and ships is unwise, since even the smallest decimal difference can mean life or death. In addition, technology makes it possible to add and multiply large groups of numbers faster than it may take to estimate the total. Nevertheless, estimation is an important tool in managing the everyday mathematics of life. SEE ALSO Financial Planner; Rounding.

Michael Ota

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Euclid and His Contributions

Euclid was an ancient Greek mathematician from Alexandria who is best known for his major work, *Elements.* Although little is known about Euclid the man, he taught in a school that he founded in Alexandria, Egypt, around 300 B.C.E.

Euclid, the best-known mathematician of classical antiquity, is considered by many to be the founder of geometry.

For his major study, *Elements*, Euclid collected the work of many mathematicians who preceded him. Among these were Hippocrates of Chios, Theudius, Theaetetus, and Eudoxus. Euclid's vital contribution was to gather, compile, organize, and rework the mathematical concepts of his predecessors into a consistent whole, later to become known as Euclidean geometry.

In Euclid's method, **deductions** are made from premises or axioms. This deductive method, as modified by Aristotle, was the sole procedure used for demonstrating scientific certitude ("truth") until the seventeenth century.

deduction conclusion arrived at through reasoning

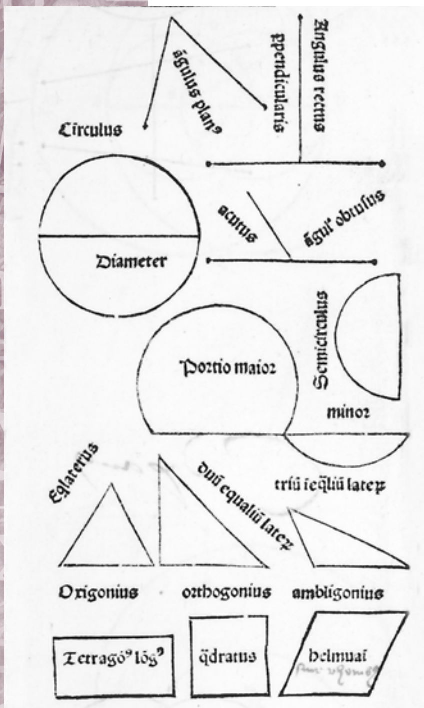
At the time of its introduction, *Elements* was the most comprehensive and logically rigorous examination of the basic principles of geometry. It survived the eclipse of classical learning, which occurred with the fall of the Roman Empire, through Arabic translations. *Elements* was reintroduced to Europe in 1120 C.E. when Adelard of Bath translated an Arabic version into Latin. Over time, it became a standard textbook in many societies, including the United States, and remained widely used until the mid-nineteenth century. Much of the information in it still forms a part of many high school geometry curricula.

Axiomatic Systems

To understand Euclid's *Elements*, one must first understand the concept of an **axiomatic system**. Mathematics is often described as being based solely on logic, meaning that statements are accepted as fact only if they can be logically deduced from other statements known to be true.

axiomatic system a system of logic based on certain axioms and definitions that are accepted as true without proof

What does it mean for a statement to be "known to be true?" Such a statement could, of course, be deduced from some other "known" statement. However, there must be some set of statements, called axioms, that are sim-



Euclidean geometry is based on Euclid's thirteen-volume *Elements.* An excerpt is shown here from an early Latin translation.

theorem a statement in mathematics that can be demonstrated to be true given that certain assumptions and definitions (called axioms) are accepted as true

ply assumed to be true. Without axioms, no chain of deductions could ever begin. Thus even mathematics begins with certain unproved assumptions.

Ideally, in any axiomatic system, the assumptions are of such a basic and intuitive nature that their truth can be accepted without qualms. Yet axioms must be strong enough, or true enough, that other basic statements can be proved from them.

Definitions are also part of an axiomatic system, as are undefined terms (certain words whose definitions must be assumed in order for other words to be defined based on them). Thus an axiomatic system consists of the following: a collection of undefined terms; a collection of definitions; a collection of axioms (also called postulates); and, finally, a collection of **theorems**. Theorems are statements that are proved by the logical conclusion of a combination of axioms, definitions, and undefined terms.

Euclid's Axioms

In the *Elements*, Euclid attempted to bring together the various geometric facts known in his day (including some that he discovered himself) in order to form an axiomatic system, in which these "facts" could be subjected to rigorous proof. His undefined terms were point, line, straight line, surface, and plane. (To Euclid, the word "line" meant any finite curve, and hence a "straight" line is what we would call a line segment.)

Euclid divided his axioms into two categories, calling the first five postulates and the next five "common notions." The distinction between postulates and common notions is that the postulates are geometric in character, whereas common notions were considered by Euclid to be true in general.

Euclid's axioms follow.

1. It is possible to draw a straight line from any point to any point.
2. It is possible to extend a finite straight line continuously in a straight line. (In modern terminology, this says that a line segment can be extended past either of its endpoints to form an arbitrarily large line segment.)
3. It is possible to create a circle with any center and distance (radius).
4. All right angles are equal to one another. (A right angle is, by Euclid's definition, "half" of a straight angle: that is, if a line segment has one of its endpoints on another line segment and divides the second segment into two angles that are equal to each other, the two equal angles are called right angles.)
5. If a straight line falling on (crossing) two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.
6. Things which are equal to the same thing are equal to each other.
7. If equals are added to equals, the wholes (sums) are equal.
8. If equals are subtracted from equals, the remainders (differences) are equal.

9. Things that coincide with one another are equal to one another.
10. The whole is greater than the part.

It was Euclid's intent that all the remaining geometric statements in the *Elements* be logical consequences of these ten axioms.

In the two millennia that have followed the first publication of the *Elements*, logical gaps have been found in some of Euclid's arguments, and places have been identified where Euclid uses an assumption that he never explicitly states. However, although quite a few of his arguments have needed improvement, the great majority of his results are sound.

Euclid's Fifth Postulate

The axioms in Euclid's list do seem intuitively obvious, and the *Elements* itself is proof that they can, as a group, be used to prove a wide variety of important geometric facts. They also, with one exception, seem sufficiently basic to warrant axiom status—that is, they need not be proved by even more basic statements or assumptions. The one exception to this is the fifth postulate. It is considerably more complicated to state than any of the others and does not seem quite as basic.

Starting almost immediately after the publication of the *Elements* and continuing into the nineteenth century, mathematicians tried to demonstrate that Euclid's fifth postulate was unnecessary. That is, they attempted to upgrade the fifth postulate to a theorem by deducing it logically from the other nine. Many thought they had succeeded; invariably, however, some later mathematician would discover that in the course of his "proofs" he had unknowingly made some extra assumption, beyond the allowable set of postulates, that was in fact logically equivalent to the fifth postulate.

In the early nineteenth century, after more than 2,000 years of trying to prove Euclid's fifth postulate, mathematicians began to entertain the idea that perhaps it was not provable after all and that Euclid had been correct to make it an axiom. Not long after that, several mathematicians, working independently, realized that if the fifth postulate did not follow from the others, it should be possible to construct a logically consistent geometric system without it.

One of the many statements that were discovered to be equivalent to the fifth postulate (in the course of the many failed attempts to prove it) is "Given a straight line, and a point P not on that line, there exists at most one straight line passing through P that is parallel to the given line." The first "non-Euclidean" geometers took as axioms all the other nine postulates of Euclidean geometry but replaced the fifth postulate with the statement "There exists a straight line, and a point P not on that line, such that there are two straight lines passing through P that are parallel to the given line." That is, they replaced the fifth postulate with its negation and started exploring the geometric system that resulted.

Although this negated fifth postulate seems intuitively absurd, all our objections to it hinge on our pre-conceived notions of the meanings of the undefined terms "point" and "straight line." It has been proved that there is no logical incompatibility between the negated fifth postulate and the

other postulates of Euclidean geometry; thus, non-Euclidean geometry is as logically consistent as Euclidean geometry.

The recognition of this fact—that there could be a mathematical system that seems to contradict our most fundamental intuitions of how geometric objects behave—led to great upheaval not only among mathematicians but also among scientists and philosophers, and led to a thorough and painstaking reconsideration of what was meant by words such as "prove," "know," and above all, "truth." SEE ALSO Postulates; Theorems and Proofs; Proof.

Naomi Klarreich and J. William Moncrief

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Euler, Leonhard

Swiss Geometer and Number Theorist
1707–1783

Leonhard Euler developed integral calculus, introduced the concept of function, and created much mathematical notation. His devotion to solving difficult equations caused him to suffer from eye trouble.

Leonhard Euler is a name well known in many academic fields: philosophy, **hydrodynamics**, astronomy, optics, and physics. His true fame comes, however, through his prolific work in pure mathematics. He produced more scholarly work in mathematics than have most other mathematicians. His abilities were so great that his contemporaries called him "Analysis Incarnate."

hydrodynamics the study of the behavior of moving fluids

Euler (pronounced "oiler") was born in Switzerland in 1707. He had originally intended to follow the career path of his father, who was a Calvinist clergyman. Euler studied theology and Hebrew at the University of Basel.

Johann Bernoulli, however, tutored Euler in mathematics on the side. Euler's facility in the subject was so great that his father, an amateur mathematician himself, soon favored the decision of his son to pursue mathematics rather than join the clergy.

Euler first taught at the Academy of Sciences in St. Petersburg, Russia, in 1727. He married and eventually became the father of thirteen children. His children provided him with great joy, and children were often playing in the room or sitting on his lap while Euler worked. It was in Russia that he lost sight in one eye after working for three days to solve a mathematics problem that Academy members urgently needed but had predicted would take months to solve.

Euler was a very productive writer, completing five hundred books and papers in his lifetime and having four hundred more published posthumously. The Swiss edition of his complete works is contained in seventy-four volumes. He wrote *Introductio in Analysin Infinitorum* in 1748. This book introduces much of the material that is found in modern algebra and trigonometry textbooks.

Euler wrote the first treatment of **differential calculus** in 1755 in *Institutiones Calculi Differentialis* and in 1770 explored determinate and **indeterminate algebra** in *Anleitung zur Algebra*. Three-dimensional surfaces and **conic sections** were also extensively treated in his writings.

Euler introduced many of the important mathematical symbols that are now in standard usage, such as $\sum$ (for summation), π (the ratio of the circumference of a circle to its diameter), $f(x)$ (function notation), e (the base of a natural logarithm), and i (square root of negative one). He was the first to develop the calculus of variations. One of the more notable equations that he developed was $\cos\theta + i\sin\theta = e^{i\theta}$, which shows that exponential and trigonometric functions are related. Another important equation he developed establishes a relationship among five of the most significant numbers, $e^{\pi i} + 1 = 0$.

All of Euler's work was not strictly academic, however. He enjoyed solving practical problems such as the famous "seven bridges of Königsberg" problem that led to Euler circuits and paths. He even performed calculations simply for their own sake.

Euler later went to Berlin to become the director of Mathematics at the Academy of Science under Frederick the Great and to enjoy a more free political climate. However, Euler was viewed as being rather unsophisticated, and Frederick referred to him as a "mathematical Cyclops." He returned to Russia when a more liberal leader, Catherine the Great, came to rule.

By 1766, Euler was completely blind but continued to dictate his work to his secretary and his children. His last words, uttered as he suffered a fatal stroke in 1783, imitated his work in eloquence and simplicity: "I die." SEE ALSO Bernoulli Family; Nets.

Laura Snyder

differential calculus the branch of mathematics primarily dealing with the solution of differential equations to find lengths, areas, and volumes of functions

indeterminate algebra study and analysis of solution strategies for equations that do not have fixed or unique solutions

conic sections the curves generated by an imaginary plane slicing through an imaginary cone (circles, ellipses, parabolas, hyperbolas)

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Exponential Growth and Decay

An exponential function is a function that has a **variable** as an **exponent** and the base is positive and not equal to one. For example, $f(x) = 2^x$ is an exponential function. Note that $f(x) = x^2$ is *not* an exponential function (but instead a basic **polynomial** function), because the exponent is a constant and not a variable. Exponential functions have graphs that are continuous curves and approach but never cross a horizontal **asymptote**. Many real-

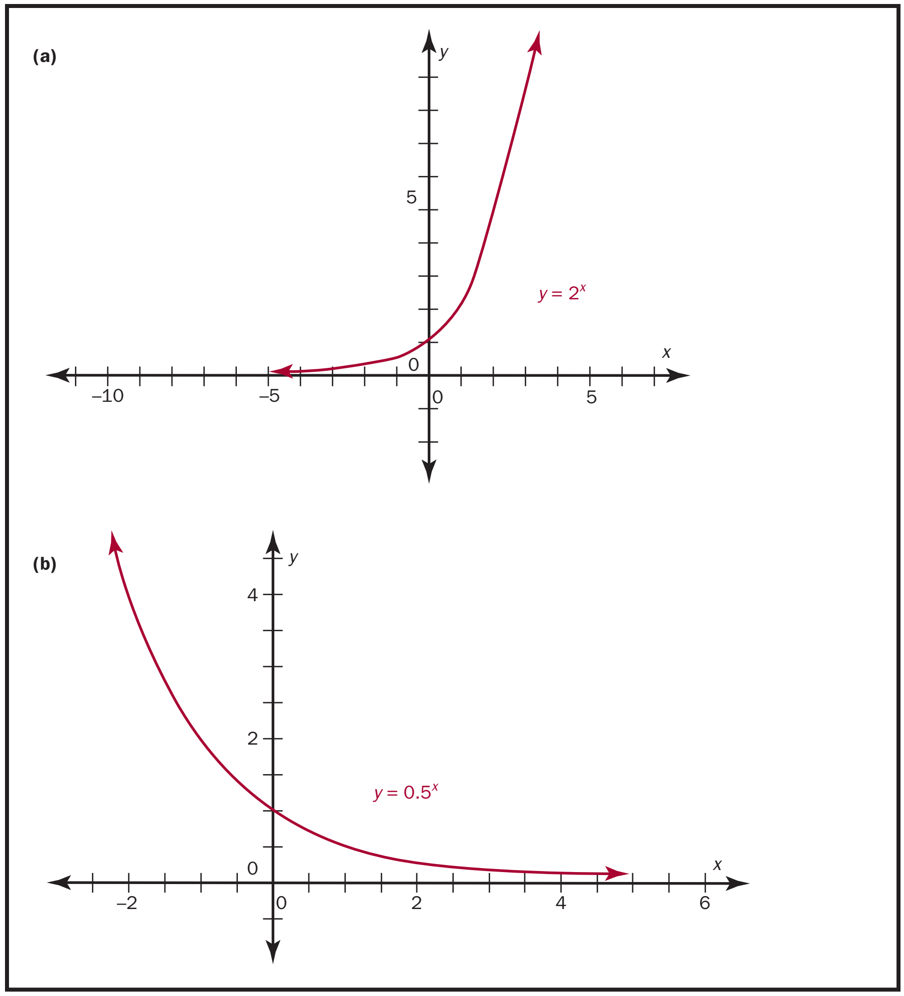
variable a symbol, such as letters, that may assume any one of a set of values known as the domain

exponent the symbol written above and to the right of an expression indicating the power to which the expression is to be raised

polynomial an expression with more than one term

asymptote the line that a curve approaches but never reaches

Graphs of a two exponential functions.



world processes follow exponential functions or their inverses, logarithmic functions.

Exponential Change

Exponential growth is a mathematical change that increases without limit based on an exponential function. The change can be in the positive or negative direction. The important concept is that the rate of change continues to increase. Exponential decay is found in mathematical functions where the rate of change is decreasing and thus must reach a limit, which is the horizontal asymptote of an exponential function. In the figure above, the asymptote is the x-axis where the rate of change approaches zero. Exponential decay may also be either decreasing or increasing; the important concept is that it progresses at a slower and slower rate.

Exponential growth and decay are modeled in many real-world processes. Populations of growing microbes, and indeed a growing population of any life when not constrained by environmental factors such as available space and nutrition, can be modeled as a function showing exponential growth. The growth of a savings account collecting compound interest is another example of an exponential growth function.

Exponential decay is seen in many processes as well. The decrease in radioactive material as it undergoes fission and decays into other atoms fits a curve of exponential decay. The discharge of an electric capacitor through a resistance can be calculated using exponential decay. A warm object as it cools to a constant surrounding temperature, or a cool object as it warms, will exhibit a curve showing exponential decay.

A Sample Problem

The following is an example of how the mathematics of exponential growth and decay can be used to solve problems. Suppose that a radioactive sample, measured after 2 days, had only 60 percent as much of the sample as it had initially. How much of the sample could be expected to remain after 5 days?

To solve this problem, it must be understood that the sample is reducing in size by exponential decay and the rate at which it is reducing must first be determined. Formulas for this exponential decay are as follows:

$$N = N_o e^{kt} \quad \text{and} \quad \ln\left(\frac{N}{N_o}\right) = kt$$

where N_o is the original value, N is the new value, k is decay constant or rate of change, and t is time. Using the second formula we see that N_o is 100 and N is 60, and t (in days) is 2, we get $\ln(\frac{60}{100}) = k2$ which is $-0.5108 = k2$ or $-0.2554 = k$. So the decay rate is -0.2554. Now that we know the decay rate, to find out what happens in 5 days we use the first formula $N = N_o e^{kt}$ where $N = 100e^{5(-0.2554)}$ and so $N = 27.89$. Thus we will have 27.89 percent of the sample after 5 days. SEE ALSO Logarithms; Powers and Exponents.

Harry J. Kuhman

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Factorial

The pattern of multiplying a positive **integer** by the next lower consecutive integer occurs frequently in mathematics. Look for the pattern in the following expressions.

$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$4 \times 3 \times 2 \times 1$

$(n + 5) \times (n + 4) \times (n + 3) \times (n + 2) \times (n + 1) \times n$

The mathematical symbol for this string of factors is the familiar exclamation point (!). This pattern of multiplied whole numbers is called n factorial and is written as $n!$ So, starting with the greatest factor, n, the factorial pattern is as follows:

$$n! = n(n - 1)(n - 2)(n - 3). . . (1).$$

So,

$3!$ is $3 \times 2 \times 1 = 6$

$5!$ is $5 \times 4 \times 3 \times 2 \times 1 = 120$

and

$1! = 1.$

Zero factorial (0!) is arbitrarily defined to be 1.

Most scientific calculators have a key (such as $x!$) that can be used to find factorial values. As n becomes larger, the value of its factorials increases rapidly. For example, 13! is 6,227,020,800.

How Factorials Are Used

Many mathematical formulas use factorial notation, including the formulas for finding **permutations** and **combinations**. For example, the number of permutations of n elements taken n at a time is $n!$, and the number of permutations of n elements taken r at a time is equal to $\frac{n!}{(n - r)!}$.

There is also a problem that involves prime and **composite numbers** which uses a formula containing factorial notation. Mathematicians have, for many years, puzzled over the question of how **prime numbers** were distributed. Notice that, in the **whole numbers** less than 20, there are eight

integer a positive whole number, its negative counterpart, or zero

permutation arrangement of a set of objects

combinations a group of elements from a set in which order is not important

composite number a number that is not prime

prime number a number that has no factors other than itself and 1

whole numbers the positive integers and zero

prime numbers (2, 3, 5, 7, 11, 13, 17, and 19). But from 20 to 40, there are only four prime numbers (23, 29, 31, and 37).

No one has yet found a formula that will generate all the prime numbers. However, the following sequence will give a string of n consecutive composite numbers (numbers that are *not* prime) for any positive integer n.

$$(n + 1)! + 2, (n + 1)! + 3, (n + 1)! + 4, (n + 1)! + 5, (n + 1)! + 6, \text{ and so on up to } (n + 1)! + (n + 1).$$

When n is 2, notice that this sequence only has two terms:

$$(n + 1)! + 2, (n + 1)! + (n + 1)$$

which is

$$(2 + 1)! + 2, (2 + 1)! + (2 + 1)$$

For the first term, $(2 + 1)! + 2$ is $3! + 2$ or $(3 \times 2 \times 1) + 2$, giving a value of 8. The second term has a value of 9.

When $n = 2$, this sequence gives two consecutive numbers that are *not* prime numbers: 8, 9. When $n = 3$, this sequence gives three consecutive numbers that are *not* prime numbers: 26, 27, 28. This relationship between the value of n and the number of consecutive numbers that are not prime numbers continues in this sequence for any whole number value for n. For a greater n, such as 300, a sequence of 300 composite numbers (that is, a list of 300 consecutive numbers with no prime number in the list) can be found. SEE ALSO Factors; Primes, Puzzles of; Permutations and Combinations.

Lucia McKay

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Factors

Factors can be thought of as the multiplying building blocks for integers. A factor is an **integer** that divides another integer without leaving a remainder. In general, an integer x is a factor of the integer y if $\frac{y}{x}$ is also an integer.

integer a positive whole number, its negative counterpart, or zero

Because 8 divides 24 evenly (as 3), with no remainder, 8 is a factor of 24. Eight factors into $2 \times 2 \times 2$. Hence, the integer 24 is made up of the factors 2, 2, 2, and 3 multiplied together.

Every integer is a factor of itself, because it divides itself evenly, with no remainder. Also, 1 is a factor of every number. All integers therefore have factors, because each integer (except 0) has at least two factors—1 and itself.

When an integer has only two factors, then it is a **prime number**. Because the only factors of 5 are 1 and 5, 5 is a prime number. When a number has more than two factors, then it is a composite number. Besides 1 and 15, 15 has two more factors—3 and 5; hence, 15 is a composite number.

prime number a number that has no factors other than itself and 1

All even numbers, 2, 4, 6, 8,. . . have 2 as a factor. By definition, an even number is a multiple of 2 and can be written as $2n$, where n is a positive integer. Odd numbers, 1, 3, 5, 7,. . . are of the form $2n + 1$. Therefore, by definition, 2 cannot be a factor of an odd number.

A factor that is a prime number is called a prime factor. For instance, 3 and 5 are prime factors of 15, and 3 × 5 = 15. The "fundamental theorem of arithmetic" states that every integer can be expressed as a unique product of prime factors. In other words, every whole number can be expressed as a product of primes (and 1) unique to it. For example, the prime factors of 20 are 2, 2, and 5, since 2 × 2 × 5 = 20, and no other set of prime factors will yield the number 20.

The Greatest Common Factor

What is the greatest common factor of 16 and 8? This question asks which factors 16 and 8 have in common and which of those factors is the greatest. For instance, 4 is a common factor of both 8 and 16. Eight is also a common factor of 8 and 16. But 8 is the greatest common factor of 8 and 16. As the name suggests, the greatest common factor of two or more numbers is the largest factor shared by them. What is the greatest common factor of 12, 8, and 4? It is 4 because 4 is the largest number that divides all three numbers. SEE ALSO Primes, Puzzles of.

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number theory the study of the properties of the natural numbers, including prime numbers, the number theorem, and Fermat's Last Theorem

analytic geometry the study of geometric properties by using algebraic operations

probability theory the branch of mathematics that deals with quantities having random distributions

loci sets of points, lines, or surfaces that satisfy particular requirements

Fermat, Pierre de

French Lawyer and Mathematician
1601–1665

Pierre de Fermat was born in 1601 in Beaumont-de-Lomagne, France, and died in 1665 in Castres, France. He was the founder of modern **number theory**, one of the initiators of **analytic geometry**, and co-founder of **probability theory**. He also invented important mathematical methods that anticipated differential and integral calculus.

Little is known of Fermat's early life. He received his law degree from the University of Orleans and served as councilor to parliament beginning in 1634. His real passion, however, was mathematics. Throughout his life he attacked difficult problems, many times with remarkable success. Unfortunately, he did not publish his results, which were known only through correspondence. Had Fermat published, his contributions would have been more influential during his lifetime.

Pierre de Fermat's disagreements with Descartes over principles of analytic geometry hurt Fermat's reputation as a mathematician, but it was Descartes who admitted in the end that Fermat's methods were correct.

Fermat applied the methods of algebra to geometry using a coordinate system in his study of **loci**. Descartes's equivalent approach earned him, not Fermat, credit for founding analytic geometry because Fermat's work was not published until after his death.

Blaise Pascal contacted Fermat in 1654, and through the ensuing correspondence in which they tried to mathematically predict the numbers that dice would show in gambling, they co-founded probability theory. In developing a method of finding tangents to curves and determining the area

bounded by curves, Fermat laid the groundwork for both differential calculus and integral calculus.

prime number a number that has no factors other than itself and 1

integral numbers integers, that is, positive whole numbers, their negative counterparts, or zero

Fermat developed numerous theorems involving **prime numbers** and **integral numbers** and, consequently, is regarded as the founder of modern number theory. The best known of these theorems, Fermat's Last Theorem, was not proved until 1994. SEE ALSO FERMAT'S LAST THEOREM; PROBABILITY, THEORETICAL.

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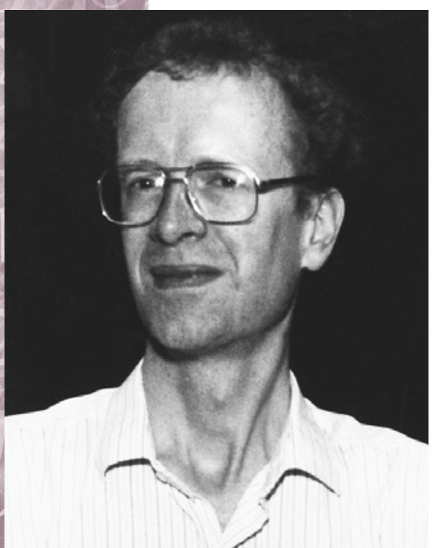
Fermat's Last Theorem



Andrew Wiles worked secretly and mostly alone to solve Fermat's Last Theorem. Despite his proof offered in 1994, no one has yet found the proof that Pierre de Fermat claimed to possess because no one has found a proof that used the mathematics and mathematical tools available to Fermat in the seventeenth century.

The proof of Fermat's Last Theorem involves two people separated by over 350 years. The first is the French lawyer and mathematician Pierre de Fermat, who, in about 1637, left a note written in the margin of a book. His note said that the equation $a^n + b^n = c^n$ has no solutions when a, b, and c are whole numbers and n is a whole number greater than 2. The note went on to say that he had marvelous proof of this statement, but the book margin was too narrow for him to write out his proof.

In the twentieth century, a 10-year-old British boy named Andrew Wiles read about this problem and was intrigued and challenged by it. No wonder: Fermat's Last Theorem has been called the world's greatest and hardest mathematical problem. Wiles's childhood dream became "to solve it myself . . . such a challenge . . . such a beautiful problem."

The Challenge

Before finding out how Wiles proved Fermat' Last Theorem, consider the equation $a^n + b^n = c^n$. Working with the Pythagorean theorem and right triangles reveals that, in every right triangle, $a^2 + b^2 = c^2$. There are also certain whole number values for a, b, and c that are called Pythagorean triples, such as 3, 4, and 5; 5, 12, and 13; 27, 36, and 45; or 9, 40, and 41. Many such triples can be found, and there are formulas that can be used to grind them out endlessly.

Notice, however, that the exponent (n) for these Pythagorean triples is 2. Fermat's Last Theorem says that such triples cannot be found for any whole number greater than 2. By the 1980s, mathematicians using computers had proven that Fermat's Last Theorem was correct for all the whole number values of n less than 600, but that is not the same as a general proof that the statement must always be true. Fermat's statement—that the equation $a^n + b^n = c^n$ has no solutions when a, b, and c are whole numbers and n is a whole number greater than 2—may be fairly simple to state, but it has not been so simple to prove.

The Solution

After 7 years of working on the problem in complete secrecy, Andrew Wiles used modern mathematics and modern methods to prove Fermat's Last The-

orem. The modern mathematics and methods he used did not, in general, entail making long and elaborate calculations with the aid of computers. Instead, Wiles, while a researcher at Princeton, used modern **number theory** and thousands of hours of writing by hand on a chalkboard as he thought, tried, and failed, and thought some more about how to prove Fermat's Last Theorem.

As Wiles worked on finding a proof, he looked for patterns; he tried to fit in his ideas with previous broad conceptual ideas of mathematics; he modified existing work; he looked for new strategies. He used the work of many mathematicians, reading their papers to see if they contained ideas or methods he could use. It took him 3 years to accomplish the first step. In spring of 1993, he felt that he was nearly there, and in May of 1993, he believed he had solved the problem.

Wiles asked a friend to review his work, and, in September of 1993, a fundamental error was found in his proof. He worked until the end of November 1993 trying to correct this error, but finally he announced that there was a problem with part of the argument in the proof. Wiles worked almost another whole year trying to correct this flaw. After months of failure while working alone, he was close to admitting defeat. He finally asked for help from a former student, and together they worked for 6 months to review all the steps in the proof without finding a way to correct the flaw.

In September of 1994, a year after the error was originally found, Wiles went back one more time to look at what was not working. On Monday morning, September 19, 1994, Andrew Wiles saw how to correct the error and complete his proof of Fermat's Last Theorem. He called the insight that completed this proof "indescribably beautiful—so simple and elegant."

The solution of Fermat's Last Theorem involves the very advanced mathematical concepts of elliptic curves and modular forms. Wiles's subsequent paper on the proof, "Modular Elliptic Curves and Fermat's Last Theorem," was published in 1995 in the *Annals of Mathematics*. SEE ALSO Fermat, Pierre de.

Lucia McKay

WHY IS IT CALLED THE "LAST" THEOREM?

Fermat's Last Theorem was not called so because it was his last work. (He apparently wrote the marginal note about this theorem in 1637, and he died in 1665.) Rather, this statement came to be called Fermat's Last Theorem because it was the last remaining statement from Fermat's mathematical work that had not yet been proved.

number theory the study of the properties of the natural numbers, including prime numbers, the number theorem, and Fermat's Last Theorem

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Fibonacci, Leonardo Pisano

Italian Number Theorist
1175–1240

Leonardo Pisano Fibonacci (c. 1175–c. 1240) is considered by many to be the greatest number theorist of the Middle Ages. The following sequence

Leonardo Pisano Fibonacci is credited with introducing the Hindu-Arabic numbering system to Western Europe, but his fame is more often associated with the development of the Fibonacci sequence of numbers.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,
233, 377, 610, 987, 1597, 2584, 4181, . . .

defined by $F_1 = 1$, $F_2 = 1$, and for $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$ is called the Fibonacci sequence in his honor. The Fibonacci sequence evolved from the following problem in Fibonacci's book *Liber abbaci.*

> A certain man put a pair of rabbits in a place surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

The answer is $F_{12} = 377$.

It is worth noting that Fibonacci did not name the Fibonacci sequence; the sequence was given the name by the nineteenth-century French mathematician, Edouard Lucas. Lucas also found many important applications of the Fibonacci sequence.✶

✶**Today, the Fibonacci Association publishes a journal, *The Fibonacci Quarterly,* whose primary focus is to promote interest in Fibonacci and related numbers.**

Fibonacci made many other contributions to mathematics. He is credited with introducing the Hindu-Arabic numerals to Europe. This is the positional number system based on the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and a decimal point.

In *Liber quadratorum* ("The Book of Squares"), Fibonacci described a method to construct **Pythagorean triples**. If he wanted to find two **squares** whose sum was a square, he took any odd square as one of the two squares. He then found the other square by adding all the odd numbers from 1 up to but not including the odd square. For example, if he took 9 as one of the two squares, the other square is obtained by adding all the odd numbers up to 9—that is, 1, 3, 5, and 7, whose sum is 16, a square. And 9 + 16 = 25, another square. Also, in this book Fibonacci proved that there are no positive integers m and n, such that $m^2 + n^2$ and $m^2 - n^2$ are both squares.

Pythagorean triple any set of three numbers obeying the Pythogorean relation such that the square of one is equal to the sum of the squares of the other two

square a quadrilateral with four equal sides and four right angles

Curtis Cooper

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Field Properties

David Hilbert, a famous German mathematician (1862–1943), called mathematics the rules of a game played with meaningless marks on paper. In defining the rules of the game called mathematics, mathematicians have organized numbers into various sets, or structures, in which all the numbers satisfy a particular group of rules. Mathematicians call any set of numbers that satisfies the following properties a *field*: closure, commutativity, associativity, distributivity, identity elements, and inverses.

Determining a Field

Consider the set of non-negative even numbers: {0, 2, 4, 6, 8, 10, 12,. . .}. To determine whether this set is a field, test to see if it satisfies each of the six field properties.

Closure. When any two numbers from this set are added, is the result always a number from this set? Yes, adding two non-negative even numbers will always result in a non-negative even number. The set of non-negative even numbers is therefore *closed under addition*.

Is the set of even non-negative numbers also closed under multiplication? Yes, multiplying two non-negative even numbers will also always result in a non-negative even number. The *closure property* applies to the set of non-negative even numbers under the two operations of addition and multiplication.

Commutativity. Notice also that, with any two numbers from this set (a, b), $a + b = b + a$ and $ab = ba$. Therefore, the *commutative property for addition and for multiplication* applies also.

Associativity. It is also true that $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$. The *associative property for addition and for multiplication* thus applies for the set of non-negative even numbers.

Distributivity. If the *distributive property* applies to the set of non-negative even numbers, $a(b + c) = ab + ac$. Since this is true for any non-negative even numbers, the set does satisfy this property.

Identity Elements. Within this set of non-negative even numbers, is there an identity element for addition? That is, is there a number n such that adding that number leaves a non-negative even number unchanged in value? Does the set contain an n such that $a + n = a$? Yes, n can be 0, so 0 is the identity element for addition in this set.

Is there a corresponding identity element for multiplication in this set? No. Here the set of non-negative even numbers fails the test. There is no number p in this set such that $ap = a$. The number p could be 1 because 1 is an identity element for multiplication, but 1 is not in the set of non-negative even numbers.

Because the *identity property* is not satisfied by the set of non-negative even numbers, the set does *not* form a field.

Inverses. The set of non-negative even numbers also does not satisfy the sixth property for a field. This set does not contain additive and multiplicative inverses for each number in the set. An additive inverse for 2 might be -2, since $2 + (-2) = 0$, but -2 is not in this set. A multiplicative inverse for 2 might be $\frac{1}{2}$, since $2\ (\frac{1}{2}) = 1$, but $\frac{1}{2}$ is also not in this set.

Numbers Sets that Are Fields

Are there sets of numbers that are fields—that is, that satisfy all six of the field properties—closure, commutativity, associativity, distributivity, identity elements, and inverses? If the set of non-negative even integers is expanded to include the negative integers (to supply the additive inverses), all the integers (so that 1 is the multiplicative identity), and all the rational numbers (such as $\frac{1}{2}$, to supply all the multiplicative inverses, or reciprocals), then the result is the set of all **rational numbers**.

rational numbers numbers that can be written as the ratio of two integers in the form a/b where a and b are integers and b is not zero

The set of rational numbers is a field because it satisfies all six properties. This set is closed because adding or multiplying any two rational numbers results in a rational number. It is commutative, associative, and distributive. It contains an additive identity, 0, and a multiplicative identity, 1. Every number in the set (except 0) has an additive inverse and a multiplicative inverse in the set.

Notice that the rules for a field do not require that 0 have a reciprocal; division by 0 is undefined.

Another set of numbers that form a field, because they satisfy all six of the field properties, is the set of all numbers on the real number line. This set of all real numbers is formed by joining the rational numbers to all the irrational numbers. Recall that an irrational number cannot be expressed as the ratio of two integers.

A third set of numbers that forms a field is the set of complex numbers. Complex numbers are all the numbers that can be written in the form $a + bi$ where a and b are real numbers, and i is the square root of -1.

There are other sets of numbers that form a field. For example, consider this set of numbers: {0, 1, 2, 3}. The operation of addition is defined in the following way. Add the two numbers in the set and, if the result is 4 or more, subtract the number 4 until a number, called the sum, remains that is in the set. This method of arithmetic is called modular arithmetic (in this case, mod 4).

Thus, $2 + 3$, for example, yields 1 (since $5 - 4 = 1$); and $1 + 3$ yields 0 (since $4 - 4 = 0$). When addition is defined in this way (in this case, as mod 4), then this set is closed under addition. The identity element for addition is 4 because, for example, $2 + 4$ yields 2, so that adding 4 leaves a number from this set unchanged.

Notice that each number in the set (other than 0) does have a multiplicative inverse, since, using mod 4 arithmetic, $1 \times 1 = 1$, and $2 \times 3 = 2$, and $3 \times 1 = 3$. The set is also closed under multiplication, using mod 4 arithmetic. It also shows associativity, commutativity, and distributivity under these definitions of addition and multiplication. There are many other finite sets that are finite fields when this kind of modular arithmetic is used. SEE ALSO Integers; Numbers, Real.

Lucia McKay

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Financial Planner

Financial planners help individuals and businesses invest money. They provide financial advice based on their knowledge of tax and investment strategies, stocks and bonds, insurance, retirement plans, and real estate.

Financial planners interview their clients to help them define their financial needs and goals and to determine how much money clients have

available for investment. Whether the goal is long-term investing (for example, retirement) or short-term investing, financial planners use information provided by a client to develop a financial plan tailored to the client's needs. Additionally, a financial planner often sells stocks, bonds, mutual funds, and insurance.

Financial planners can work for credit unions, credit counseling companies, banks, and companies that specialize in offering financial advice. The vast majority of financial planners have bachelor's or master's degrees. A college degree in business administration or finance is useful.

Financial planners often use mathematics on the job. They calculate which proportion of a client's money may go into a particular investment. For example, does the client want half of his money to be invested in stocks and half in real estate?

Financial planners review stocks and bonds to determine which have the best profits. For example, a stock that is bought for $10 a share and becomes worth $15 a share shows a 50 percent return on the investment. They must calculate how much a purchase of multiple shares of a stock will cost. Four shares at $20 per share will cost $80.

For long-term investments, a financial planner may need to calculate compound interest. He or she must project and add all the investments over time to assess if enough money will be available for the client's retirement. SEE ALSO Economic Indicators; Stock Market.

Denise Prendergast

Financial planners often discuss investment strategies with clients over the telephone. Their pay, or commission, is often a percentage of the dollar amount of the financial product sold.

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Flight, Measurements of

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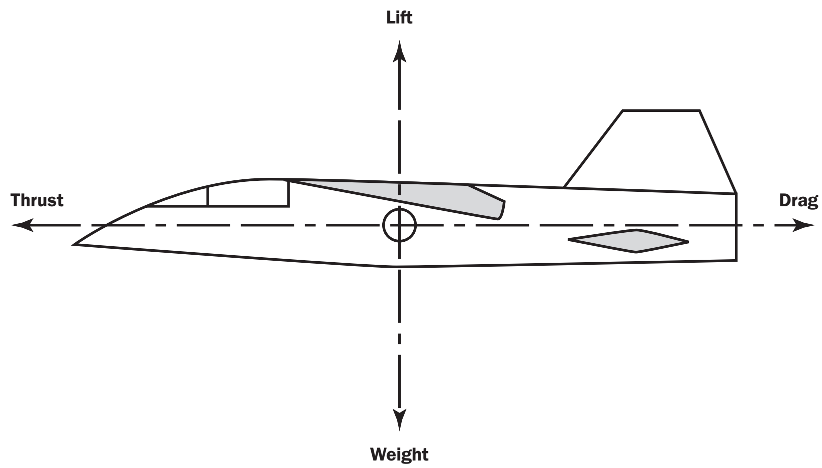
MATHEMATICS AND FLYING

High school students who plan to attend a college with a strong flight program should complete 4 years of high school mathematics, including trigonometry and pre-calculus. Some students will have taken calculus in high school, but students may take calculus in their first year of college. Good grades in trigonometry and precalculus, in which vectors are studied, meet the prerequisites for enrolling in aerodynamics and simultaneously beginning flight training. Soon after beginning, first-year college students in aviation programs become accustomed to the many measurements critical to flying and to using mathematics to make sense of the measurements.

Level Flight

On an aircraft, a propeller or jet engine provides thrust, which moves the plane forward through the air. Thrust acts parallel to the direction of flight. The weight of the aircraft is the force of gravity acting toward the center of Earth. The aerodynamic forces of lift and drag result from air pressure. Lift, the vertical component, acts to oppose gravity. Drag acts opposite to the flight path of the aircraft.



An aircraft in level flight must be in a condition of equilibrium, meaning the forces (vector quantities) acting upon the aircraft (thrust, drag, weight, and lift) are in balance. To be in balance, lift must equal weight and engine thrust (power) must equal drag.

During flight, pilots continuously scan the cockpit instrument panel. They pay particular attention to the instruments that display measurements of altitude, speed, attitude, and direction.

Altitude

Pilots are concerned with three types of altitude: true-altitude, pressure-altitude, and absolute-altitude. They read true-altitude (height above mean sea level) and pressure-altitude (height above a standard reference plane) from a pressure-altimeter, which is an instrument dependent upon air pressure. A pilot uses a radar-altimeter to read absolute altitude, which is the height above the terrain directly below the aircraft.

Prior to takeoff, pilots set local barometric pressure into the pressure-altimeter to monitor true-altitude during takeoff and climb. For flight below 18,000 feet, pilots continue to monitor true-altitude. Every 100 miles, they obtain local pressure from air controllers on the ground and update their altimeter. For flight above 18,000 feet pilots reset their altimeter to the standard pressure of 29.92 inches of mercury and monitor pressure-altitude. Consistent use of standard air-pressure at high altitudes by all aircraft permits maintenance of necessary vertical separation.

Altimeter readings do not provide pilots with information about how high they are above terrain features. Instead, pilots monitor their position using a variety of electronic aids, including the global positioning system (GPS). Below 5,000 feet they can monitor their ground height using a radar altimeter. For example, if the altimeter reads 3,000 feet and the aircraft is over a 2,000-foot plateau, the pilot can "ping" Earth with a radar signal. The radar altimeter translates the return signal to an altitude above ground level of 1,000 feet.

Airspeed

knot nautical mile per hour

The airspeed indicator converts airflow to an airspeed reading in **knots**. The airspeed indicator receives information about airflow from a device that is

mounted on the outside of the aircraft. Pilots or computers correct for the air density at altitude to determine true airspeed. As altitude increases, air density decreases. At 18,000 feet the air density is about one-half of sea-level density and creates considerably less impact on the aircraft.

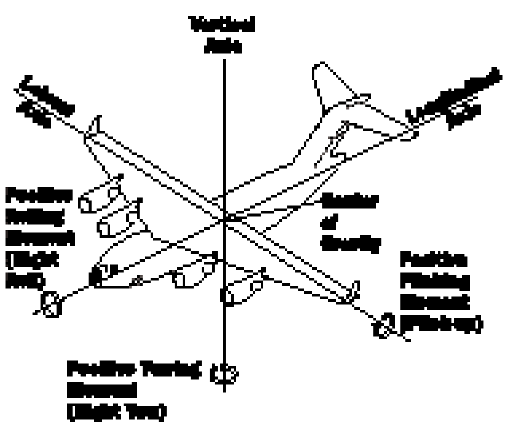
To determine true airspeed, pilots must make corrections for the airspeed indicator reading. If an airspeed indicator reads 100 knots in an aircraft that is traveling at an altitude of 5,000 feet with outside air temperature of 10 degrees Celsius, a pilot can consult a conversion chart that shows the indicated airspeed should be multiplied by 1.09 to obtain true airspeed. In this example, the computed true airspeed would be 109 knots using the conversion chart, or 110 knots using a rule of thumb. For quick mental estimations, pilots use a rule of thumb in which they add 2 percent of indicated airspeed for each 1,000 feet of altitude to get true airspeed. Pilots can compute groundspeed, critical for determining flight time and fuel needs, by adding **tailwind** or subtracting **headwind** from true airspeed.

tailwind a wind blowing in the same direction as that of the course of a vehicle

headwind a wind blowing in the opposite direction as that of the course of a vehicle

Attitude

Attitude refers to the position of an aircraft with reference to the horizon. Attitude measurements of pitch, roll, and yaw are measured on a right hand, three-dimensional axis system, with the origin representing the center of gravity for the aircraft. Attitude is measured in degrees from level flight.



An attitude indicator on the control panel provides pitch and roll information by displaying a symbol that represents aircraft wings on a "moveable horizon." The moveable horizon is an equator-like horizontal line drawn on the sphere in the horizon indicator. The sphere is attached to a spinning **gyroscope**, which remains in a vertical position relative to Earth. The hemispherical region above the horizon line represents the sky, and below represents the ground. A symbol of an aircraft is drawn on the clear circular case over the sphere. When the aircraft changes attitude, the aircraft symbol changes position relative to the horizon on the sphere. Pilots can determine the amount of pitch or roll by comparing the position of the wing of the aircraft symbol with the horizon line.

gyroscope a device typically consisting of a spinning wheel or disk, whose spin-axis turns between two low-friction supports; it maintains its angular orientation with respect to inertial conditions when not subjected to external forces

Pitch is rotation about the y-axis or lateral axis. A nose-up aircraft position results in positive degrees of pitch. To determine the pitch of the aircraft, pilots consult the position of the aircraft symbol with respect to the

horizon line on the attitude indicator. When an aircraft is climbing, the wings of the symbol in the display are positioned above the horizon line. A nose-down aircraft position results in negative degrees of pitch.

Roll is rotation about the positive *x*-axis or longitudinal axis. If a pilot causes the left wing to roll downward 30 degrees about the longitudinal axis, the aircraft is said to be in a 30 degree left bank.

Direction

Yaw is the term for rotation about the vertical or *z*-axis. An aircraft is said to yaw when it changes direction. Pilots fly a magnetic heading that they read from the heading indicator. This tool consists of a flat circular card that is mounted underneath a clear cover with an aircraft drawn on it. The circular card is marked with 360 degrees similar to a circular protractor, with 0 degrees representing north.

The heading indicator illustrated above shows the aircraft's heading in degrees from magnetic North. Here the aircraft is headed due North. Note that each number must be multiplied by 10 to yield the actual degree reading.

As an aircraft (and the aircraft symbol) turns and rotates, a gyroscope attached to the circular card keeps the card stabilized and fixed in space. The nose of the aircraft symbol continually points in the direction the aircraft is heading.

A gyroscope does not "know" the direction of magnetic north: therefore, pilots initially read a magnetic heading from a magnetic compass and set the heading indicator. Gyroscopic drift causes heading indicators to become inaccurate during flight. Therefore, heading indicators must be periodically corrected with information from a magnetic compass.

SCALARS AND VECTORS

Scalar and vector are two types of measurements important to the study of flight. Scalar quantities have size or magnitude only, whereas vector quantities consist of magnitude and direction. For example, if an aircraft travels 100 miles, the distance is a scalar. If an aircraft travels 100 miles to the east, the displacement is a vector.

Take-off

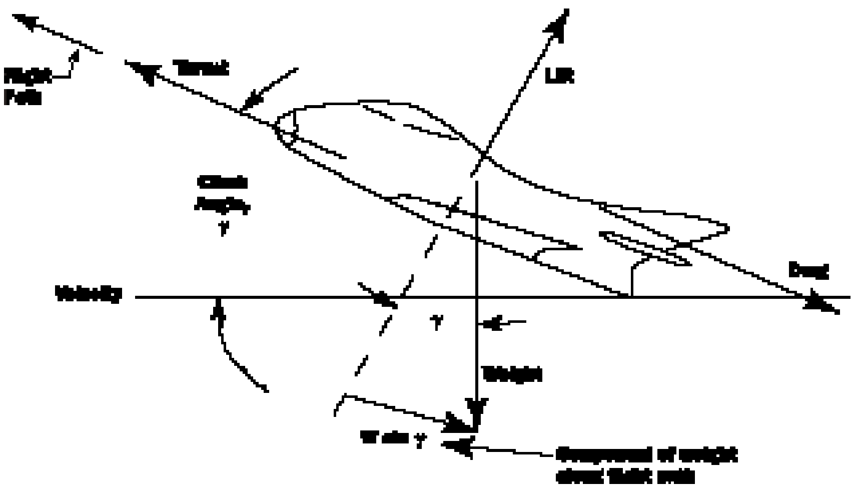
During take-off, engine thrust accelerates the aircraft to the critical velocity necessary for lift to overcome drag, rolling friction, and the weight of the plane. Pilots continually monitor airspeed during take-off, aware that once the aircraft exceeds what is termed "refusal speed" there is no longer enough runway to stop the aircraft and the pilot is committed to take-off.

Gross weight of an aircraft is a critical factor in take-off. Runway distance required for take-off varies with the square of the gross weight of the

aircraft. For example, consider a Boeing 707 taking off at sea level at standard temperature and barometric pressure. If the aircraft weighs 172,500 pounds without cargo or passengers, it requires a take-off speed of 112 knots and 1,944 feet of runway distance to take-off. If the aircraft were loaded with 100,000 pounds of cargo and passengers, the aircraft would then weigh 272,500 pounds, require a 156 knot take-off speed, and 5,500 feet of runway to take-off. The 58 percent increase in weight required a 39 percent increase in take-off speed and a 183 percent increase in take-off distance.

At take-off, the pilot guides the aircraft to the intended altitude. To maintain a steady **velocity** climb, forces must be in equilibrium. During the climb, the weight vector (W) is resolved into a vector perpendicular to the flight path ($W\cos\gamma$) and a vector parallel to the flight path ($W\sin\gamma$), where γ is the climb angle, and cos and sin are two trigonometric functions.

velocity distance traveled per unit of time in a specific direction



To balance the forces along the flight path, the thrust force must equal the drag force plus $W\sin\gamma$. A reasonable climb angle for a Boeing 707 aircraft is 6 degrees. A take-off weight of 172,500 pounds and a 6-degree angle of climb requires approximately 78,100 pounds of jet engine thrust. The four engines on a Boeing 707 each can supply 22,000 pounds of thrust. The take-off angle is the result of optimizing the pounds of fuel needed for climb and the time and ground distance traveled before the aircraft reaches its desired flight altitude. During take-off the pilot monitors the vertical velocity indicator to assure that the aircraft maintains the climb angle. Ascending at 4,325 feet per minute results in an approximate climb angle of 6 degrees.

Landing

When pilots travel, Federal Aviation Agency (FAA) air traffic controllers monitor their flight. Pilots fly between points on airways, which can be thought of as highways in the sky. When an aircraft comes within a specified radius of the airport at which the pilot plans to land, FAA controllers pass the aircraft to an approach controller. Approach controllers "vector the aircraft" for approach and landing. This means they give pilots an airspeed and direction to fly toward the glide slope. The glide slope is an angle, with direction, on which the aircraft descends for its landing. It is generally about 3 degrees.

In a ground directed approach, approach controllers direct pilots with heading corrections and rates of descent in feet per minute so the aircraft will stay on the glide slope. In an instrument landing approach, pilots monitor direction, attitude, and vertical-velocity indicators to maintain the aircraft on the glide slope until touch down. SEE ALSO Angles, Measurement of; Global Positioning System; Magnetic and Geographic Poles; Navigation; Vectors.

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Form and Value

Two mathematical expressions may take different form and yet have the same value. For example, $\sqrt{4}$ and 2 look different but are the same number. Likewise, the number 1 can be expressed as 3^0 or $\frac{5}{5}$.

In the decimal system, for example, the fraction $\frac{1}{2}$ is also expressed as 0.5. The word "fraction" refers to the form of a number. The form of $\frac{8}{2}$ is a fraction, but its value is 4, which is an integer. Similarly, $2\frac{1}{2}$ has the form of a mixed fraction, but it can be expressed as the fraction $\frac{5}{2}$, or the decimal number 2.5.

monomial an expression with one term

binomial an expression with two terms

Algebraic expressions also have form and value. The value of $3(x + 2)$ and $3x + 6$ is the same, but $3(x + 2)$ is in the **monomial** form, and $3x + 6$ is in the **binomial** form. Numbers can be also written using both monomial and binomial form.

In monomial form, even numbers are expressed as $2x$ for $x = 1, 2, 3,\ldots$ generating all even numbers 2, 4, 6,. . . . In binomial form, even numbers are expressed as $2x + 2$ for $x = 0, 1, 2,\ldots$, which again generates all the even numbers 2, 4, 6,. . . .

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Fractals

The term "fractal" was coined by Benoit Mandelbrot to describe a "self-similar" geometrical object that looks much the same on many different scales of measurement. This property contrasts with the property of a circle, for example, which loses its structure when viewed on a different scale and becomes almost a straight line when any arc is greatly magnified.

Fractals are representations of objects with an **infinite** amount of detail. When magnified, fractals do not become simpler, but instead remain as complex as they were without magnification. This is why fractals seem to describe natural objects in a better way than simple geometric figures like triangles, rectangles, or circles.

infinite a quantity beyond measure; an unbounded quantity

A coastline is a classical example of self-similarity in nature. From the air, a sea coast looks irregular by virtue of its bays and headlands. A closer look will reveal the same structure yet on a different scale. Each bay has its own bays and headlands. An even closer look will show even more bays and promontories within the larger bays. Even a beach will have small bays, capes, and peninsulas. On a much smaller scale in nature, a microscope will reveal a self-similar structure even within a grain of sand, which will have indentations and extrusions.

Constructing Geometric Fractals

Any mathematically created fractal can be made by the iteration, or repetition, of a certain rule. There are three basic types of iteration:

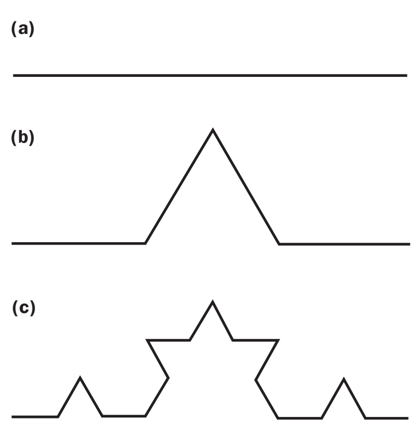
- generator iteration, which is repeatedly substituting certain geometric shapes with other shapes;
- IFS (Iterated Function System) iteration, which is repeatedly applying geometric **transformations** (such as rotation and reflection) to points; and
- formula iteration, which is repeating a certain mathematical formula or several formulas.

transformation changing one mathematical expression into another by translation, mapping, or rotation according to some mathematical rule

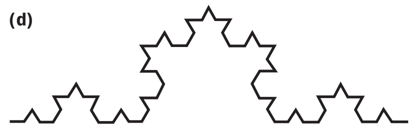
The property of self-similarity holds true for the majority of mathematically created fractals.

The figure below illustrates the geometric construction of the Koch Curve, named after Helge von Koch, a Swedish mathematician who introduced this curve in a 1904 paper. First, begin with a straight line, as shown in (a). This initial object can be called the initiator. Partition this into three equal parts, then replace the middle third by an **equilateral** triangle and take away its base, as shown in (b). These steps are repeated with each resulting segment, as shown in (c). The repetition of steps is known as iteration. The curve shown in (d) is the result after three iterations, and the curve in (e) is after four iterations. The actual Koch Curve cannot be shown because it is theoretical, resulting from an infinite number of iterations.

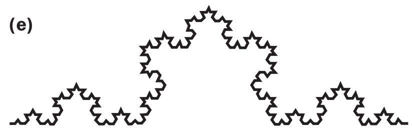
equilateral having the property that all sides are equal; a square is an equilateral rectangle



(d)

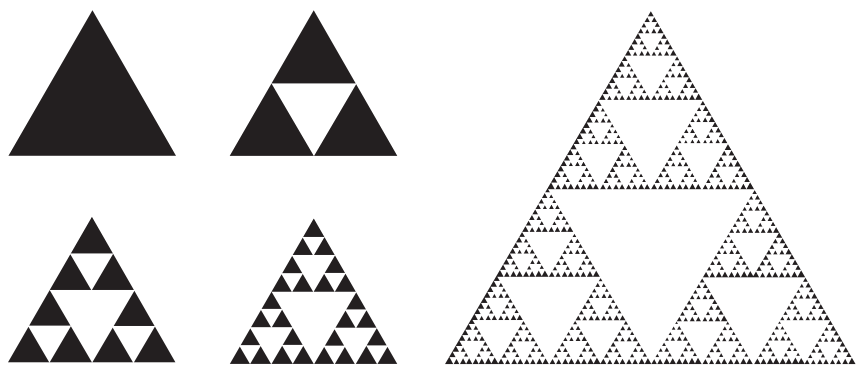


(e)



✶The Sierpinski Gasket is named after Polish mathematician Waclaw Sierpinski (1882–1969).

Other geometric fractals can be created using the same method. Using a triangle as the initiator, the Sierpinski Gasket✶ is constructed as shown below. With each iteration, the figure becomes more complex as scaled copies build upon identical scaled copies, as shown by the small triangles on the left. The large image on the far right shows the results after six iterations.



Characteristics of Fractals

finite having definite and definable limits; countable

When showing images of fractal figures, approximations given by a **finite** number of steps are displayed because these approximations—as in the case of the Koch Curve—will yield a curve with finite length. The actual fractal curve will have infinite length.

perimeter the distance around an area

Because actual fractal figures like the Koch Curve have infinite length, they have interesting properties uncommon in simpler geometric figures. For example, a fractal closed curve such as the Koch Snowflake can enclose a figure with infinite **perimeter** and finite area. Although not shown here, the Koch Snowflake is constructed from an equilateral triangle, using a Koch Curve to initiate each of its sides. The snowflake has an infinite perimeter because the geometric pattern in (d) of the previous unnumbered figure comprises the snowflake's outer border and can be repeated an infinite number of times at increasingly smaller scales.

Fractal Dimensions. Because of their complexity, fractal objects cannot be assigned a dimension as can a line or a square. For example, the Koch Curve cannot have dimension 1, as a line, nor can it have dimension 2, as a square. So there must be other ways of calculating its fractal dimension.

The calculation of fractal dimensions is related to (1) the number of pieces into which a structure can be divided and (2) the reduction factor. The Koch Curve, in general, has 4^k pieces with a reduction factor of $\frac{1}{3^k}$. So

when it has four pieces, the reduction factor is $\frac{1}{3}$, and when it has sixteen pieces the reduction factor is $\frac{1}{9}$. Similarly, the Sierpinski Gasket, in general, has 3^k pieces with a reduction factor $\frac{1}{2^k}$. So with three pieces the reduction factor is $\frac{1}{2}$, and with nine pieces the reduction factor is $\frac{1}{4}$.

This idea was used in 1919 by the German mathematician Felix Hausdorff to define a fractal dimension that agrees with the usual dimension on the usual spaces. Although it is too complicated to be presented here, it is interesting to know that the dimension of the Koch Curve is approximately 1.2619 (or $\frac{\log 4}{\log 3}$) and the Sierpinski Gasket has a dimension close to 1.585 (or $\frac{\log 3}{\log 2}$).

For shapes that are not as regular as the Koch Curve or the Sierpinski Gasket, such as clouds or coastlines, this method of determining the fractal dimension does not work. Fractals that are not composed of a certain number of identical versions of itself require other methods for determining the fractal dimension.

Julia and Mandelbrot Sets

Complex numbers are numbers of the form $a + bi$, where $i = \sqrt{-1}$. By representing the complex number $a + bi$ with the point (a, b) in the **Cartesian plane**, a graphical representation of the complex numbers known as the **complex plane** is obtained. Complex numbers can be added, multiplied, and divided, just as **real numbers**. However, it is important to bear in mind that $i^2 = -1$. So functions can be defined using complex numbers as input, and the output of these functions will be, in general, complex numbers.

Cartesian plane a mathematical plane defined by the *x* and *y* axes or the ordinate and abscissa in a Cartesian coordinate system

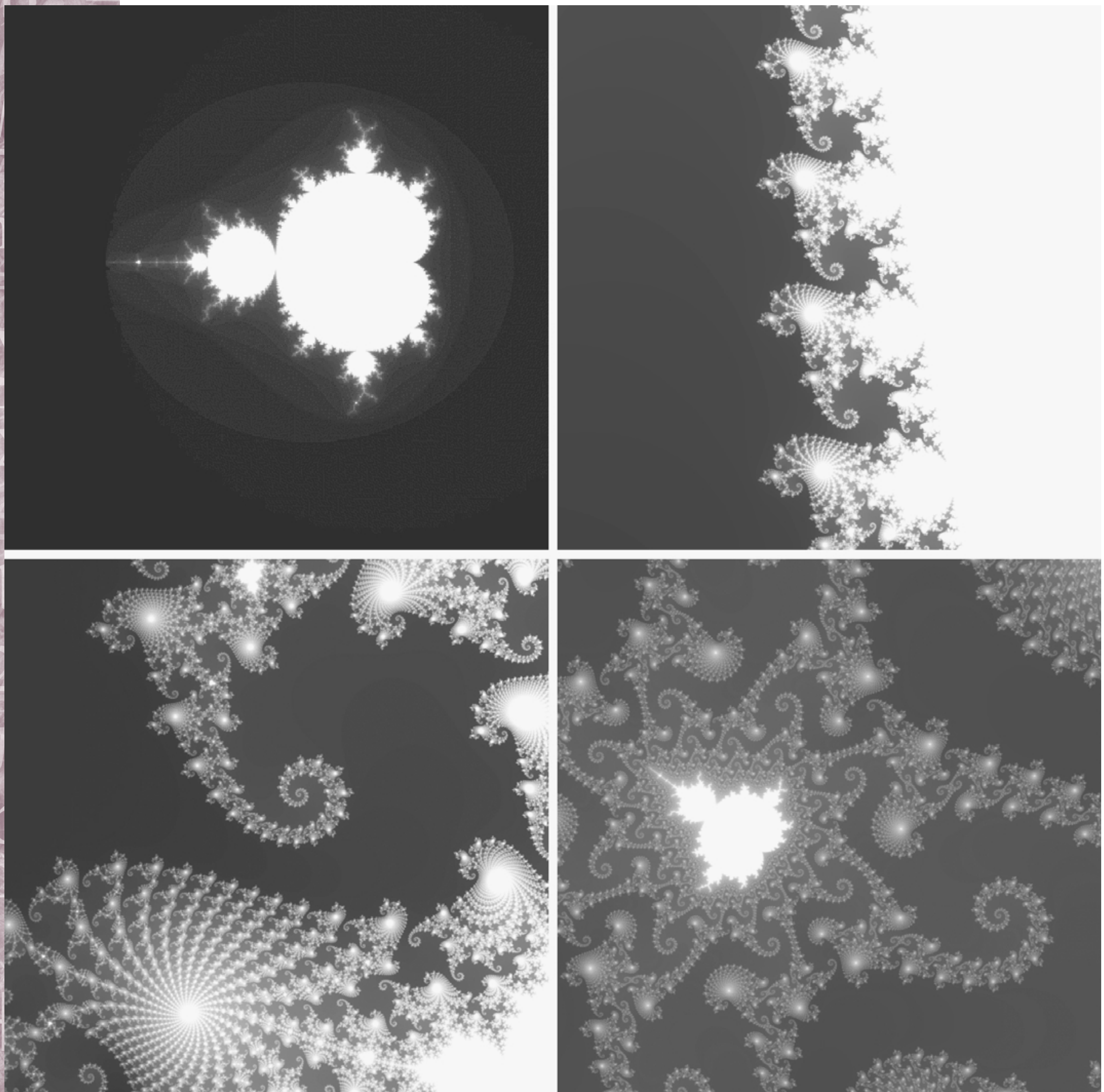
complex plane the mathematical abstraction on which complex numbers can be graphed; the *x*-axis is the real component and the *y*-axis is the imaginary component

real number a number that has no imaginary part; member of a set composed of all the rational and irrational numbers

Gaston Julia (1893–1978) investigated what happens when functions in the complex plane are iterated. Consider, for example, the function $f(z) = z^2 + c$, where c is a complex number. For real numbers, it is not difficult to evaluate this function. If $c = 1 + i$, and one wants to evaluate the function for $z = 2$, then $f(2) = 2^2 + 1 + i = 4 + 1 + i = 5 + i$. Squaring complex numbers is just a little bit more difficult, but it is enough to realize that when a function like this takes a complex number as input, it yields another complex number as output. If this function is iterated (that is, if the output becomes the input), and the function is evaluated again and again, one of two things can happen. Either the output numbers will begin to grow and to go farther from the origin, or somehow they will stay close to the origin, even if the function is iterated many times.

For example, select $c = -0.125 + 0.75i$, and evaluate f for $z = 0$. Evaluating the function again using as the input the output of $f(0)$, and continuing this repetition of using each output value as the next input value yields a sequence of complex numbers different than the sequence of complex numbers that would result from evaluating the same function with an initial value of $z = -0.5 + 0.5i$. The difference is that for the initial value $z = 0$, the resulting sequence of complex numbers remains bounded; that is, the sequence remains close to the origin. On the other hand, the sequence given by $z = -0.5 + 0.5i$ quickly goes far away from the origin.

The collection of complex numbers, represented as points on the complex plane, that lead to sequences that stay always close to the origin is called the prisoner set for c, whereas the collection of points that lead to unbounded sequences is called the escape set for c. The Julia Set is the boundary between the two sets.

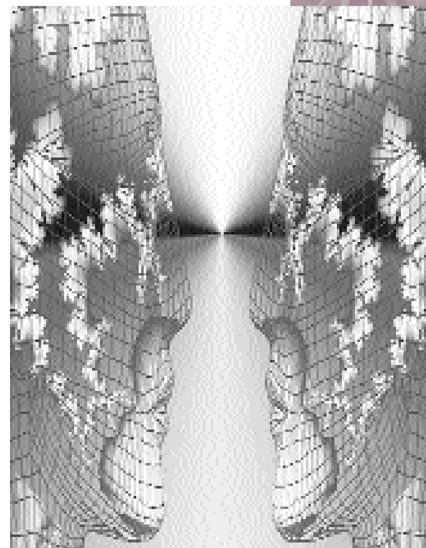


The elegance of the Mandelbrot Set (top left) becomes evident in three magnifications, starting at a very small portion of the set's border in the center of the first frame. Between the top right and lower left frames, the magnification increases by a factor of 5. The lower right frame is a much greater magnification of the edge of the spiral in the middle of the previous frame. Note the miniature Mandelbrot Set (in white) embedded in this final magnification.

Although not shown here, the prisoner set for $c = -0.125 + 0.75i$ and its bordering Julia Set is considered connected because it appears in one piece. On the other hand, the Julia Set for $c = -0.75 + 0.125i$ is disconnected because it consists of pieces that are separated from each other. If all those values c in the complex plane that have connected Julia Sets are colored black, the result is known as the Mandelbrot Set, named in honor of

Benoit Mandelbrot. It is not surprising that this set has a complexity that placed it beyond the reach of mathematicians until computers were used to study it. Mandelbrot studied Julia's work extensively and used computer graphics to render the Julia Sets and the Mandelbrot Set.

Self-similarity in the Mandelbrot Set is of a different nature than in the Koch Curve and Sierpinski Gasket because it arises from iterations of quadratic functions rather than from generator iterations or IFS iterations, as described above. In the Mandelbrot Set, identical pictures cannot be seen right away. But as the four-frame image shows, under increasing magnifications, the borders will reveal hidden complexities and even tiny copies of the Mandelbrot Set.



This image entitled "Futuristic Heads" illustrates the creative use of fractals. The irregular patterns on the surface of each head would reveal tiny details and complexities if magnified.

Fractals in Science and Art

Before Mandelbrot, none of the mathematical pioneers thought that their theoretical speculations about iterative processes and their relation to extremely unusual sets would end up being the best tools to describe nature. And yet fractals have proven to be a rich subject of study. They have been used to describe nature and are used frequently by scientists of different disciplines to explore very diverse phenomena. Fractal structures can be found in the leaves of a tree, in the course of a river, in the shape of a broccoli, in our arterial system, and on the surface of a virus.

The earliest applications of fractals, and perhaps the most widely seen by nonscientists, occur in the arts and in the film industry, where **fractal forgery** has been used to create landscapes for science fiction movies. Using fractals, convincing simulations of clouds, mountains, and surfaces of alien worlds have been created for our amusement.

fractal forgery creating a natural landscape created by using fractals to simulate trees, mountains, clouds, or other features

In the 1970s, a young scientist, Loren Carpenter, made a computer movie of a flight over a fractal landscape. This brought him to the attention of Lucasfilm Ltd, whose graphic division, Pixar, immediately hired him. His work with fractals was used to create the geography of the moons of Endor and the outline of the Death Star in the movie *Return of the Jedi.* Fractals were also used to generate the landscape of the Genesis planet in the movie *Star Trek II: Wrath of Khan.* Carpenter has received awards for his contributions to the film industry, and his work in these two movies triggered the extended use of fractals for special effects and to simulate landscapes and other irregular shapes in three-dimensional (3-D) computer games.

The study of fractals is still a young branch of mathematics, and more applications are yet to be revealed. SEE ALSO MANDELBROT, BENOIT B.; NUMBERS, COMPLEX.

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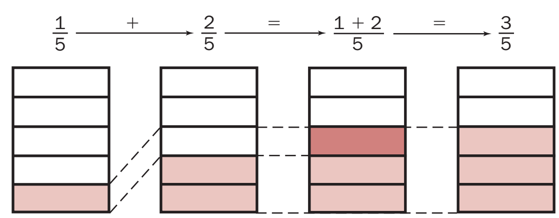
Fraction Operations

A fraction compares two numbers by division. To conduct basic operations, keep in mind that any number except 0 divided by itself is 1, and 1 times any number is itself. That is, $\frac{5}{5} = 1$, and $1 \times 5 = 5$. Thus, any number divided or multiplied by a fraction equal to one will be itself. For example, $5 \times \frac{3}{3} = 5$ and $5 \div \frac{481}{481} = 5$.

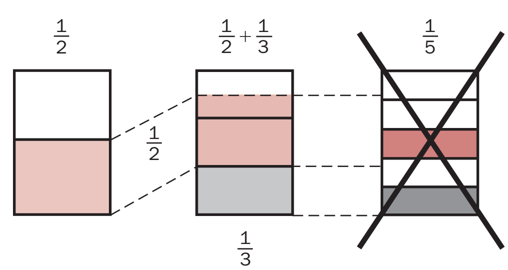
When multiplying fractions, the numerators (top numbers) are multiplied together and the denominators (bottom numbers) are multiplied together. So $\frac{5}{7} \times \frac{8}{9} = \frac{(5 \times 8)}{(7 \times 9)} = \frac{40}{63}$. And $\frac{4}{5} \times 6 = \frac{4}{5} \times \frac{6}{1} = \frac{24}{5}$.

To divide fractions, rewrite the problem as multiplying by the reciprocal (multiplicative inverse) of the divisor. So $\frac{5}{7} \div \frac{8}{9} = \frac{5}{7} \times \frac{9}{8} = \frac{45}{56}$.

To add fractions that have the same, or a common, denominator, simply add the numerators, and use the common denominator. The figure below illustrates why this is true.



However, fractions cannot be added until they are written with a common denominator. The figure below shows why adding fractions with different denominators is incorrect.



To correctly add $\frac{1}{2}$ and $\frac{1}{3}$, a common denominator must first be found. Usually, the least common multiple of the denominators (also called the least common denominator) is the best choice for the common denominator. In the example below, the least common multiple of the two denomi-

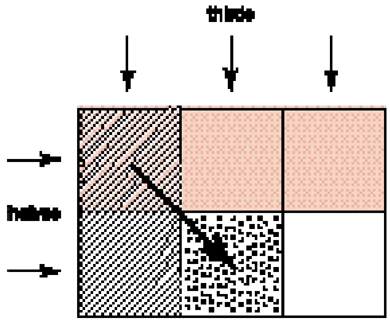
nators—2 and 3—is 6, so the least common denominator is 6. To convert the fractions, multiply $\frac{1}{2}$ by $\frac{3}{3}$ (which is equivalent to 1) to get $\frac{3}{6}$. Similarly, multiply $\frac{1}{3}$ by $\frac{2}{2}$ (which is equivalent to 1) to get $\frac{2}{6}$.

First $\frac{1}{2} \times \frac{3}{3} = \frac{3}{6}$

And $\frac{1}{3} \times \frac{2}{2} = \frac{2}{6}$

So $\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$, or $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

To model this problem visually, divide a rectangle into halves horizontally, then into thirds vertically, creating six equal parts (see the figure below). Shade one-half in color to show $\frac{1}{2}$, and then shade one-third in gray to show $\frac{1}{3}$. Since, as the figure shows, the upper left square has been shaded twice, it must be "carried" (see arrow). Now five of six squares are shaded; therefore, $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.



Subtraction of fractions is similar to addition, in that the fractions being subtracted must have a common denominator. So $\frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$. SEE ALSO Fractions.

Stanislaus Noel Ting

Fractions

Fractions are usually thought of as $\frac{1}{2}$, $\frac{3}{4}$, or maybe the decimal fraction 0.5. Perhaps some think of a fraction as a part of a circle or 5 parts out of 10 parts. The word "fraction" itself is related to a word meaning broken, as in "fracture."

Historically, fractions have been thought of and written in many different ways. Apparently the Babylonians, about 4,000 years ago, wrote fractions in a way similar to our decimal fractions, but instead of our **base-10**, they used **base-60**. They also used a space instead of a decimal point. This relatively simple system, however, does not reappear in common usage until about 3,600 years later in Europe.

base-10 a number system in which each place represents a power of 10 larger than the place to its right

base-60 a number system used by ancient Mesopotamian cultures for some calculations in which each place represents a power of 60 larger than the place to its right

Understanding Fractions

One mathematics dictionary defines a fraction as "a number less than 1." But this definition is too simple. The number $\frac{5}{3}$ is certainly not "less than one," but most people would still call it a fraction. Indeed, this same dic-

> **WHOLES AND PARTS**
>
> The Greek mathematicians (around 600 B.C.E. to 600 C.E.) had some problem working with fractions because the number 1 had a mystical significance as an indivisible unity. They did not want to break unity into parts, so they used ratios instead.

tionary calls $\frac{5}{3}$ an improper fraction or "a fraction whose numerator is larger than the denominator."

Another book defines a fraction as "a numeral representing some part of a whole." But what whole is $\frac{5}{3}$ representing part of, according to this definition? If a circle or a rectangle is divided into three equal parts, the denominator, or bottom, of the fraction is 3. But how many of those parts, or thirds, are represented by the fraction $\frac{5}{3}$? The numerator, or top, of the fraction tells you that $\frac{5}{3}$ represents five of the parts, each of which is one-third of a whole.

Another way to define a fraction is as a number that can be expressed in the form $\frac{a}{b}$, where a and b are **whole numbers** and b is not equal to 0. But this definition, taken from another mathematics dictionary, has problems too. The whole numbers are 0, 1, 2, 3, 4, 5. . .. So does this definition mean that $-\frac{2}{3}$ is not a fraction? Although -2 is not a whole number, clearly $-\frac{2}{3}$ is a fraction.

whole numbers the positive integers and zero

A better definition—and one that holds up under scrutiny—is that a fraction is a numeral written in the form $\frac{a}{b}$ where a can represent any number and b can represent any number except 0. By this definition, $\frac{4.5}{-6.2}$ is a fraction, as are $\frac{(2.5)}{7}$, $\frac{100}{3}$, and $\frac{6}{2}$, as well as $\frac{\pi}{2}$. This definition names a fraction according to the form. This means that $\frac{100}{2}$ is a fraction, and 50 is not a fraction. The values of the two numerals are the same, but the form of the numerals is not.

However, it is useful to be able to talk about the numbers that can be expressed as the ratio, or quotient, of two **integers**, such as $\frac{2}{3}$, $\frac{6}{3}$, and $-\frac{8}{9}$. These numbers, which can be negative, are called **rational numbers**. All rational numbers can be written in the form of fractions; however, not all fractions are rational numbers. For example, the fraction $\frac{\pi}{2}$ is not a rational number because π is not an integer. SEE ALSO Decimals; Form and Value; Fraction Operations; Numbers, Irrational; Numbers, Rational; Ratio, Rate, and Proportion.

integer a positive whole number, its negative counterpart, or zero

rational number a number that can be written in the form a/b where a and b are integers and b is not 0

Lucia McKay

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Functions and Equations

In mathematics, function is a central idea. Imagine a machine that takes numbered balls from 1 through 26 and labels them with the English alphabet letters A through Z. This machine mimics a mathematical function. A function takes an object from one set $\boldsymbol{A}$ (the input) and maps it to an object in another set $\boldsymbol{B}$ (the output). In mathematics, $\boldsymbol{A}$ and $\boldsymbol{B}$ are usually sets of numbers. In symbols, this relationship is written as $f\colon \boldsymbol{A} \rightarrow \boldsymbol{B}$.

So, a function f is the name of a relationship between two sets. Functions are usually denoted by the letters f, g, or h. $\boldsymbol{A}$ is called the **domain** (input), and $\boldsymbol{B}$ is called the **range** (output). If the elements of the domain

domain the set of all values of a variable used in a function

range synonymous with range, the set of all values of a variable in a function mapped to the values in the domain of the independent variable

are denoted by x, and the elements of the range are denoted by y, then a function can also be written as $y = f(x)$. This is read as "y is a function of x." Notice that this notation does not mean that f is multiplied by x. Instead, the value of f depends on the value of x.

Examples of Functions

A simple example of a function is $y = f(x)$, where $f(x) = x + 2$. To each number x, add 2 to get y. When x is 3, y is 5, and when x is 4, y is 6. The value y of the function, $f(x)$, depends on the choice of x. The input, or x, is called the independent variable, and the output, or y, is called the dependent variable.

Another example is a relationship between the positive **integer** set (domain) and the even number set (range). To each positive integer n, the function $f(n)$ assigns a value of $2n$. In symbols, $f(n) = 2n$.

In a function, each element of the domain must map to exactly one element of the range. However the opposite is not true. For example, $f(x) = |x|$ is a function. Each value of $f(x)$ corresponds to two values of x.

Now consider a function g with the **real number set** as the domain set. To each number x, g assigns 3 times x. That is, $g(x) = 3x$.

Function Notation and Graphs

Functions are visualized geometrically by plotting their graphs on a **Cartesian plane**. You can plot a function by taking a few numbers from the domain sets and finding their functional values. For example, $g(x) = 3x$ would yield the points $(-1, 3)$, $(0, 0)$, and $(1, 3)$. These points can be connected by a straight line.

In functions such as $f(x) = 3x$, $g(x) = x + 2$, or $h(x) = (\frac{1}{2})x$, the **power** of the independent variable, x, is 1. Such functions are called **linear functions**. Plotting the graph of linear functions always produces straight lines. In contrast, consider the function $f(x) = x^2$; its graph is not a straight line but rather a **parabola**. SEE ALSO Mapping; Mathematical.

Rafiq Ladhani

integer a positive whole number, its negative counterpart, or zero

real number set the combined set of all rational and irrational numbers; the set of numbers representing all points on the number line

Cartesian plane a mathematical plane defined by the x and y axes or the ordinate and abscissa in a Cartesian coordinate system

power the number of times a number is to be multiplied by itself in an expression

linear function a function whose graph on the x-y plane is a straight line or a line segment

parabola a conic section; the locus of all points such that the distance from a fixed point called the focus is equal to the perpendicular distance from a line

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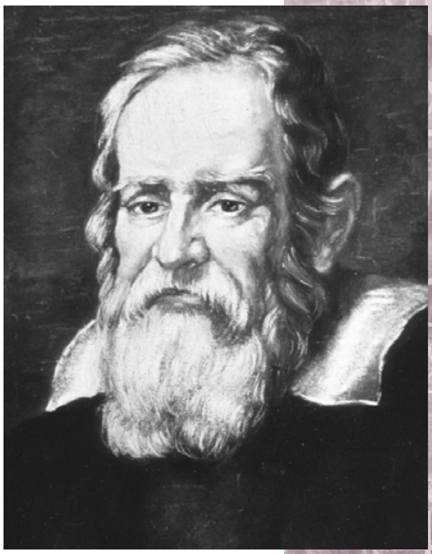
Galileo Galilei

Italian Astronomer, Physicist, and Mathematician
1564–1642

Galileo Galilei is a pivotal figure in intellectual and scientific history. His ideas and activities were integral to the Scientific Revolution, which resulted in world-changing advances in science and technology, and in fundamental changes in the way reality is perceived.

Galileo was born in Pisa, Italy, in 1564. In 1581, he entered the University of Pisa, where his father wanted him to study medicine. But Galileo was interested in mathematics and philosophy, and he left the university without a degree. In 1589, he taught mathematics at the university, but lost his job by challenging Aristotelian teachings held by the university and the Catholic Church. However, he immediately became professor of mathematics at the University of Padua.



Galileo's many important discoveries put him in direct opposition to the Catholic Church, the ruling body of the time. Only centuries later would Galileo be cleared of heresy.

Galileo was among the first to perceive that the natural world acts in a regular manner that can be interpreted and understood mathematically. Applying this approach, he developed the concept of acceleration and discovered the law of falling bodies, explaining the movement of projectiles, pendulums, and objects moving on an inclined plane.

Galileo accepted the Sun-centered model of the solar system that had been proposed by Copernicus. This model was in opposition to the Earth-centered model of Ptolemy that was accepted by scholars and the Catholic Church. Soon after the first telescope was invented, Galileo built his own version in 1609 and improved its magnification power. He was the first to use a telescope to study the heavens, obtaining, through these investigations, proof of the Copernican system. He discovered sunspots, valleys and mountains on the Moon, satellites circling Jupiter, and the phases of Venus.

In 1610, Galileo published his observations and interpretations in *The Starry Messenger*, refuting Aristotle and Ptolemy and supporting Copernicus. Opposition came immediately from scholars and churchmen, who accused him of heresy.

In 1612, Galileo published a book on hydrostatics based on observations, measurements, and mathematical analysis. He was again attacked by churchmen and university scholars for not adhering to the accepted Aristotelian approach. He openly argued that physical evidence and mathemat-

GALILEO'S BELIEF IN EMPIRICISM

Galileo believed that the development of new ideas and understanding was stifled by blind adherence to the authority of the Catholic Church and the writings of the ancient Greeks. Rather than simply accepting the statements of authorities, he believed investigators should rely on their own observations, measurements, and calculations.

ical proofs should not be made dependent on interpretations of scripture but that such interpretations should be subject to change when new evidence becomes available. Not surprisingly, the Catholic Church issued an edict in 1616 banning Copernicanism and censored Galileo's writings.

Undaunted, in 1632 Galileo published *Dialogue on the Two Chief World Systems* contrasting the planetary models of Ptolemy and Copernicus, with clear preference for Copernicus. He was called to Rome and tried for heresy. Convicted, he was forced to publicly retract his ideas and placed on permanent house arrest. His works were banned, but this order was essentially ignored outside of Italy. His ideas spread rapidly, gaining support throughout Europe.

Galileo continued his work, and his last book, *Discourses Concerning Two New Sciences*, was published in Leiden in 1638. This classic volume presented a mechanical mathematical physics that eventually led to the development of what would be called Newtonian physics. Galileo died in 1642, the year of Isaac Newton's birth. In 1992, the Catholic Church rescinded its 1633 conviction of Galileo as a heretic—350 years after his death. SEE ALSO NEWTON, SIR ISAAC; SOLAR SYSTEM GEOMETRY, HISTORY OF.

J. William Moncrief

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Games

Have you ever played Tic-Tac-Toe? Did you win? Did you know that each Tic-Tac-Toe game will always end in a tie unless one player makes a mistake?

There are nine squares on a Tic-Tac-Toe game board, so the first player has nine choices for the first move. The second player has eight choices for the second move (one square is taken), which makes a total of 72 possible arrangements after the first two turns. After five turns, there are 15,120 possible arrangements.

Someone could possibly win on the sixth turn if the other player played really badly, so the nine possible winning positions must be subtracted from the total, leaving 60,471 arrangements after only six moves. No one can keep track of that many combinations, but that is not necessary. The successful Tic-Tac-Toe strategy is simply to block the other player's moves while hoping the other player makes a mistake.

A good chess player must use a sequence of opening moves that will yield the best possible position. In chess, the player in control of the white pieces (White) always moves first. Since only the eight pawns and two knights can move on the first move, there are twelve possible first moves for White. (The two knights can each move to two different positions).

There are also twelve different opening moves for the player controlling the black pieces (Black), so after only two moves, there are 144 differ-

ent possible arrangements of pieces on the board. Each move opens up other pieces that can move, so after only four moves, there are about 70,000 different possible arrangements of pieces on the chessboard. Not every arrangement is of equal value, but with 70,000 different positions to consider, it is difficult for even good players to keep track of all possibilities. So good chess players remember patterns of pieces and learn to recognize certain patterns that give them an advantage over their opponents.

A great deal of mathematics is therefore involved in most games. Poker players must calculate the probabilities of certain card arrangements in order to win. (Never draw to an inside straight!) Bridge players must use probability to calculate the possible arrangements of cards in their opponents' hands in order to decide which strategy to use in playing winningly.

Card games such as bridge and poker use arithmetic and probabilities.

Bridge Strategies

Bridge players use mathematics in evaluating their hands. In one popular system, an ace is worth four points, a king is worth three points, a queen is worth two points, and a jack is worth only one point. Players also count distribution points. Having no cards of one suit is worth three points, a singleton is worth two points, and only two cards of a suit is worth one point. Evaluating the hand this way allows a player to determine if a hand is "biddable."

Once play starts, math is used to determine how best to play the cards. For example, in a typical hand, one side may have nine cards of the trump suit, which means the other side has four trump cards. Suppose the missing cards are the queen, 8, 6, and 3. How are those cards likely to be distributed between the opponents' two hands? One opponent could have all four trump cards, one could have three cards, and the other opponent one card, or each opponent might have two cards. The mathematics of probabilities shows that the most likely arrangement is for one opponent to have three of the missing trump cards. So a player cannot depend on capturing the missing queen by simply leading the ace and king. SEE ALSO PROBABILITY, THEORETICAL.

Elliot Richmond

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Gaming

Recent years have indicated a change in the locality of legalized casino gambling across the United States. Until 1978 gaming, as it is commonly called, was available only in Nevada. Today, however, the casino business has expanded to other states. One reason for the increase in the number of gambling places is that many people enjoy playing these games of chance. Another critical factor is the profit margins realized by entrepreneurs in the casino industry.

Casinos offer a wide range of games to meet the interests of their clientele. Blackjack, roulette, baccarat, and craps are among the popular games. All casino games share two important characteristics that are critical to their

The gaming industry continues to grow because the odds are in favor of the house.

continued success. On the one hand, each game is exciting because the unpredictable nature of probabilistic outcomes makes it possible for a player to be lucky and win in the short run. On the other hand, the significant "house edge" associated with each of these games ensures that in the long run the bettor will lose and the casino will win money. To better understand the risks of casino gambling, consider the chances of winning in roulette and blackjack.

Roulette

The simplest of casino games to analyze from the perspective of theoretical probability is roulette. In this game a small ball is rolled onto a revolving tray containing thirty-eight numbered compartments. The ball randomly comes to rest in one of these thirty-eight compartments. In addition to the compartments numbered from 1 to 36, there are two extras numbered 0 and 00. It is these last two numbers that provide the casino with their edge.

The bettor has several options when playing roulette. If a particular number is chosen, the player will win $35 for every dollar bet if that number occurs. In this case, if a winning bet of $2 is placed the bettor will receive $72 in return, 35 to 1 on each dollar bet, plus the original $2 bet. The bettor may also choose to bet on the color of the number hit (eighteen are red and eighteen are black), whether the number is even or odd (0 and 00 are considered neither even nor odd), or whether the number hit will be 1 through 18, or 19 through 36.

In each of these bets, the probability of winning is 18/38 because in each case there are eighteen favorable results out of the total number of thirty-eight possible outcomes. Finally, the player can get 2 to 1 odds by betting that the number will be from 1 to 12, 13 to 24, or 25 to 36.

One interesting aspect of roulette is that the "strategy" used in choosing from the aforementioned betting options has absolutely no effect on the expected value of the win or loss on that bet. To show this, the expected value of each of these options can be calculated. In a nutshell, the expected

value is the average amount that a player would win or lose on a bet. A positive expected value indicates that the game is favorable and that on average the player will win money, whereas a negative expected value indicates that the player will on average lose money.

If 0 and 00 were not included on the roulette wheel, the expected value on any wager would be 0, indicating that in the long run, the average win or loss for the bettor would be zero. To calculate the expected value, multiply the probability of each of the expected outcomes by the amount won or lost by the player given that outcome. Thus without 0 and 00, a player betting on even or odd would expect to win 50 percent of the time and lose the other 50 percent of the time.

If a bet of X dollars is made, the expected value is $(0.5)(-X) + (0.5)(+X) = 0$. Similarly, on the 36 number wheel, the expected win or loss on a bet of \$X on an individual number is 0. The player wins $\frac{1}{36}$th of the time and loses $\frac{35}{36}$ths of the time, making the expected value $(\frac{35}{36})(-X) + (\frac{1}{36})(+35X) = 0$. Any game with an expected value of 0 is referred to as a fair game and will not be found in any casino!

The green numbers 0 and 00 provide the casino with its edge. If a bet is made on an individual number, the probability of winning is $\frac{1}{38}$, since there are now 38 equally likely outcomes that can occur. Calculating the expected value for betting \$X on an individual number yields the following equation: $(\frac{37}{38})(-X) + (\frac{1}{38})(+35X) = (-\frac{2}{38})X = (-\frac{1}{19})X$. Thus the expected value is approximately -0.053 times the amount of the wager. This means that the player on average will lose a little more than 5 cents on every dollar bet.

An even or odd bet has a probability of $\frac{18}{38}$ of winning since 0 and 00 are considered neither even nor odd. Calculating the expected value for the player betting \$X on even or odd yields the following equation: $(\frac{20}{38})(-X) + (\frac{18}{38})(+X) = (-\frac{2}{38})X = (-\frac{1}{19})X$. Thus the expected value on this type of bet is exactly the same as that of betting on an individual number. The reader is encouraged to perform the appropriate calculations to verify that betting on 1 through 12 also yields an expected value of -0.053 multiplied by the amount of the bet. In roulette, therefore, no matter what strategy is used, the disadvantage to the player remains constant.

Some gamblers, however, feel they can win by trying to determine what is "due" to happen on a particular trial. For example, if the previous four numbers have been red, some feel that black is more likely to occur the next time. They reason that because attaining red five times in a row is extremely unlikely, the next result is more likely to be black. The fallacy here is that although the probability of red occurring five times in a row is indeed very unlikely, the unlikely event of four consecutive reds has already occurred.

Because successive trials of a roulette wheel are independent of one another, the next number has the same chance of being red or black as in previous trials, $\frac{18}{38}$. Thus playing this "hunch" strategy does not make the player any more (or less) likely to win as the expected value for each trial remains -0.053 times the amount bet.

Blackjack

Blackjack is perhaps the most complex of casino games and for this reason has been the subject of considerable analysis by gamblers. Calculating the

expected value of a hand of blackjack is an extremely difficult task for several reasons. First, there are many ways in which a hand can be dealt to the player and the dealer. Second, almost every casino offers slightly different rule variations of the game. Finally, unlike roulette, the decisions made by the player throughout the course of a hand dramatically affect the chances of winning.

In the game of blackjack, picture cards have a value of 10, aces can be counted as 1 or 11, and all other cards take on the value of the card. The game begins with the player and the dealer each being dealt two cards. The player's goal is to come closer to 21 than the dealer without going over, known as busting. After seeing both of their cards and one of the dealer's cards (commonly called the "dealer's up card"), the player must decide how to proceed. The player can "stay" with the present total, take another card ("hit"), double the wager and be given one and only one more card, or split a pair by placing another bet equal to the original and then proceeding with two separate hands.

In the majority of cases, the only reasonable decision is to take a hit or stand with the current total. Should the player take a hit, he or she can take additional hits until either the total of the cards exceeds 21 or the player does not want any more cards. The player loses immediately if the sum of the cards exceeds 21. If the player stops taking cards without exceeding 21, it is the dealer's turn to act. In casino versions of blackjack, the dealer does not make decisions but instead the dealer's play is fixed. The dealer must hit whenever her total is 16 or less and must stay when her total is 17 or more.

Many books on blackjack provide a basic strategy that is considered optimal for the player. Optimal basic strategy is determined by considering every possible combination of the player's hand and the dealer's up card and making the play that yields the best expected value in each case. Depending on the rule variations, most authors claim that by playing a sound basic strategy the house advantage will only be about 2 to 3 percent. Although this disadvantage is less than that of roulette, we must keep in mind that it is based on the player making the correct decisions at all times. Unlike roulette, the quality of the decisions made in the course of playing has a dramatic effect on the chances of winning. In practice, most players do not consistently make the optimal decisions and for this reason, many blackjack players lose at a much faster rate than they otherwise should.

How the House Stays Ahead

A simple example will show how the casino gains its advantage as the number of games played increases. Suppose we consider a situation in which the player has a 40 percent chance of winning and a 60 percent chance of losing each game with the amount won or lost in each game being equal. If three games are played, the player must win two or three games in order to be ahead. The probability of this occurring is $3(.4)(.4)(.6) + 1(.4)(.4)(.4) =$ 35 percent, because there are three ways in which the player can win two out of three games. That is, the probability of this occurring is $(.4)(.4)(.6)$ for each of the three ways, and there is one way of winning all three games, the probability of which is $(.4)(.4)(.4)$. Similarly, the probability of winning three or more games in five is

$$10(.4)(.4)(.4)(.6)(.6) + 5(.4)(.4)(.4)(.4)(.6) + 1(.4)(.4)(.4)(.4)(.4) = 32 \text{ percent.}$$

Further analysis of more games played can be done using the **binomial distribution**. The successive probabilities of the player winning when the game is played 7, 9, 11, 13, 15, 17 and 19 times are 0.29, 0.266, 0.246, 0.228, 0.212, 0.198, and 0.187, respectively.

binomial distribution in statistics, a function whose values yield the probability of obtaining a certain number of successes in independent trials wherein the probability of a successful outcome is constant from one trial to the next

The probability of winning decreases with the duration that a gambler plays. In a casino, a gambler may play hundreds of hands of blackjack or games of roulette consecutively, yet the chances of the player being ahead after a lengthy session are extremely small. Furthermore, from the casino's perspective, thousands of hands of blackjack and games of roulette are being played every hour by their numerous patrons, nearly around the clock. Given this huge number of trials, the casinos are assured of making large profits. SEE ALSO Probability and the Law of Large Numbers; Probability, Theoretical.

Robert J. Quinn

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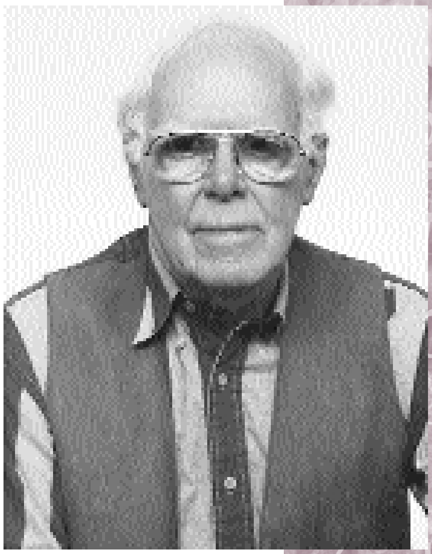
Gardner, Martin

American Author
1914–



Martin Gardner is a prolific writer and an expansive, stimulating thinker. Many of his columns and books are about mathematics.

One of the most well-known creators of mathematical puzzles is Martin Gardner. From 1957 to 1982, he wrote a column for *Scientific American* called "Mathematical Recreations." He presented intriguing problems, discussed the mathematics of various games, and demonstrated recreational aspects of mathematical discoveries. He always aimed to entertain and stimulate his readers, which ranged from high school students to college professors.

Early Work

Born in 1914 in Tulsa, Oklahoma, Martin Gardner became fascinated with mathematics in high school when he took Pauline Baker's geometry course. She communicated a love for the subject that he readily absorbed. Gardner also had other academic interests. He graduated from the University of Chicago in 1936, with a major in philosophy, and did graduate work in the philosophy of science.

Most of Gardner's early writing had little to do with philosophy or mathematics. Before World War II, he worked as a reporter for the *Tulsa Tribune.* After serving in the U.S. Navy, he supported himself as a freelance writer, working for eight years as a contributing editor to *Humpty Dumpty's Magazine*. This ended in 1957 when, drawing on his interest in magic, he sold his first article to *Scientific American.*

Fascinated when a magician showed him a paper toy called a hexaflexagon, Gardner contacted the inventor, John Tukey, a mathematician at Princeton, and with Tukey's permission and help, he wrote an article about hexaflexagons and the mathematics behind them. Delighted, the *Scientific American* editors published it and asked for more. Martin Gardner scoured

New York City for old books on recreational mathematics and found enough material to get the column going. Shortly thereafter, he began to draw material from recreational mathematics journals.

Gardner had a gift for simplifying ideas and communicating them wittily in a warm, playful spirit. His writing was so well received that mathematicians whose work had recreational aspects—Solomon Golomb, John Conway, Roger Penrose, and Frank Harary, among others—shared their discoveries with him. Through these contacts his columns became more sophisticated, and he enabled mathematicians to present their work to a much larger audience.

Popular Columns

Central to his work was the belief that mathematics, whether formal or recreational, is enormously interesting and of vital importance to humankind. Mathematics is the solving of puzzles. Good puzzles, even if they appear to be of trivial importance, open the door to all sorts of useful interconnections, often leading to "better and better answers to puzzles posed by nature."

One of his most popular columns was on John Conway's Game of Life, a population simulation game. A few counters are placed on a large checkerboard. Counters are born or die according to these rules:

(1) a counter survives to the next round if it has two or three neighboring counters;

(2) a counter dies if it has four or more neighbors or one or zero neighbors; and

(3) each empty cell with exactly three neighbors will give birth to a new counter in the next round.

The game is fascinating because of the great variety of population behaviors that arise from different beginning arrangements of counters.

Another favorite article was on a method of encoding messages called trapdoor functions, which are functions whose inverses—the key to decoding the message—are computationally impossible to discover in thousands of years. Martin Gardner's article was the first to discuss the work of Ron Rivest, who combined trapdoor functions with prime number factorization, creating coding systems that could be used in the electronic transmission of information over the Internet.

The Works of Martin Gardner

Over the years Martin Gardner published challenging problems in his columns, the answers of which can be found in the books listed here:

Aha! Gotcha: Paradoxes to Puzzle and Delight.

Aha! Insight.

Fractal Music, Hypercards, and More: Mathematical Recreations from Scientific American Magazine.

Hexaflexagons & Other Mathematical Diversions: The First Scientific American Book of Puzzles and Games.

The Incredible Dr. Matrix.

Knotted Doughnuts and Other Mathematical Entertainments.

Mathematical Carnival.

Mathematical Circus.

Mathematical Magic Show.

Mathematics, Magic, and Mystery.

My Best Mathematical and Logic Puzzles.

New Mathematical Diversions from Scientific American.

The Numerology of Dr. Matrix.

Penrose Tiles and Trapdoor Ciphers.

Riddles of the Sphinx and Other Mathematical Puzzle Tales.

The 2nd Scientific American Book of Mathematical Puzzles and Diversions.

Sixth Book of Mathematical Diversions from Scientific American.

Time Travel and Other Mathematical Bewilderments.

The Unexpected Hanging and Other Mathematical Diversions.

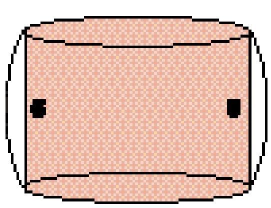
The Universe in a Handkerchief: Lewis Carroll's Mathematical Recreations, Games, Puzzles, and Word Plays.

Wheels, Life, and Other Mathematical Amusements.

The math puzzles Gardner presented to the public were enjoyed by people of all ages, and offered a variety of problems for readers to solve. Some examples include:

A boy and a girl were talking. "I'm a boy," said the one with black hair. "I'm a girl," said the one with red hair. If at least one of them is lying, who has black hair? [From *Wheels, Life, and Other Mathematical Amusements*, 1983]

A cylindrical hole six inches long is drilled right through the center of a solid sphere as shown below. Determine the volume remaining in the sphere. [From *Hexaflexagons & Other Mathematical Diversions: The First Scientific American Book of Puzzles and Games*, 1988]

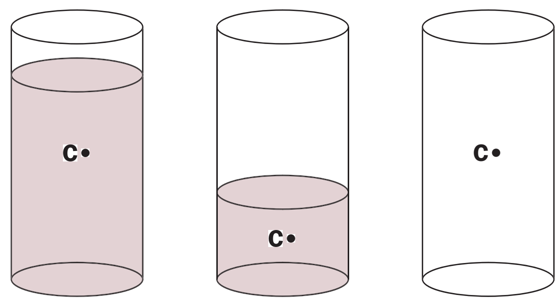


Dissect an isosceles triangle *ABC* with a 120° angle into five triangles similar to *ABC*. [From *Wheels, Life, and Other Mathematical Amusements*, 1983]

A worm is at the end of a 1 kilometer rubber rope. It crawls forward for one second, covering 1 centimeter and then the length of the rope is increased by 1 kilometer. This process is continued indefinitely. When the rope is stretched it is pulled from both

ends. Show that the worm can reach the end of the rope. [From *Time Travel and Other Mathematical Bewilderments*, 1988]

Consider the figure below. As you drink a soda, the center of gravity *C* drops, but when the can is empty, the center of gravity has risen back to its starting point. Assuming that the can is 8 inches high, weighs 1.5 ounces empty and 13.5 ounces full, determine the lowest point reached by the center of gravity *C*. [From *Wheels, Life, and Other Mathematical Amusements*, 1983]



The game of Sim: put six dots on a paper, forming the vertices of a regular hexagon. Each player in turn connects two of the dots; one player uses a blue color, the other a red color. The first player forced to form a triangle of his own color loses. What is the best strategy? [From *Knotted Doughnuts and Other Mathematical Entertainments*, 1987]

A paradox: a man who always keeps his promises tells his wife that "tomorrow for your birthday I will give you an unexpected gift. You have no way of guessing what it is. It is the gold bracelet we saw at the jewelry store." Will his wife be surprised or not? [From *The Unexpected Hanging and Other Mathematical Diversions*, 1991]

Gardner also created the mysterious Dr. Matrix as a foil for playing with numerology. Here, for example, is Dr. Matrix's proof that William Shakespeare helped translate the King James Bible: In Psalm 46, the 46th word from the beginning is SHAKE and the 46th word from the end is SPEAR. Furthermore, the King James Version was completed in 1610 when Shakespeare was 46 years old. SEE ALSO Puzzles, Number.

Don Barry

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Notes on Martin Gardner. The Recreational Math Pages. <http://www.citlink.net/citlink/d/dmn1/gardner.htm>.

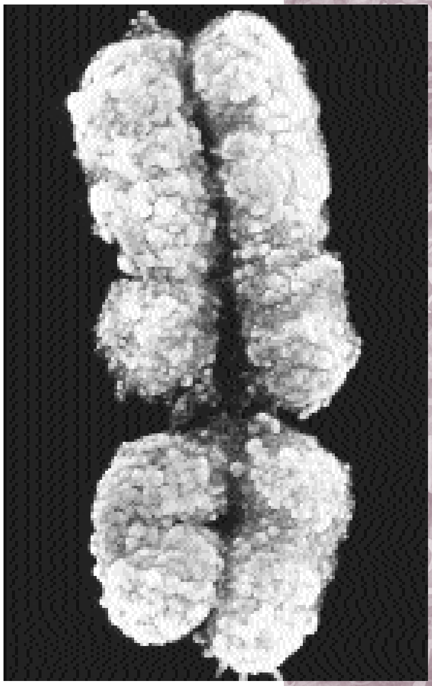
Genome, Human

The word "genome" means the totality of all the genetic information present in the cells of an organism. Most of this genetic information is contained in chromosomes. A chromosome consists of deoxyribonucleic acid

(DNA) molecules wound up into a compact bundle. Humans have 46 chromosomes in their cells, organized into 23 corresponding pairs.✶

✶If a single strand of human DNA were unwound and stretched out, it would be over two meters long.

The DNA in the chromosomes of an organism consists of a double-stranded molecule formed from four basic units that are repeated many times. The four basic units are called nucleotides. Each nucleotide is made of a sugar, a phosphate group, and a nucleotide base. The sugars in the nucleotides stack up and link together to form a backbone for one strand of the DNA molecule, leaving the nucleotide bases projecting. The four kinds of nucleotide bases found in DNA are adenine, cytosine, guanine, and thymine. A fifth nucleotide base, uracil, is found in ribonucleic acid (RNA) in place of thymine.



Scanning electron micrograph of a human X chromosome.

The nucleotide bases of DNA are usually designated as A, C, G, and T (or U) respectively. In the double-stranded DNA molecule, each nucleotide base forms a bond with a nucleotide base on the other strand of the molecule, such that A always pairs with T, and C with G. These two strands then wrap around each other, forming a structure known as a double helix. The two strands are antiparallel so that the sequence of bases starting from one end on one strand is repeated, starting from the opposite end on the other strand.

A gene is a unique sequence of bases that occupies a particular position on a chromosome. A single strand of DNA contains hundreds of individual genes. However, there are several different types of genes included in a DNA molecule. Structural genes code for particular amino acid sequences. These are the genes that contain instructions on how to build proteins. Operator genes control the structural genes and regulate their output. Regulatory genes may produce repressor proteins that turn the operator genes on or off or they may act like punctuation marks, signaling the beginning and end of coding sequences. Suppressor genes may suppress the actions of other genes and can reverse the effects of a harmful mutation. Kinetic genes regulate the chromosomes themselves. The 30,000 or so genes in every human cell code for more than one million different proteins including albumin and hemoglobin; brain chemicals like dopamine and serotonin; hormones like insulin, testosterone, and estrogen; and the countless enzymes that keep us alive.

Mathematics of the DNA Code

Structural genes responsible for the production of the amino acid sequences used to build proteins are the best understood of the different kinds of genes. To produce a protein, a section of DNA becomes "unzipped," exposing the nucleotide bases. A molecule of messenger RNA (mRNA) is synthesized by pairing up nucleotide bases. Each group of three nucleotide bases on the RNA molecule, called a codon, codes for a particular amino acid. Since there are four choices (A, C, G, and U) for each of the three-nucleotide bases, each codon can be one of 64 different "words" ($4 \times 4 \times 4 = 64$). However, there are only 20 different amino acids. Several different codons produce the same amino acid. For example, UUU and UUC both produce the amino acid phenylalanine. There are also three codons, UAA, UGA, and UAG, that act as "stop" signals ending the protein chain.

A protein contains around 100 amino acids, so a structural gene must contain at least 300 nucleotide bases. Using this logic, it might then be ex-

THE IMPORTANCE OF MAPPING THE HUMAN GENOME

Many genetic diseases can result from a misspelling in the DNA sequence. Since each DNA codon codes for a particular amino acid, a change in one nucleotide base can result in a different protein being produced. If that protein is essential to health, a genetic disease results. For example, normal hemoglobin differs from the hemoglobin of sickle-cell anemia by only one amino acid out of hundreds.

The genetic maps being developed by the Human Genome Project are available on the Internet and are updated frequently, making accurate genetic information accessible to every researcher and opening up great potential for improving human health through research.

pected that the human genetic code would contain 10 million genes. However, DNA also contains stop and start codons, other regulatory molecules, some sequences (called introns) that are removed before protein synthesis begins, duplicate genes, non-coding sequences, redundant genes, and other non-functioning bits of DNA. As a result, there are probably only around 30,000 individual genes among the 3 billion nucleotide units of human chromosomes.

The Human Genome Project

The U.S. Human Genome Project (HGP) is a joint effort of the Department of Energy (DOE) and National Institute of Health (NIH). The goal of the Human Genome Project is to decipher human heredity through the creation of maps for each of the 23 human chromosomes. The first step in this process is to determine the actual DNA code. In June, 2000, then-U.S. President Clinton, leaders of the Human Genome Project, and officers of Celera Genomics (a private biotechnology firm) jointly announced that the rough draft of the human genetic code was ready for publication. The February 16, 2001 issue of *Science* published articles related to the work of Celera Genomics, and the February 15, 2001 issue of *Nature* published articles relating to the work of the Human Genome Project.

As of July 30, 2001, only the two smallest human chromosomes, 21 and 22, were completely sequenced to the level of accuracy specified by the protocols established by the Human Genome Project. At the time this article was written, 47.1 percent of human DNA had been mapped to final standards and 51.4 percent had been mapped to preliminary standards for a total of 98.5 percent mapped.

The final publication of the human genome map is expected by 2003. However, this final draft will include only the human genome code. Still to be determined are the exact number and locations of genes, how genes are regulated, how the DNA sequence is organized, how chromosomes are organized, which parts of the DNA are redundant or noncoding, how gene expression is coordinated, how genetic information is conserved, and many other concepts essential to understanding the human genetic code. SEE ALSO Human Body.

Elliot Richmond and Marilyn K. Simon

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Internet Resources

DOE Human Genome Project. <http://www.ornl.gov/hgmis/>.

Genome Information on the World Wide Web. <http://cib.nig.ac.jp/others/genome-info.html>.

Geography

Geography is the study of the physical and geopolitical aspects of the surface of Earth. Physical geography describes the different surface and climatic conditions around the world. Political geography is concerned with the division of the world into various levels of government, human activity, and production. Geography is not confined to merely describing Earth as it is now, but also understanding how it has evolved and how it may change in the future.

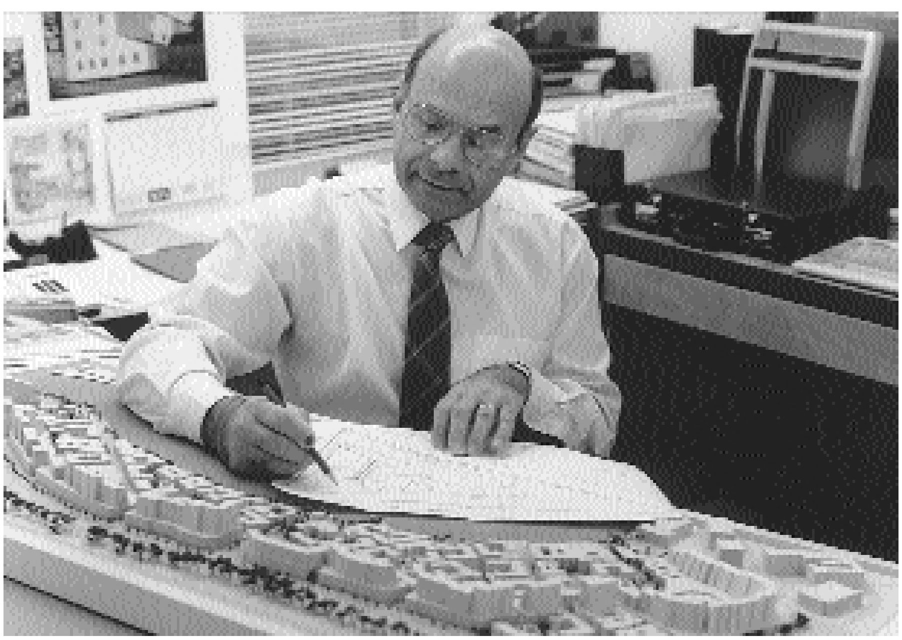
The problems that faced humankind at the dawn of history stimulated both geography and mathematics. In fact, much of early mathematics was concerned with making measurements of the land; so much so that a whole branch of mathematics became known as Earth-measurement, which in Greek is *geometry*. Geometry as a mathematical study is less concerned with its practical roots, but for geographers, geometry and trigonometry are invaluable tools.

Outside of the activities associated with mapmaking, geography was for many years mainly descriptive. The information collected about the physical and social characteristics of the world was reported in a narrative form with little attempt to analyze the data that had been collected. In the late 1940s and early 1950s, a revolution took place in geography when the acceptance of any theory within the science became subject to mathematical analysis. As with other sciences, geography began to use mathematics as the language to describe relationships in the discipline.

The Geographic Matrix

An observation made by a geographer has two main attributes—a location and a physical attribute associated with that location. Each place may have more than one characteristic and each characteristic may be found at more than one location. These data can be recorded in a **matrix** with rows representing characteristics and columns the places where the

matrix a rectangular array of data in rows and columns



City planners and architects must have an understanding of mathematics and physical and political geography in order to optimize their designs. Shown here is a planner using a scale model.

observations have been taken. The organization of data into a matrix greatly aids geographers with the mathematical analysis of the information they gather.

sampling selecting a subset of a group or population in such a way that valid conclusions can be made about the whole set or population

Before geographers collect data, they must select the locations within the region where they will measure the characteristics of interest. This requires an understanding of the **sampling** techniques found in mathematical statistics. From statistical sampling theories the geographer calculates how many locations will be required, which will consequently reveal the number of columns in the matrix. In establishing the appropriate number of locations, it is necessary to ensure that there is sufficient data so that the samples are representative of the whole region.

Three main types of sampling systems are used in selecting locations. The first type is a totally random sample, in which each location in the study is selected at random from all possible points in the region. The second type of sample is a systematic sample, in which an initial point is chosen at random and all other points are determined by fixed intervals from the randomly chosen point. The third type of sample is a stratified sample, in which the region is subdivided into subregions. Within the subregions, points are chosen by either using a totally random sample, or a stratified sample, or by dividing into further subdivisions. This process can continue until the degree of accuracy required matches the number of sampling points. For example, in studying a country, a geographer may first break the country into regions. Then the regions may subdivide by using political divisions such as a state, and this may go further by using counties, and at this level there may be a random selection of sampling points.

Ultimately, the selection of sampling locations should permit a rapid, accurate, and economical amount of calculation in order to analyze the data. The selection process should also be such that final analysis is comparable to data collected in other regions so that regional comparisons may be made. In addition, consideration needs to be given to national and international standards, and to enabling comparisons with data collected over time.

Analysis of the Geographical Matrix

When the collected data have been placed in a geographic matrix, an analysis of a region can proceed in many ways. One common method of analysis is an examination of how a characteristic is distributed over a region by examining the row of the matrix for that characteristic. For example, attention may be focused on the way in which the rural population is distributed over an area such as the Great Plains of the United States.

Secondly, a geographer may try to get an understanding of the complexity of a location by identifying its characteristics. In other words, the column for that location may be analyzed. For instance, interest may be in the rainfall, soil type, or most successful crop production at a location in order to make recommendations for other places with similar characteristics. If the location is an urban area, a geographer might try to connect transportation access data, raw material availability, and expert labor supply, in order to explain why a particular industry is successful at that location.

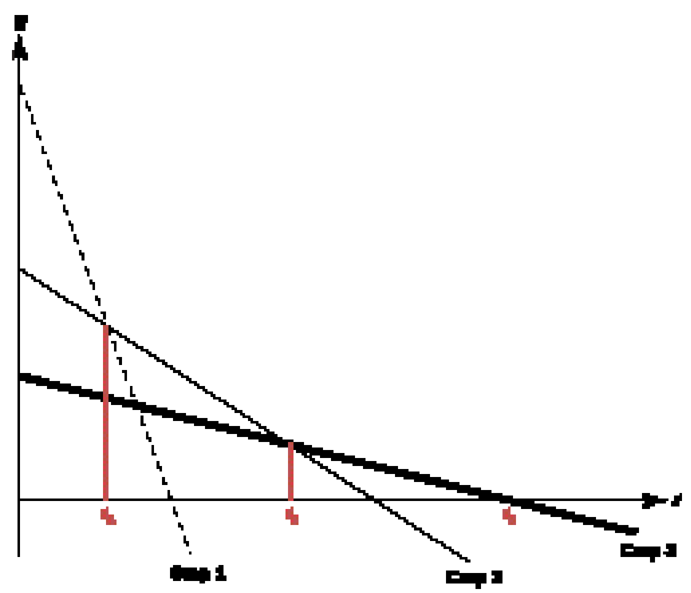
A third way to make comparisons is between rows. This enables an understanding of which characteristics are found together or separately, or to what degree they might mix. For example, looking at common characteristics for two economically successful locations can show why they contribute to the locations' success.

A fourth option for a method of analysis is to make a comparison of columns. This allows the geographer to describe which locations are similar and which are very different. For instance, by analyzing locations where the weather data are the same, a geographer can classify climates that are similar. All of these analyses require the statistical techniques of correlation and regression (defining and characterizing relationships among data).

Optimization Problems Solved by Geographers

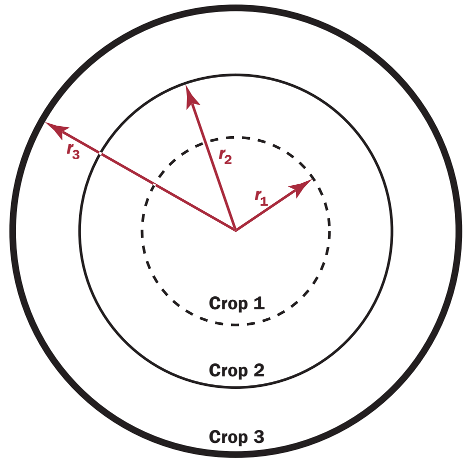
Although mathematics has become an essential tool of modern geography, it was also present in the geography of the nineteenth century. In 1826, Von Thünen collected data on land values in agricultural communities; he also collected data on how farmers used land. His data were centered on a town that was the main market for a region.

Von Thünen found that for each particular type of crop the costs of getting the produce to market was a product of the distance from town, r, the volume of the crop produced in a unit area of land, v, and the cost of transportation per unit of distance, c. If the crop sells at a price of p and the fixed costs of producing the crop are a, then the net profit is expressed as $R = (p - a)v - rcv$.



Von Thünen constructed graphs of profit R plotted against the distance from town, r for various crops. The figure above shows the graphs for three crops. Crop 1 produces the highest profit as long as it is inside a distance of r_1 of the market. Between a distance of r_1 and r_2 crop 2 is the most profitable, and between r_2 and r_3 crop 3 is the most profitable. At r_3 all three crops become unprofitable. Von Thünen suggested that the land around a market be used to reflect these rings, and that there should be no cultiva-

tion of these crops beyond r_3 at all, as there was no profit in farming at this distance (see below).



The modern equivalent of this geographical distribution model is the understanding of why the location of a shopping center or a factory affects each one's success or failure. From the geographical matrix the locations of various resources that are required by a manufacturing plant can be established. Given the locations of different raw materials, labor resources, and transportation of raw materials to the factory, the location for the optimum manufacturing plant can be calculated and compared to an existing plant.

A geographer's data can also be used to support or refute the location of a manufacturing plant at a particular location. However, this is only part of the solution, for once the goods have been manufactured they have to be distributed to market centers, and this has an associated cost that can affect the decision concerning a manufacturing plant's location. By weighting distances with regard to cost, an optimum location can be found by finding the equivalent of the center of gravity of the system.

In the location of a factory, one of the problems that has to be tackled is the distribution of the product to market. Here another branch of mathematics aids the analysis. Graphs and trees deal with the analysis of networks, and can be employed in finding solutions to this part of the problem. One of the classic problems of networks, the travelling salesman problem, is concerned with the most efficient route for a travelling salesman to take in order to cover all the customers. This is also the route that the supply trucks will be interested in following. The full understanding of this problem is still the object of mathematical research.

Calculus in Geography

A major problem of geography is the modeling of population change. Change in populations implies that geographers are interested in data that are time dependent. Therefore any data collection has to be repeated at various intervals, the most familiar way being the United States census that is required every 10 years by law.

The census gives data over long periods of time, but annual sampling is necessary in order to monitor more detailed changes. The data can then be matched to a mathematical model. The most common model for population growth results in the construction of a differential equation in which the change in population, with respect to time, varies directly with time and can be solved through calculus.

Calculus also helps model the way in which the profile of a hill develops. Another application of calculus gives a mathematical model of the freezing of water in a lake. If air above a lake maintains temperatures below the freezing point of water for a prolonged period of time, the thickness of the ice will continue to increase. The rate of advance of the ice depends on the rate at which heat can be carried away from the surface by convection currents in the water below the ice surface. The model leads to a **differential equation**.

differential equation an equation that expresses the relationship between two variables that change in respect to each other, expressed in terms of the rate of change

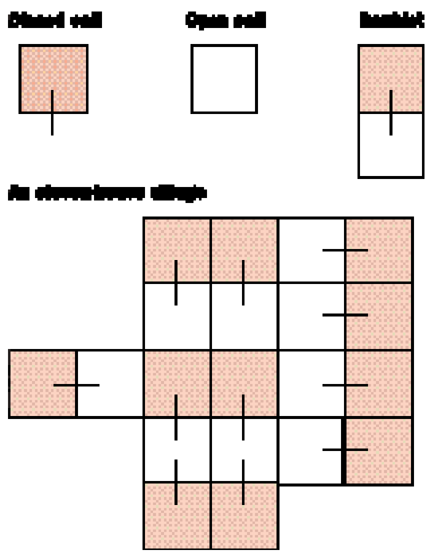
Fractals and River Watersheds

The way in which rivers begin their life as a collection of small springs or gullies that collect rain and spill into brooks or streams has been better understood in recent times by the analysis offered by mathematics through **fractals**. In river systems, fractal scaling can be seen in the organization of the river network at various levels of observation; that is, they conform to the fractals first described by Benoit B. Mandelbrot. Research around 1990 described the scaling properties of the geometry of several river systems and a calculation was made of their fractal dimension.

fractal a type of geometric figure possessing the properties of self-similarity (any part resembles a larger or smaller part at any scale) and a measure that increases without bound as the unit of measure approaches zero

Probability and the Layout of Villages

Probability leads to an understanding of the way villages develop over a long period of time, given that there has been no deliberate planning. The model requires the description of two objects, a closed cell with an entrance, and an open cell (see figure below).



These cells are joined together to form a doublet so that the entrance always faces onto an open cell—corresponding to a house opening out onto the public space. In the modeling process, the doublets are allowed to accumulate with the condition that each new doublet that joins the village does so with its open cell having at least one edge common with another open cell. Which open cell a new doublet joins is chosen at random. This modeling process has been successful in describing a number of old villages in which town planning did not influence the layout. SEE ALSO Cartographer; Fractals; Global Positioning System; Mandelbrot, Benoit B.; Maps and Mapmaking; Probability, Theoretical.

Phillip Nissen

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Geometry Software, Dynamic

Tucked in with the business news of the day was this headline from the December 9, 1996 issue of *The Wall Street Journal* newspaper: "Teen Math Whizzes Go Euclid One Better." High-schoolers David Goldenheim and Daniel Litchfield had revisited a 2,000-year old challenge from the Greek mathematician Euclid and solved it in a new way. Given an arbitrary segment, the freshmen found a geometric recipe for dividing its length into any number of equal parts. The mathematics community hailed the students' work as "elegant" and "significant."

Goldenheim and Litchfield devised their segment-splitting technique through old-fashioned conjecturing and reasoning. Yet there was nothing traditional about their geometric tools of choice. The duo conducted their experiments without the aid of a compass or even a ruler. Instead, they turned to technology and a new breed of computer software programs known collectively as "dynamic geometry."

At first glance, the word "dynamic" might sound like an odd way to describe geometry. The dictionary defines "dynamic" as "characterized by vigorous activity and producing or undergoing change," but the images in geometry textbooks are immobile, forever frozen in place.

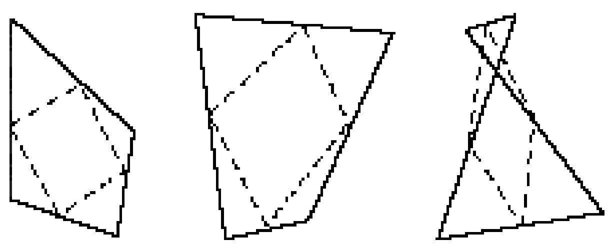
Consider a picture of a triangle. Any illustration represents a particular triangle with specific side lengths and angle measures. Triangles, however, can be small, large, narrow, or wide. No single image captures this generality.

By contrast, a triangle drawn with dynamic geometry software possesses more freedom. With a click and a drag of the mouse, you can tug a corner of the triangle and watch the object adjust before your eyes. The shape remains triangular, but its sides and angles grow and shrink in a smooth, con-

tinuous motion. The effect is similar to an animated movie, only here you are the one controlling the movement.

In science, you devise experiments and then test your theories. Mathematics classes often have fostered a more hands-off approach: textbooks state what to prove. With dynamic geometry software, mathematics regains its rightful place as a laboratory science. Any object constructed on the screen allows you to roll up your sleeves and search for patterns.

Suppose you draw an arbitrary quadrilateral, find the midpoints of its four sides, and then connect these points to their adjacent partners (see the leftmost picture below). By doing so, you form a new quadrilateral (represented by dashed segments) nested inside the original. Measuring its sides and angles with the software reveals a surprise. Opposite sides are equal in length and parallel—the very qualities of a parallelogram. A coincidence?



A mouse tug to any corner of the outer quadrilateral reconfigures the construction into new positions. Within seconds, you can view hundreds of quadrilaterals, each with their midpoints connected. In every one, the dashed quadrilateral remains a parallelogram. Even a twisted pretzel shape cannot disturb the parallel sides (see the rightmost picture). Such visual evidence is quite compelling; it is not a watertight proof, and it does not explain why we should expect to see a parallelogram, but it is a useful start.

A Brief History

As director of Swarthmore College's Visual Geometry Project, Eugene Klotz was among the founders of the dynamic geometry movement. His original goals for geometry software were relatively modest. Bothered by the cumbersome nature of ruler-and-compass constructions, Klotz imagined a software package that would make it easier to draw shapes like lines and circles. He comments:

> Basic motor skills were keeping students from being able to draw. I thought we needed to have something that allowed people to make the basic constructions. So to me, our software was a drawing tool. You'd make a geometric drawing that was precise and accurate, and scroll over the page to see what was going on.

This vision of geometry software was a non-interactive one: once drawn, objects on the screen could not be reshaped via mouse dragging. The missing "dynamic" element was to come from a student, Nicholas Jackiw, whom Klotz advised during Jackiw's freshman year at Swarthmore.

Jackiw was perhaps an unlikely choice for a mathematics project. He had steered away from mathematics in high school and college, focusing instead on English and computer science. Still, when Jackiw viewed Klotz's

geometry proposal, he sensed something was missing. Jackiw's interest in programming computer games provided an unexpected source of inspiration. If a computer game could immerse players in an interactive world, then why not geometry? Jackiw says:

> It's the video game aspect that gives me my sense of interactivity when dealing with geometry . . . Looking at the input devices of video games is a tremendously educational experience. In the old days, you had games with very interesting controls that were highly specific. . .The video game Tempest had a marvelous input device. . .The types of games they would write to suit this bizarre and unique device were always interesting experiments in what does this hand motion transport you to in your imagination. I wanted to have a good feel in all of my games.

The mouse of a Macintosh computer was not an ideal input device for games, but it was suitable for virtual environments where objects could be dragged. The illustration program MacDraw, in particular, contained the rudimentary features of dynamic geometry, as one could draw and move a segment with the mouse and change its length. When Jackiw took the basic premise of MacDraw and applied it (with considerable reworking) to geometry, Klotz found the results striking:

> I remember how shocked I was when I first saw it. Jackiw had played with a Macintosh long enough to know that you should be able to drag the vertex or a side of a triangle and protrude the figure. I was flabbergasted. I mean, he made the connection, and I didn't.

One of the earliest programs to showcase the graphical capabilities of the computer was Ivan Sutherland's "Sketchpad." A hand-held light pen allowed the user to draw and manipulate points, line segments, and arcs on a cathode ray tube monitor. In honor of Sutherland's work, Klotz named their new program, "The Geometer's Sketchpad." The product received its commercial release in the spring of 1991.

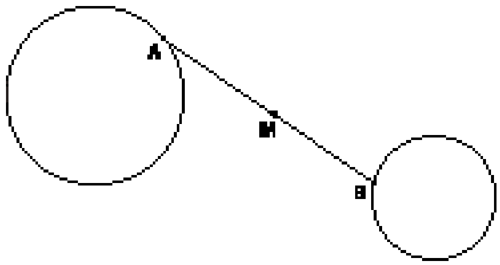
Interestingly, the Swarthmore group was not alone in its thinking. Working in France, Jean-Marie Laborde and his programming team simultaneously developed the software package Cabri Geometry, which also featured dynamic movement. Initially, neither the Sketchpad nor Cabri people knew of the other's existence. When Laborde and Klotz finally met, they marveled at the similarities in their software. Klotz says:

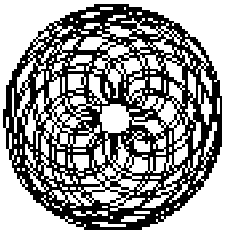
> We had just that Fall got into our dragging bit, and were very proud of what we had. We thought, God, people are going to really love this. But Cabri had scooped us, and we had scooped them. It was one of these, you know, just amazing things where . . . maybe you can sort out the exact moment, maybe there was a passing meteor, or something.

Student Exploration

Dynamic geometry software programs are great for learning geometry, and they can also be fun. The top picture below shows two circles and a segment *AB* connecting them. As points *A* and *B* spin around their respective circles, what path does point *M*, the midpoint of segment *AB*, trace? Dy-

namic geometry makes this investigation simple to perform. The result, shown in the lower picture below, is an attractive spiral.





Other applications of the software display its versatility. Students can build a working clock, model planetary motion, create a spinning Ferris Wheel, and investigate algebra in a geometric way.

Since words alone cannot convey the experience of using dynamic geometry software, students can try programs themselves. Free demonstration copies of various software programs are available on the Internet. SEE ALSO Computer-Aided Design; Computer Simulations.

Daniel Scher

Internet Resources

Cabri Geometry. <http://www-cabri.imag.fr/index-e.html>.

The Geometer's Sketchpad. Key Curriculum Press. <http://www.keypress.com>.

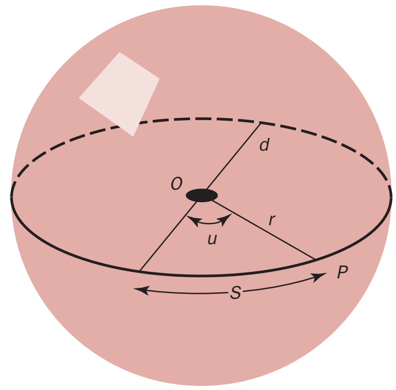
Geometry, Spherical

Spherical geometry is the three-dimensional study of geometry on the surface of a sphere. It is the spherical equivalent of two-dimensional planar geometry, the study of geometry on the surface of a plane. A real-life approximation of a sphere is the planet Earth—not its interior, but just its surface. (Earth is more accurately called an "oblate spheroid" because it is slightly flattened at the ends of its axis of rotation, the North and South Poles.) The surface of a sphere together with its interior points is usually referred to as the spherical region; however, spherical geometry generally refers only to the surface of a sphere.

As seen in the figure on the next page, a sphere is a set of points in three-dimensional space equidistant from a point O called the center of the sphere. The line segment from point O (at the center of the sphere) to point P (on the surface of the sphere) is called the radius r of the sphere, and the radius r extended straight through the sphere's center with ends on opposite points of the surface is called the diameter d of the sphere (with a value of $2r$;

✶The average length of Earth's diameter is d = 6,886 miles.

that is, two times the value of the radius). As an example, the line that connects the North Pole and the South Pole on Earth is considered a diameter✶.

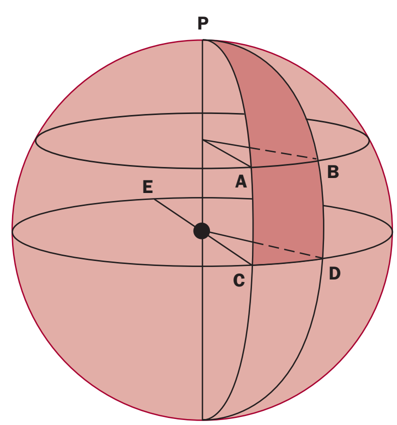


An infinite line that intersects a sphere at one point only is called a tangent line. An infinite plane can also intersect a sphere at a single point on its surface. When this is the case the plane is also considered tangent to the sphere at that point of intersection. For example, if a basketball were lying on the floor, the floor would represent a tangent plane because it intersects the ball's surface (the sphere) at only one point.

Great and Small Circles

The shortest path between two points on a plane is a straight line. However, on the surface of a sphere there are no straight lines. Instead, the shortest distance between any two points on a sphere is a segment of a circle. To see why this is so, consider that a plane can intersect a sphere at more than one point. Whenever this is the case, the intersection results in a circle. A great circle is defined to be the intersection of a sphere with a plane that passes through the center of the sphere. For example, see the circle containing points C and D in the illustration below. Similar to a straight line on a plane, the shortest path between two points on the surface of a sphere is the arc of a great circle passing through the two points.

The size of the circle of intersection will be largest when the plane passes through the center of the sphere, as is the case for a great circle. If the plane does not contain the center of the sphere, its intersection with the sphere is known as a small circle. For example, see the circle containing points A and B in the illustration below.



As a real-world example, assume a cabbage is a sphere, and is cut exactly in half. The slice goes through the cabbage's center, forming a great circle. However, if the slice is off-centered, then the cabbage is cut into two unequal pieces, having formed a small circle at the cut.

Spherical Triangles

Consider a circle of radius r. A portion of the circle's circumference is referred to as an arc length, and is denoted by the letter s. The first illustration of this article shows a circle of radius r and arc length s. The angle θ is defined as $\theta = \frac{s}{r}$. Rearranging this equation in terms of s yields $s = \theta r$. So the arc length s of a great circle is equal to the radius r of the sphere times the angle **subtended** by that arc length.

subtend to extend past and mark off a chord or arc

Connecting three nonlinear points on a plane by drawing straight lines using the shortest possible route between the points forms a triangle. By analogy, to connect three points on the surface of a sphere using the shortest possible route, draw three arcs of great circles to create a spherical triangle. A triangle drawn on the surface of a sphere is only a spherical triangle if it has all of the following properties:

(1) the three sides are all arcs of great circles;

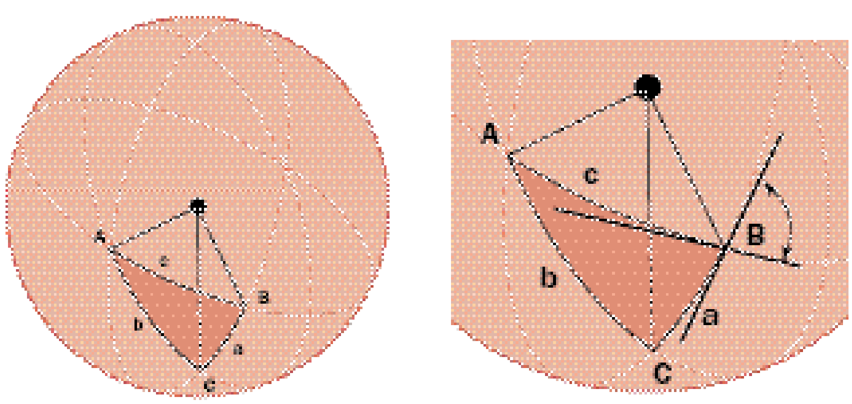
(2) any two sides, summed together, is greater than the third side;

(3) the sum of the three interior angles is greater than 180°, and

(4) each spherical angle is less than 180°.

In the second illustration of the article, triangle PAB is not a spherical triangle (because side AB is an arc of a small circle), but triangle PCD is a spherical triangle (because side CD is an arc of a great circle).

The left portion of the figure directly below demonstrates how a spherical triangle can be formed by three intersecting great circles with arcs of length (a,b,c) and vertex angles of (A,B,C).



The right portion of the figure directly above demonstrates that the angle between two sides of a spherical triangle is defined as the angle between the tangents to the two great circle arcs for vertex angle B.

The above illustration also shows that the arc lengths (a,b,c) and vertex angles (A,B,C) of the spherical triangle are related by the following rules for spherical triangles.

sine if a unit circle is drawn with its center at the origin and a line segment is drawn from the origin at angle (so that the line segment intersects the circle at (x, y), then y is the sine of θ

cosine if a unit circle is drawn with its center at the origin and a line segment is drawn from the origin at angle θ so that the line segment intersects the circle at (x, y), then x is the cosine of θ

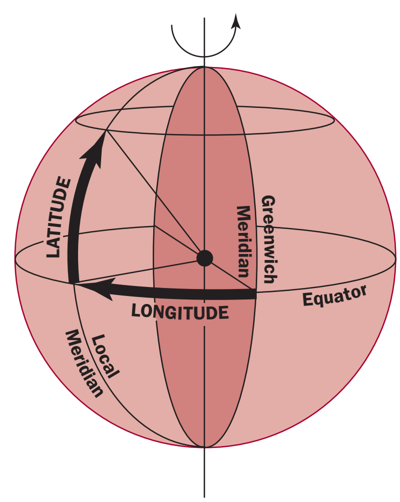
Sine Rule: $(\frac{\sin a}{\sin A}) = (\frac{\sin b}{\sin B}) = (\frac{\sin c}{\sin C})$

Cosine Rule: $\cos a = (\cos b \cos c) + (\sin b \sin c \cos A)$.

Spherical Geometry in Navigation

Spherical geometry can be used for the practical purpose of navigation by looking at the measurement of position and distance on the surface of Earth. The rotation of Earth defines a coordinate system for the surface of Earth. The two points where the rotational axis meets the surface of Earth are known as the North Pole and the South Pole, and the great circle perpendicular to the rotation axis and lying halfway between the poles is known as the equator. Small circles that lie parallel to the equator are known as parallels. Great circles that pass through the two poles are known as meridians.

Measuring Latitude and Longitude. The two coordinates of latitude and longitude can define any point on the surface of Earth, as is demonstrated within the diagram below. Great circles become very important to navigation because a segment along a great circle provides the shortest distance between two points on a sphere. Therefore, the shortest travel-time can be achieved by traveling along a great circle.



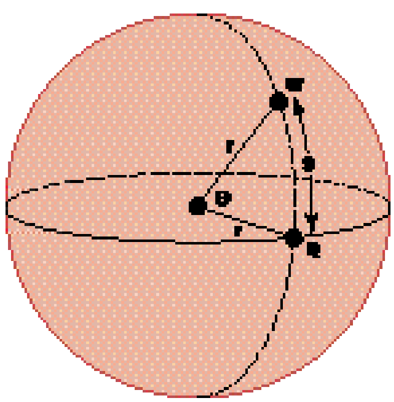
The longitude of a point is measured east or west along the equator, and its value is the angular distance between the local meridian passing through the point and the Greenwich meridian (which passes through the Royal Greenwich Observatory in London, England). Because Earth is rotating, it is possible to express longitude in time units as well as angular units. Earth rotates by 360° in 24 hours. Hence, Earth rotates 15° of longitude in 1 hour, and 1° of longitude in 4 minutes.

The latitude of a point is the angular distance north or south of the equator, measured along the meridian, or line of longitude, passing through the point.

Measuring Nautical Miles. Distance on the surface of Earth is usually measured in nautical miles, where 1 nautical mile (nmi) is defined as the distance subtending an angle of 1 minute of arc at the center of Earth. Since there are 60 minutes of arc in a degree, there are approximately 60 nautical miles in 1 degree of Earth's surface. A speed of 1 nautical mile per hour (nmph) is known as 1 knot and is the unit in which the speed of a boat or an aircraft is usually measured.

A Case Study in Measurement. As noted earlier, Earth is not a perfect sphere, so the actual measurement of position and distance on the surface of Earth is more complicated than described here. But Earth is very nearly a true sphere, and for our purposes this demonstration is still valid.

The terms and concepts that have been developed can be applied to a real-world example. Consider a voyage from Washington, D.C. ("W" in diagram below) to Quito, Ecuador ("Q" in diagram below), which is nearly on the equator at 0° latitude, 77° West longitude. The latitude and longitude of Washington, D.C. is about 37° North latitude, 77° West longitude. If the entire voyage from Washington, D.C. to Quito (on the equator) is along the great circle of longitude 77°, we can use the equation $s = \theta r$ to find the distance s that the airplane travels from Washington D.C. to Quito.



For this example, $\theta = 37°$ (the angle between W and Q). Knowing that 2π **radians** equals 360° (one complete revolution around a great circle), we now convert the angle from degrees to radians: $(37°)(\frac{2\pi \text{ radians}}{360°}) = 0.628$ radians. Denoting the radius of Earth as r, we use the "arc-length" equation developed earlier, that is $s = \theta r$, to compute the arc length between Washington, D.C. and Quito.

radian an angle measure approximately equal to 57.3°, it is the angle that subtends an arc of a circle equal to one radius

Placing the values of $\theta = 0.628$ radians and $r = 3{,}443$ nautical miles (nmi) (the average radius-value for Earth) into the equation yields: $s = \theta r = (0.628 \text{ rad} \times 3{,}443 \text{ nmi}) = 2{,}163$ nmi. Therefore, along the arc of the great circle of longitude 77°, from Washington D.C. to Quito, Ecuador, our trip covers a distance of 2,163 nmi. SEE ALSO Triangles; Trigonometry.

William Arthur Atkins (with Philip Edward Koth)

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Geometry, Tools of

Plane (or Euclidean) geometry is the branch of mathematics that studies figures (such as points, lines, and angles) constructed only with the use of the straightedge and the compass. It is primarily concerned with such problems as determining the areas and diameters of two-dimensional figures. To determine geometric designs four important tools of geometry—compass, straightedge, protractor, and ruler—are used. Technically a true geometric construction with Euclidian tools, originally used by the ancient Greeks, uses only a compass or a straightedge. The ruler and protractor were later inventions. Today, the study of geometry is an essential part of the training of such professionals as mathematicians, engineers, physicists, architects, and draftspersons.

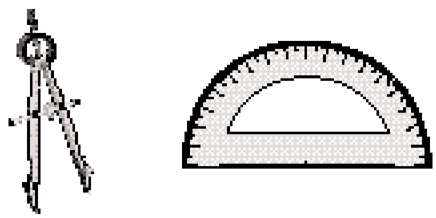
As early as 2000 B.C.E. geometers were concerned with such problems as measuring the sizes of fields and irrigation systems, and laying out accurate right angles for corners of buildings and monuments. Greek mathematician Euclid (c. 300 B.C.E.) and other geometers formalized the process of building geometrical figures with the use of specific tools. The ancient Greeks introduced construction problems that required a certain line or figure to be constructed by the use of the straightedge and compass alone. Simple examples are the construction of a line that will be twice as long as another line, or of a line that will divide a given angle into two equal angles.

Basic Tools

Straightedge. A straightedge is a geometric tool used to construct straight lines. It contains no marks. As the name says, it is a "straight edge." A ruler can be used as a straightedge by simply ignoring the measuring marks on it.

Ruler. A ruler is a geometric tool used to measure the length of a line segment. A ruler is basically a straightedge with marks usually used for measuring either inches or centimeters. To use a ruler, place the zero mark on the point to begin the measurement. To stop measuring, look at the mark on the ruler that lies over the point at which the measurement is to end.

Compass. A compass is a V-shaped tool used to construct circles or arcs of circles. (See sketch below on left.) One side of the "V" holds a pencil and the other side is a point. The point anchors the compass at one location on the paper as the tool is turned so the pencil can trace circles, arcs, and angles. A compass is adjustable: the setting determines how far away from the point the arc would be located. Once the user determines the correct setting the compass is turned around its anchoring point so that the pencil creates a mark, an arc, or a full circle.



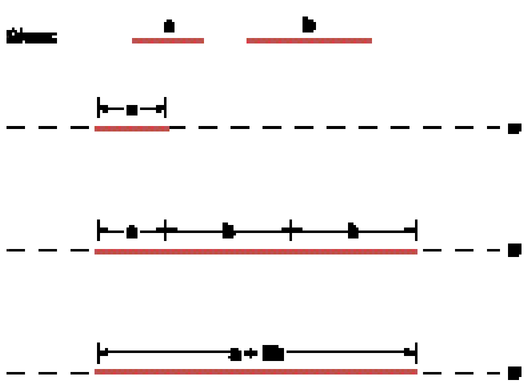
Protractor. A protractor is a geometric tool in the shape of a semicircular disk, as shown above on the right. It is used to measure the size of an angle in degrees—usually from 0 to 180 degrees. To use a protractor, lay the protractor on the angle to be measured. There will be a mark on the bottom of the protractor (the straight edge of it) indicating its middle. Place this mark over the origin of the angle, and align the straight edge of the protractor with one side of the angle. Where the other side of the angle intersects the protractor there will be a mark with a number next to it. This measurement is the measure of the angle.

Solving Construction Problems

Construction problems are generally solved by following six steps.

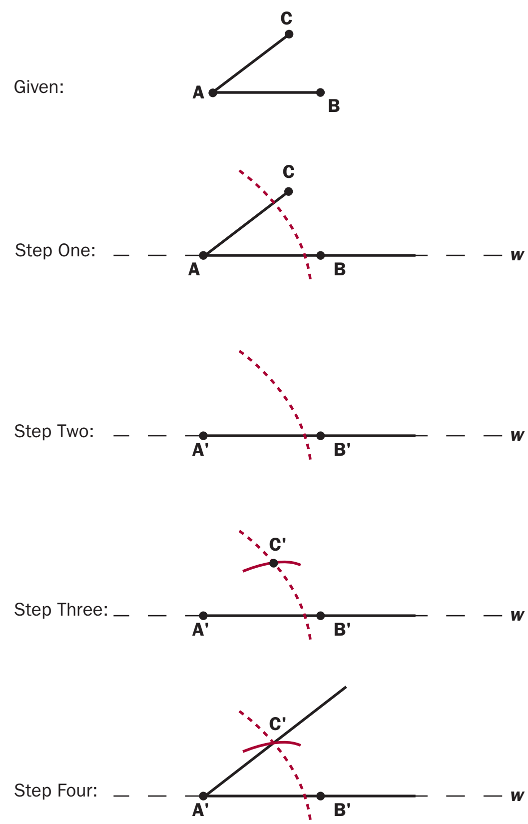
1. Provide a general statement of the problem that describes what is to be constructed.
2. Draw a figure representing the given parts.
3. Provide a statement of what is given in step 2.
4. Provide a specific statement of the result to be obtained.
5. Develop the construction, with a description (reason) for each step.
6. Provide a statement proving that the desired result was obtained.

As an example using a straightedge: Given line segments a and b, construct the line segment $c = a + 2b$.



This example is solved as follows. On working line w, draw a line segment equal to the length of line segment a. At the right end of a draw a line segment equal to length of line segment b. At the right end of line segment $a + b$, draw a second line segment equal to length of b. The resulting line segment is the desired line segment $c = a + b + b = a + 2b$.

As an example of using a compass: Construct an angle equal to the given figure $\angle BAC$ with line segments AB and AC.



This example is solved as follows. In Step One, place the compass pivot point at A and adjust the compass width to be between A and B, then construct an arc that intercepts both line segments AB and AC. Leaving the compass width unchanged, draw a line **w** and place the pivot point at an arbitrary point A'. Pivot the compass from line **w**, forming an arc like in Step One (as shown in Step Two). The arc's intersection with line **w** is denoted B'.

Referring to Step One, now adjust the compass width to equal the distance between the arc's intersection with AB and AC. With this width, move the compass down to Step Two, place the pivot point at B'; and make an arc that intersects with the first arc, as in Step Three. The line segments $A'C'$ and and $A'B'$, as shown in Step Four, complete the new angle $\angle B'A'C'$ that is equal to the original angle $\angle BAC$.

William Arthur Atkins (with Philip Edward Koth)

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Germain, Sophie

French Mathematician
1776–1831

Sophie Germain's foundational work on Fermat's Last Theorem stood unmatched for more than 100 years.

Sophie Germain is remembered for her work in the theory of numbers and in mathematical physics. Germain was born in Paris to a father who was a wealthy silk merchant. She educated herself by studying books in her father's library, including the works of Sir Isaac Newton and the writings of mathematician Leonhard Euler.

When the École Polytechnique opened in 1794, even though women were not allowed to attend as regular students, Germain obtained lecture notes for courses and submitted papers using the pseudonym M. LeBlanc. One of the instructors, noted scientist Joseph-Louis Lagrange, became her mentor.

In 1804 Germain began to correspond with German mathematician Carl Friedrich Gauss, sending him discoveries she made in number theory. Among these was a limited proof of Fermat's Last Theorem, her best known contribution to mathematics. This theorem was finally proved in 1994 using her approach. Germain also corresponded with mathematician Adrien Marie Legendre, who used her suggestions in one of his publications.

In mathematical physics, Germain is known for her work in acoustics and elasticity. She won a prize from the French Academy of Sciences in 1816 for the development of mathematical models for the vibration of elastic surfaces. Subsequently, she was invited to attend sessions of the Academy of Sciences and the Institut de France, but because she was a woman, she could never join either group. SEE ALSO Euler, Leonhard; Fermat's Last Theorem; Newton, Sir Isaac.

J. William Moncrief

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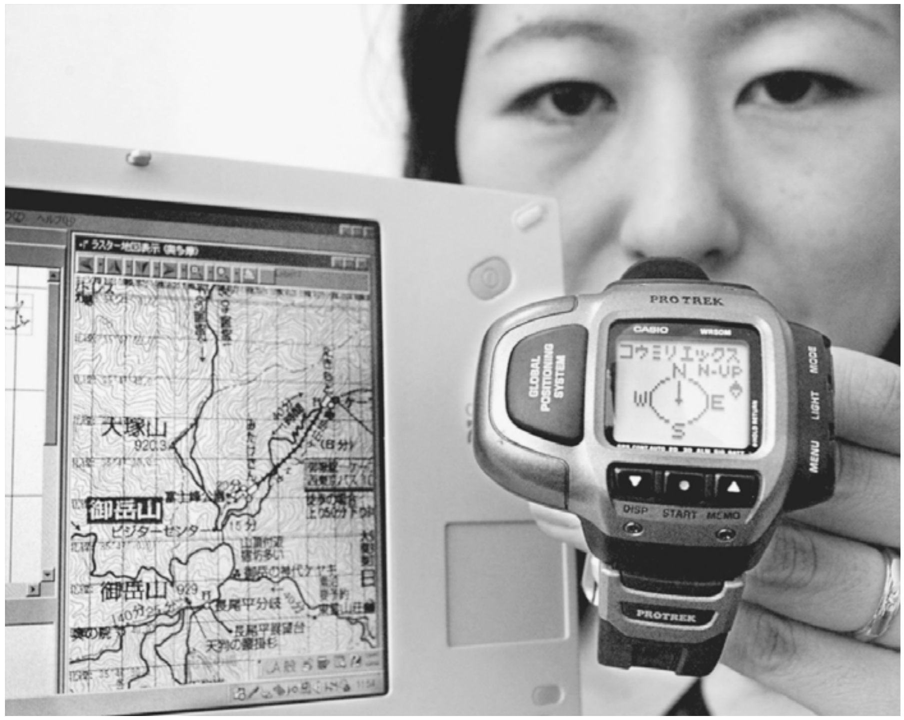
Global Positioning System

Most people have been lost at one time or another, but what if it were possible to know where you are, anywhere on Earth, 24 hours a day? The Global Positioning System (GPS) can give that information, and it is free to anyone with the proper equipment and a basic knowledge of mathematics.

In the 1980s, the U. S. Department of Defense designed GPS to provide the military with accurate, round-the-clock positional information. Twenty-seven satellites orbiting over 10,000 miles above Earth regularly send information back to Earth. A small piece of equipment, called a GPS receiver, uses this information to compute its position to within a few yards. GPS receivers used for surveying can find positions to within less than one centimeter.

The "constellation" of satellites above the Earth is constantly changing; each orbits Earth twice a day. At any given time there are enough satellite



This 84-gram wristwatch receives transmissions from twenty-seven Earth-orbiting Global Positioning System (GPS) satellites. The wearer's location can be precisely determined and portrayed on the digital map of a computer screen.

signals to accurately locate oneself in three dimensions: latitude, longitude, and elevation.

GPS is rapidly becoming a common technology, but it is still a mathematical wonder. Ancient sailors looked to the heavens to estimate their position in the vast oceans. Modern sailors also look to the sky for information, but the modern positioning information they receive is so accurate that any errors are less than the width of the pencil they use to mark their map.

Triangulation

The basic concept of GPS is triangulation. Suppose a person is standing in a valley surrounded by several towering mountain peaks. By using a compass to measure the direction to each peak, this person could locate his or her exact location on a map by using triangulation. After writing down the three measurements (remembering that there are 360 degrees in a circle), a line should be drawn from each peak in the opposite direction just measured.

Then 180 degrees is added or subtracted so that the direction the lines are drawn from each peak will fall between 0 and 360 degrees. For example, if one of the measurements is 270 degrees to peak A, the line from peak A back to the person's position would be 90 degrees. The point at which the three lines intersect is the point at which the person is standing.

The GPS satellites are like mountain peaks; they are known points in space from which lines can be drawn in order to specify a location. Each satellite transmits a radio signal that can be received on Earth and recognized by a GPS receiver. Rather than measure direction, however, a GPS receiver uses the time it takes for each satellite's beacon to reach it and calculates a distance.

Because radio waves travel at the speed of light, the receiver divides the time the signal takes to reach the receiver by the speed of light (186,000 miles per second) and determines the distance. These distances can be used to form spheres around the satellites that will intersect at a specific position just as the lines drawn from the mountain peaks will intersect at a specific position.

Understanding GPS Measurements

Assume a GPS receiver is sitting in Nebraska. Once activated it begins to collect signals from GPS satellites 1, 2, 3, 4, 5, and 6. The distance to each satellite can be determined using the distance formula $d = rt$ (distance, d, equals rate, r, multiplied by time, t, or distance equals velocity multiplied by time). Although all the satellites are 10,900 miles from the surface of the Earth, the distances to each one will vary according to its position in orbit. For example, all the street lights in a city may be 15 feet in the air but they are not all 15 feet from a specific point in the city.

The formula for determining these distances may be simple, but the calculations themselves are anything but simple. The satellites must be precisely timed so that each is synchronized with the other satellites in the constellation and with base stations on Earth. Although each satellite will be at a different distance from a particular point, the time it takes to cover those distances at the speed of light does not seem significant. In order to calculate distance, however, this time is significant to the GPS receiver.

Consider that a signal 10,900 miles from a receiver reaches that receiver in 0.058602 seconds. A signal 10,926 miles away, however, reaches the receiver in 0.0587419 seconds. A 26-mile difference translates into less than fourteen one hundred-thousandths (0.00014) of a second. Clearly, the GPS receiver has some very precise mathematics to work with, further complicated by the fact that the satellites are always moving.

After making these complex measurements, the distance to each satellite will be the hypotenuse of a right triangle created by the receiver's position, the satellite's position, and the position on Earth directly under the satellite. Once these distances are known, spheres can be created surrounding each satellite. Each sphere has a radius equal to the computed distance between the satellite and the receiver. The first sphere, around satellite 1, will have an infinite number of points along its surface so the receiver's position could fall anywhere on that sphere, including points in outer space.

Next, the sphere around satellite 2 is introduced, and the two spheres create an intersection that forms a circle. Now the GPS receiver could be anywhere on that circle, even points in space that, of course, it is not occupying. The third sphere, around satellite 3, intersects the first two spheres and limits the receiver's possible position to two points. The receiver is located at one of the two points. The other point is either in the air above the receiver or in the ground directly below the receiver. If the altitude of the receiver is known, then it is possible to determine which one of the two points is correct. The sphere around satellite number 4 will also reduce the two points to only one. It is amazing, yet basically simple, how one receiver and four satellites can reduce an infinite number of possible locations to only one.

APPLICATIONS OF GPS

Vehicle tracking is one of the fastest-growing GPS applications. GPS-equipped fleet vehicles, public transportation systems, delivery trucks, and courier services use receivers to monitor their locations. Many public service units are using GPS to determine the police car, fire truck, or ambulance nearest to an emergency.

Mapping and surveying companies use GPS extensively. GPS-equipped balloons are monitoring holes in the ozone layer. Buoys tracking major oil spills transmit data using GPS.

These are just a few examples. New applications will continue to be created as GPS technology continues to evolve.

Simulating GPS. To simulate the process of the Global Positioning System all that is needed is some string, scissors, tape, several coins, and four stationary points (the corners of a room will work). At any three-dimensional point in the room (on a desk, for example) a coin should be placed. The end of the string should then be taped into the corner of the room, with the other end pulled to the coin, cut, and then placed back in the corner. This process is repeated for the remaining three corners, and the extra coins are placed elsewhere in the room.

During this preparation, a volunteer waits outside the room. The volunteer should then enter the room and be alerted to the availability of the strings. The volunteer can then start pulling the cut ends of the string outward from the corners beginning with any two. By adding the third and fourth strings and finding where they all intersect, the volunteer should be able to eliminate all the extra coins and find the original coin. This is how GPS works in its most basic form.

Advantages and Disadvantages of GPS

Atmospheric inconsistencies can create inaccuracies in the positions computed by a GPS. Additionally, GPS is a "line of sight" system. Although a user cannot actually see the satellites in space, he or she does need an unobstructed view of the sky in order to utilize GPS. This poses serious challenges to those who choose to use GPS in canyons, cities, or other situations where large, solid objects mask out portions of the sky. When working where obstructions exist, careful planning must be done to ensure enough satellites are in "in view" for proper positioning.

Fortunately, GPS has a built-in feature, the almanac, to aid in identifying the location of satellites. Each satellite "knows" the location and direction of every other satellite. Along with the signal used to provide positions, satellites also transmit the almanac to a GPS receiver. Common GPS planning software can use the almanac to plot the entire constellation of satellites so users can plan ahead for their needs.

For example, if one needed to work in a canyon, planning software may indicate the only feasible time would be from noon until 2:00 P.M. Only during that time will the receiver have an unobstructed path to a sufficient number of satellites, all very high above the horizon, from within the canyon. SEE ALSO Flight, Measurements of; Maps and Mapmaking; Navigation.

Elizabeth Sweeney

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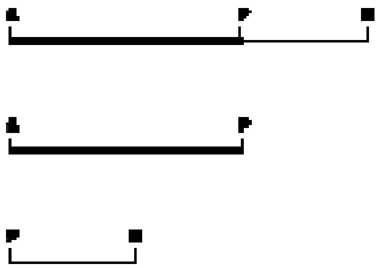
Golden Section

To understand what constitutes a golden section, consider the top line segment shown on the next page. The line segment has three of its points marked. Point P partitions the line segment AB into two smaller segments:

from left endpoint A to point P (AP), and from P to right endpoint B (PB). The line segments AP and PB are also shown individually.



A "golden ratio" can now be formed as the ratio of one line segment to another line segment. Point P divides the entire line segment AB into a golden section if the following equation is valid:

$$AP/AB = PB/AP.$$

In other words, the length of line segment AP divided by the length of the entire line segment AB is a ratio, or number, and that number is the same as the ratio of the shortest line segment PB and the segment AP. It turns out that these ratios are equal to the **irrational number** 0.61803. . .; that is:

$$AP/AB = PB/AP = 0.61803\ldots$$

Note that the decimal places continue indefinitely. The number 0.61803. . .is called the golden ratio or golden number, whereas the term "golden section" refers to the line segments formed by a point, such as P, whose resulting ratio is the golden number.

irrational number a real number that cannot be written as a fraction of the form a/b, where a and b are both integers and b is not 0

Significance of the Golden Ratio

The golden section and the golden ratio are important to mathematics, art, and architecture. The construction of the golden section goes back at least as far as the ancient Greeks, to the Pythagoreans (ca. 550 B.C.E.–300 B.C.E.). The Pythagoreans believed that numbers were the basis of nature and man, and therefore number relationships like the golden ratio were of utmost importance.

Besides line segments, the golden ratio also appears in many geometric figures. For example, a rectangle is said to be in golden section if its width and length are in the golden ratio (that is, if the width divided by the length equals 0.61803. . .).

Some scholars believe that various temples of the ancient Greeks, like the Parthenon in Athens, were purposefully produced with various dimensions in the golden ratio. Many of the artists and architects of the Renaissance period are believed to have incorporated the golden ratio into their paintings, sculptures, and monuments. A prime example is Leonardo da Vinci (1452–1519), who extensively used golden ratios (or their approximations) in his paintings. In Leonardo's famous drawing "Vitruvian Man," the distance ratio from the top of the head to navel, and from the navel to the soles of his feet approximates the golden ratio.

Many people feel that geometric forms and figures incorporating the golden ratio are more beautiful than other forms. Psychological studies pur-

ALTERNATIVE DEFINITION OF GOLDEN SECTION

Some dictionaries and textbooks define golden section (and number) as the inverse of the definition shown in this article. The formula then becomes (again referring to line segment AB) $AB/AP = AP/PB = 1.61803\ldots$.

Sometimes the larger value is denoted by the Greek letter "Phi" (i.e., Phi = 1.61803. . .) while the smaller value of the golden number is denoted by a "small" phi (i.e., phi = 0.61803. . .). Note that Phi = 1 + phi.

portedly show that people find golden-ratio rectangles more appealing than rectangles of other proportions. SEE ALSO Architecture; Leonardo da Vinci; Numbers, Forbidden and Superstitious; Pythagoras.

Philip Edward Koth (with William Arthur Atkins)

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Internet Resources

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Grades, Highway

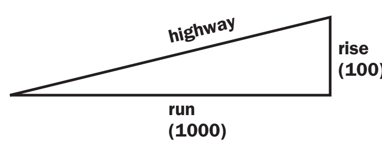
WHAT DEFINES A HIGHWAY?

A highway is a road with a hard surface, such as asphalt or concrete, that is open for the general public to drive upon. Highways connect towns, cities and industrial areas to one another. Most highways follow the "lay of the land." In other words, highways must rise and fall as hills and valleys are encountered.

The term "grade" is used in several different ways with respect to roads and highways. The "sub-grade" lies beneath a roadway and is used as a supporting base. "Grading" a road means to smooth out the roadbed with earth-moving equipment during the construction phase. In this article, the grade of a road is defined as a measure of the road's steepness as it rises and falls along its route. In other words, it is the magnitude of its incline or slope.

The grade of a highway is a measure of its incline or slope. The amount of grade indicates how much the highway is inclined from the horizontal. For example, if a section of road is perfectly flat and level, then its grade along that section is zero. However, if the section is very steep, then the grade along that section will be expressed as a number, usually a percentage, such as 10 percent.

The illustration below shows a highway in profile (from the side). Notice that a right triangle has been constructed in the diagram. The elevation, or height, of the highway increases in the sketch when moving from left to right. The bottom of the triangle is the horizontal distance this section of highway covers. This horizontal distance, sometimes called the "run" of the highway, indicates how far a vehicle would travel on the road if it were level. However, it is apparent that the road is not level but rises from left to right. This "rise" is a measure of how much higher a vehicle is after driving from left to right along the road.



To calculate the grade of a section of highway, divide the rise (height increase) by the run (horizontal distance). This equation, used to calculate the ratio of rise-to-run for highway grades, is the same ratio as the slope "y/x" encountered in a **Cartesian coordinate system**. In the example above, the rise of the highway section is 100 feet, while the run is 1,000 feet. The resulting grade is thus 100 feet divided by 1,000 feet, or 0.1.

Cartesian coordinate system a way of measuring the positions of points in a plane using two perpendicular lines as axes

Highway grades are usually expressed as a percentage. Any number represented in decimal form can be converted to a percentage by multiplying

that number by 100. Consequently, a highway grade of 0.1 is referred to as a "10 percent grade" because 0.1 times 100 equals 10 percent. The highway grade for a section of highway that has a rise of 1 kilometer and a run of 8 kilometers is $\frac{1}{8}$, or 0.125. To convert the highway grade into a percentage, multiply 0.125 by 100, which results in a grade of 12.5 percent.

Applications of Grade

When a new highway or road is being planned, its grade over its various sections is one of the key aspects that must be determined in advance. The grade of the road is very important for the safety of the motorists who will be using it. Most of us have experienced how a rolling bicycle has a tendency to pick up speed as it goes downhill, and how when going uphill the bicycle will slow down unless the cyclist pedals harder. The same situation must be considered when constructing modern highways.

Highway engineers plan carefully for the grade to be used in road construction.

If a grade is too steep, vehicle operators must use excessive braking when going downhill. In contrast, vehicle operators going uphill will have to slow down severely, possibly affecting traffic flow adversely. Grades are of great concern for vehicles carrying or pulling heavy loads, like semi-tractor trailers, or a family car pulling a camper or boat. To avoid these problems the highway engineers who plan and design highways pay close attention to road grades, and design into the highway's construction smaller grades rather than larger ones.

The concept of highway grades can also be applied to other types of pathways that are used by people or vehicles. For instance, residential roads are constructed to have a slight grade from the curb to the center of the road. A grade of 1 percent is considered the minimum required for proper drainage during a rainstorm. For a run of 10 feet, a 1 percent grade means one-tenth of a foot rise from the curb to the middle of the road.

Another situation where grade is an important consideration is in the construction of ramps for the disabled. Wheelchair ramps must have a fairly low grade, because if a ramp is too steep, people may be unable to use it. Nationwide, a generally accepted maximum grade for wheelchair ramps is 8.3 percent, or a ratio of 1:12. SEE ALSO Fraction; Percent; Slope.

Philip Edward Koth (with William Arthur Atkins)

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Gradient *See Circles, Measurement of.*

Graphs

A graph is a pictorial representation of the relationship between two quantities. A graph can be anything from a simple bar graph that displays the

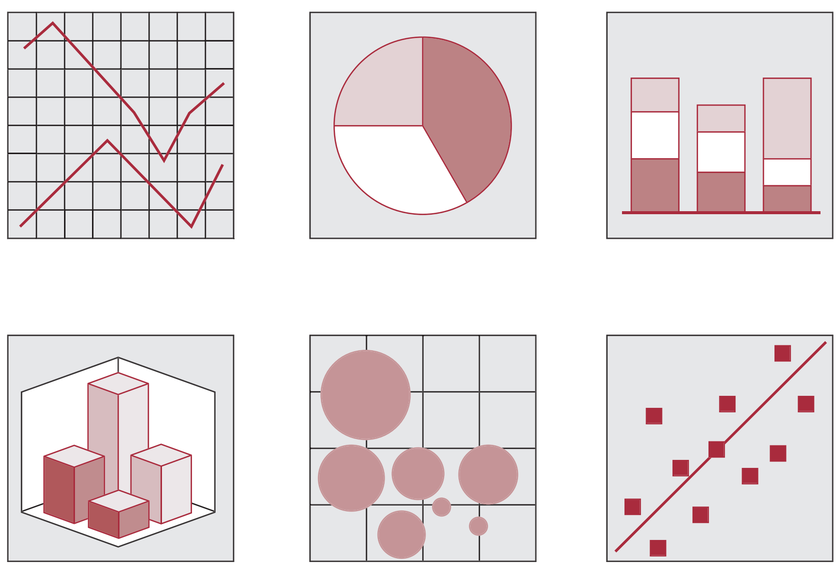
measurements of various objects to a more complicated graph of functions in two or three dimensions. The former shows the relationship between the kind of object and its quantity; the latter shows the relationship between input and output. Graphing is a way to make information easier for a viewer to absorb.

Types of Graphs

The simplest graphs show the number of many objects. For example, a bar graph might name the months of the year along a horizontal axis and show numbers (for the number of days in each month) along a vertical axis. Then a rectangle (or bar) is drawn above each month. The height of the bar might indicate the number of days in that month on which it rained, or on which a person exercised, or on which the temperature rose above 90 degrees. See the generic example of a bar graph below (top right).

Another simple kind of graph is a circle graph or pie graph, which shows fractions or percentages. In this kind of graph, a circle is divided into pie-shaped sectors. Each sector is given a label and indicates the fraction of the total area that goes with that label. See the generic example of a pie graph below (top middle).

A pie chart might be used to display the percentages of a budget that are allotted to various expenditures. If the sector labeled "medical bills" takes up two-tenths of the area of the circle, that means that two-tenths, or 20 percent, of the budget is devoted to medical expenses. Usually the percentages are written on the sectors along with their labels to make the graph easier to read. In both bar graphs and pie graphs, the reader can immediately pick out the largest and smallest categories without having to search through a chart or list, making it easy to compare the relative sizes of many objects simultaneously.



Often the two quantities being graphed can both be represented numerically. For example, student scores on examinations are often plotted on a graph, especially if there are many students taking the exam. In such a graph, the numbers on the horizontal axis represent the possible scores on

the exam, and the numbers on the vertical axis represent numbers of students who earned that score. The information could be plotted as a simple bar graph. If only the top point of each bar is plotted and a curve is drawn to connect these points, the result is a line graph. See the generic example of a line graph on the pevious page (top left). Although the points on the line between the plotted points do not correspond to any pieces of information, a smooth line can be easier to understand than a large collection of bars or dots.

Graphs for Continuous Data

Graphs become slightly more complicated when one (or both) of the quantities in the graph can have continuous values rather than a **discrete** set. A common example of this is a quantity that changes over time. For example, a scientist might be observing the rate of growth of bacteria. The rates could be plotted so that the horizontal axis displays units of time and the vertical axis displays numbers (how many bacteria exist).

discrete composed of distinct elements

Then, for instance, the point (3,1000) would mean that at "time 3" (which could mean three o'clock, or three seconds after starting, or various other times, depending on the units being used and the starting point of the experiment) there were one thousand bacteria in the sample. The rise and fall of the graph show the increases and decreases in the number of bacteria.

In this case, even though only a finite set of points represent actual data, the remaining points do have a natural interpretation. For instance, suppose that in addition to the point (3,1000), the graph also contains the point (4,1500) and that both of these points correspond to actual measurements. If the scientist joins all of the points on the graph by a line, then the point (3.5,1200) might lie on the graph, or perhaps the point (3.5,1350).

There are many different lines that can be drawn through a collection of points. Looking at the overall shape of the data points helps the scientist decide which line is the most reasonable fit. In the previous example, the scientist could estimate that at time 3.5, there were 1200 (or 1350) bacteria in the sample. Thus graphing can be helpful in making estimates and predictions.

Graphs for Predictions

Sometimes the purpose for drawing a graph may not be to view the data already known but to construct a mathematical model that will allow one to analyze data and make predictions. One of the simplest models that can be constructed from a set of data is called a best-fit line. Such a line is useful in situations in which the data are roughly linear—that is, they are increasing or decreasing at a roughly constant rate but do not fall precisely on a line. (See graph on the previous page, bottom right.)

A best-fit line can be a very useful tool for analyzing data because lines have very simple formulas describing their behavior. If, for instance, one has collected data up to time 5 and wishes to predict what the value will be at time 15, the value 15 can be inserted into the formula for the line to derive an estimation. One can also determine how good an estimate is likely to be by computing the **correlation** factor for the data. The correlation

correlation the process of establishing a mutual or reciprocal relation between two things or sets of things

factor is a quantity that measures how close the set of data is to being linear; that is, how good a "fit" the best-fit line actually is.

Graphs for Functions

One of the most common uses of graphs is to display the information encoded in a function. A function, informally speaking, is an operation or rule that can be applied to numbers. Functions are usually graphed in the cartesian plane (that is, the x,y-plane) with the horizontal or x-axis representing the input variable and the vertical or y-axis representing the output variable. The graph of a function differs from the other types of graphs described so far in that all the points on the graph represent actual information. A concrete relationship, usually given by a mathematical formula, connects the two objects being analyzed.

For example, the "squaring" function takes numbers and squares them. Thus an input of the number 1 corresponds to an output of 1; an input of 2 corresponds to an output of 4; an input of -7 corresponds to an output of 49; and so on. Therefore, the graph of this function contains the points (1, 1), (2, 4), (-7, 49), and infinitely many others.

Does the point (10, 78) lie on this graph? To determine the answer, examine which characteristics all the points on the graph have in common. Any point on the graph of a function represents an input-output pair, with the x-coordinate representing input and the y-coordinate representing output. With the squaring function, each output value is the square of the corresponding input value, so on the graph of the squaring function, each y-coordinate must be the square of the corresponding x-coordinate. Because 78 is not the square of 10, the point (10, 78) does not lie on the graph of the squaring function.

Graphing Notation

It is traditional to name graphs with an equation rather than with words. The equation of any graph, regardless of whether it is the graph of a function, is meant to be a perfect description of the graph—it should tell the viewer the relationship between the x- and y-coordinates of the numbers being graphed.

For example, the equation of the graph of the squaring function is $y = x^2$ because the y-coordinate of any point on the graph is the square of the x-coordinate. The line that passes through the point (0, 3) and slants upwards with slope 4 (that is, at a rate of four units up for every one unit to the right) has equation $y = 4x + 3$. This indicates that for every point on the graph, the y-coordinate is 3 more than 4 times the x-coordinate.

An equation of a graph has many uses: it is not only a description of the graph but also a mechanism for finding points on the graph and a test for determining whether a given point lies on the graph. For example, to find out whether the point (278, 3254) lies on the line $y = 4x + 3$, simply insert (278, 3254), resulting in the inequality $3254 \neq 4(278) + 3$. Because these numbers are not equal, the point does not lie on the line. However, the equation shows that the point (278, 1115) does lie on the

line. SEE ALSO Data Collection and Interpretation; Graphs and Effects of Parameter Changes; Statistical Analysis.

Naomi Klarreich

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Graphs and Effects of Parameter Changes

The two-dimensional **Cartesian coordinate system** may be used to graph a variety of equations in the form of straight and curved lines. One way to graph an equation is to determine a number of different values for the variables and plot them on the graph. It can be helpful, however, to understand how a change in the **parameters** of an equation affects the resulting line.

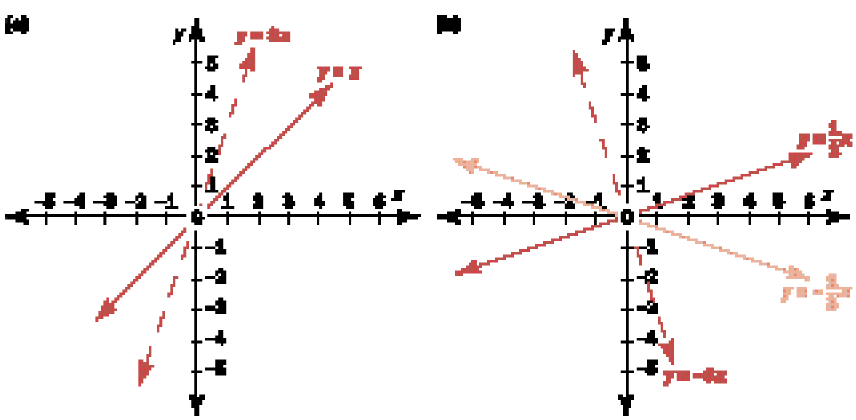
Cartesian coordinate system a way of measuring the positions of points in a plane using two perpendicular lines as axes

parameters independent variables which can be used to rewrite an expression as two separate functions

Graphs of Straight Lines

The graph of the simple equation $y = 1x$, or $y = x$, is graphed in (a) below. The line passes through every coordinate point where $x = y$, such as (2, 2) or (-3, -3).

Notice what happens to the graph of the equation when the parameters of the equation are changed by using a different coefficient for x, such as $y = 3x$. As part (a) shows, the new line has a steeper slope. If the coefficient is further increased, the slope will become even steeper, and the line will become closer to vertical.

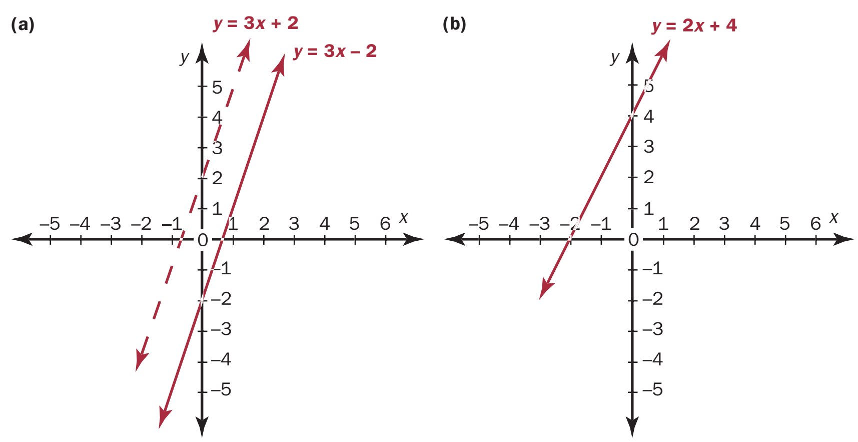


Conversely, if the coefficient for x is decreased, a different change in the line occurs. As part (b) above shows, the line for $y = \frac{1}{3x}$ has a more gradual slope than the line for $y = x$. If the coefficient were to be further decreased, the line would become flatter, and it would appear more like a horizontal line.

If the coefficient is a negative number, an interesting line results. For instance, the graph of the equation $y = -3x$ is shown in part (b) above. Comparing this line with the graph for $y = 3x$ shows that the graph of $y = -3x$ moves down from the left to right (a negative direction) and the graph of $y = 3x$ moves up from left to right (a positive direction). These two lines therefore are described as having a negative and a positive slope, respectively. From this example, it can be seen that the coefficient of x in the equation of a line indicates the slope of the line.

Lines Not Passing through the Origin. All of the lines described above pass through the origin (0, 0), but many lines do not. The line for the equation $y = 3x + 2$ will pass *above* the origin, as shown in part (a) below. The line has the same slope as $y = 3x$, but has been shifted two units *up* the y-axis. The two lines are thus parallel because they have the same slope.

The equation $y = 3x - 2$, also graphed in (a), passes through the y-axis at -2. The slope of the line is 3, which is the same the slope for $y = 3x$. But compared to $y = 3x$, the line for $y = 3x - 2$ has been shifted two units *down* the y-axis.



Understanding the role of the coefficient of x in indicating the slope, and the role of the constant in indicating the point where the line will intersect the y-axis, can make it easier to graph an equation. The equation $y = 2x + 4$ will pass through the y-axis at 4 and has a slope of 2, as shown in part (b) above.

The equation $y = 2x + 4$ may be expressed in different forms. For example, $x = \frac{1}{2}y - 2$. The form $y = 2x + 4$, written in general terms as $y = mx + b$, is called the slope intercept form. In this form, the coefficient m is the slope, and the constant b indicates where the line intersects the y-axis. The intersection point is called the y-intercept. The graph of the equation $y = 3x - 4$ has a slope of 3 and intersects the y-axis at -4.

Graphs of Curved Lines

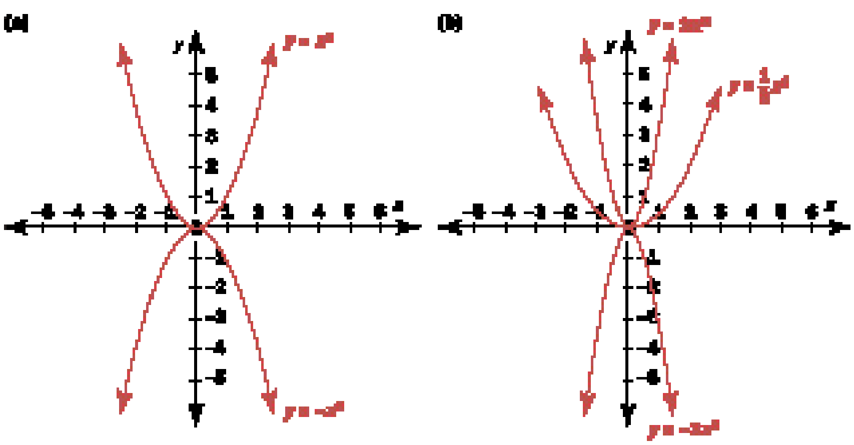
Changes in the parameters of higher-degree equations (that is, higher than first-degree) will result in patterns of changes in their graphs similar to those for straight lines. For example, the graph of $y = x^2$, as shown in part (a) below, is a curve known as a parabola. The coefficient indi-

cates whether the parabola opens facing up or facing down, and whether it is narrow or broad.

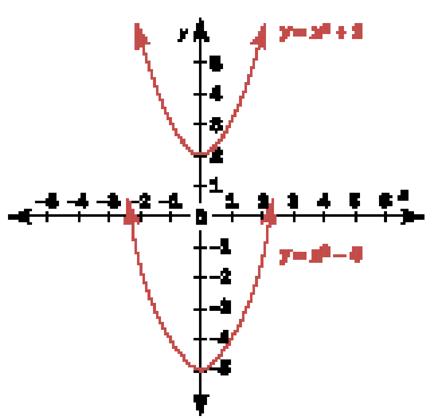
Changing the coefficient of x^2 to a negative value results in a parabola that is a mirror-image of the parabola for $y = x^2$. The parabola for $y = x^2$ opens upward, and the parabola for $y = -x^2$ opens downward.

If the coefficient of x^2 is changed so the equation is $y = 3x^2$, notice the changes to the graph of the equation, as depicted in part (b) below. The parabola still passes through the origin, but it is narrower than $y = x^2$. If the coefficient of the graph were increased even more, then the parabola would become even narrower. Conversely, if the coefficient of the original equation were decreased, the resulting parabola would still pass through the origin, but would become broader, as depicted in (b) below as $y = \frac{1}{2}x^2$.

Compare the parabola in (b) below for $y = 3x^2$ to the parabola for $y = -3x^2$. As with the equations $y = x^2$ and $y = -x^2$, one parabola is the mirror image of the other. The two parabolas open in opposite directions.



Curves Not Passing through the Origin. Changing a parameter of the equation $y = x^2$ to $y = x^2 + 2$ will change the point at which the parabola crosses the y-axis. In this case, the parabola does not pass through the origin, but passes through the y-axis at 2, as shown below. The parabola has been shifted two units *up* the y-axis. Similarly, the parabola of $y = x^2 - 5$ is shifted 5 units *down* the y-axis, and passes through the y-axis at -5.



In the general equation for a parabola, $y = ax^2 + b$, the coefficient a indicates the shape of the parabola and in which direction the parabola opens, upward or downward. The value of b indicates where the parabola intersects the y-axis.

More complex equations and their graphs will also show patterns that result from changes in the parameters of the equations. Knowing how changes in the various parameters of an equation affect the graph of the equation is helpful for drawing, interpreting, and applying equations and their graphs. SEE ALSO Graphs.

Arthur V. Johnson II

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Heating and Air Conditioning

H

In the late nineteenth and early twentieth centuries, heating a home was a matter of tossing another log or more coal into the fireplace or stove. "Air conditioning" was the shade of a backyard tree. Twenty-first century homes use far more sophisticated technological tools—tools that a homeowner is likely to take for granted.

A heating and air conditioning contractor has to be as much of a scientist as a builder or technician. The course requirements of typical heating and air conditioning programs at community or junior colleges place a heavy emphasis on chemistry, physics, engineering, and mathematics, including **geometry** and **algebra**. Good heating and air conditioning contractors are able to compute formulas and equations in order to arrive at volumes, pressures, and degrees. They must be able to accurately measure distances, angles, circles, arcs, temperatures, weights, and volumes. They also must identify and interpret geometric figures, graphs, scales, and gauge indications. Further, they must know the scientific principles that are central to their work, including heat transfer, **combustion**, temperature, pressure, electricity, and magnetism. They should also know the physical and chemical properties of commonly used substances such as **refrigerants** and **hydrocarbons**.

geometry the branch of mathematics that deals with the properties and relationships of points, lines, angles, surfaces, planes, and solids

algebra the branch of mathematics that deals with variables or unknowns representing the arithmetic numbers

combustion chemical reaction combining fuel with oxygen accompanied by the release of light and heat

refrigerant fluid circulating in a refrigerator that is successively compressed, cooled, allowed to expand, and warmed in the refrigeration cycle

hydrocarbon a compound of carbon and hydrogen

exergy the measure of the ability of a system to produce work; the maximum potential work output of a system

anergy spent energy transferred to the environment

The Science of Cooling

The science of cooling is rooted in the Second Law of Thermodynamics, which states that heat only flows from higher to lower temperature levels, and never the other way around. Using this law, physicists can explain exactly how an air conditioner (or a refrigerator, which uses the same process) works. They can also use related principles, such as those of **exergy** and **anergy**, to design better, more efficient cooling systems.

How does a typical air conditioner work? It lowers the temperature by continuous extraction of heat energy using a thermodynamic cycle. The most common of these cycles is called the vapor-compression refrigeration cycle, sometimes called the Rankine cycle. In this cycle, a substance known as the "working fluid," or refrigerant, goes through cyclical changes of state in a closed loop. This loop is made up of four parts: an evaporator, a compressor, a condenser, and a throttle valve. The evaporator is installed in the space to be cooled while the other parts are installed outside of the space.

Before it enters the evaporator, the working fluid is a liquid or liquid-vapor mixture. Its pressure is low, and its temperature is below that of the space to be cooled, also called the "cold room." In the evaporator, the fluid takes up heat from the cold room because of the temperature difference. The fluid-vapor is then brought from low to high pressure in the compressor, which increases its temperature. In a well-designed system, the temperature of the fluid-vapor leaving the compressor should be above that of the surroundings, providing the temperature difference necessary for removing heat from the fluid to the surroundings. This occurs in the condenser, where the fluid undergoes a phase change from vapor to liquid because heat has been removed. The loop is closed by the throttle valve, where the fluid is expanded from the high condenser pressure to the low evaporator pressure.

One issue that scientists struggle with is finding a good working fluid. Water would be, at first glance, the ideal fluid because it is inexpensive and safe. Its thermodynamic properties, though, prevent it from being the best choice. Water vapor has a very low density, so using it would require huge piping volumes and a lot of work on the part of the compressor. **Chlorofluorocarbons** (CFCs) work better, but unfortunately they contribute to ozone depletion in the stratosphere. In large industrial refrigeration plants, ammonia (NH_3) is the most common choice.

chlorofluorocarbons compounds similar to hydrocarbons in which one or more of the hydrogen atoms has been replaced by a chlorine or fluorine atom

Contractors who install air-conditioning systems are probably not thinking about the vapor-compression refrigeration cycle. They are, however, thinking about what size air conditioner is needed to keep you comfortable on hot summer days. Factors that must be measured and taken into account in deciding on the size of an air-conditioning system are:

- the geographical location, and therefore average temperatures during the cooling season;
- the length of walls in the rooms, including walls not exposed to direct sunlight, those that are exposed to direct sunlight, and interior walls;
- the type of wall frame construction (framing, masonry, etc.);
- the ceiling height;
- the ceiling area, and the presence and amount of insulation above the ceiling;
- the space's floor area;
- the width of doors and arches;
- the window area, the orientation of the windows (north, south, etc.), and the type of glass (single-pane, double-pane, block);
- the number of people who normally occupy the room (giving off body heat);
- the amount of heat given off by lights and appliances; and
- the hours of operation.

Taken together, these factors determine the cooling capacity that is needed, usually measured in British thermal units (BTUs); one BTU is the amount of energy needed to raise the temperature of a pound of water one degree Fahrenheit.

The Science of Heating

In many respects the science of heating a home is much simpler than cooling: Unless you have solar heat or a windmill, something somewhere gets burned, and the heat is transferred into your living space either directly (as is in the case of natural gas, propane, or heating oil) or indirectly (as in the case of electricity). When a furnace runs, it ignites the fuel with burners that heat up the heat exchanger. A blower moves air across the heat exchanger, and the warm air is then circulated through the living area by a duct system. Fumes from the burned fuel are expelled through a flue (a pipe designed to remove exhaust gases from a fireplace, stove, or burner).

To determine how big a furnace needs to be, the same information listed for air conditioning is necessary. The result of these calculations is what a heating contractor calls a "heat load," which is measured in BTUs. For example, a contractor might determine that a house's heat load is 61,000 BTUs and that an 80,000 BTU furnace running at 80 percent efficiency, providing 64,000 BTUs, is a good approximate fit. A problem with a higher efficiency furnace is that it can often "short cycle," meaning that it will turn off before all the cold air in the home has cycled through the furnace.

The simplest type of furnace is called a "single-stage" furnace. This means that the furnace is either on or off and the fan blower is adjusted to a single setting that provides the optimum amount of heat based on the home's heat load. Most furnaces, however, are "two-stage" furnaces, and these provide more comfort and typically a higher efficiency.

When a two-stage thermostat senses that a room is cold and sends an electrical signal to the furnace to run, the furnace operates at two-thirds strength. If after a set amount of time the thermostat is still calling for heat, the furnace switches to 100 percent capacity. This gives the furnace's blowers time to circulate warm air throughout the house. If the furnace operated at 100 percent right away, it might circulate enough hot air to satisfy the thermostat but not enough to warm colder pockets in other areas of the house.

Michael J. O'Neal

FUZZY LOGIC AND THERMOSTATS

Early generations of electronic controls, including the thermostats on furnaces and air conditioners, were always either on or off. Today's controls are computerized, thereby enabling them to use "fuzzy logic," or a computer's ability to "learn" and "think" in shades of gray rather than black-and-white.

Thus, instead of turning on a furnace all the way when the temperature in a building falls to a certain point, a fuzzy logic thermostat turns the heat on just a little as the temperature approaches a specific setting. This feature helps maintain the temperature at a steady, constant value and avoids cycles of chilliness followed by blasts of hot air.

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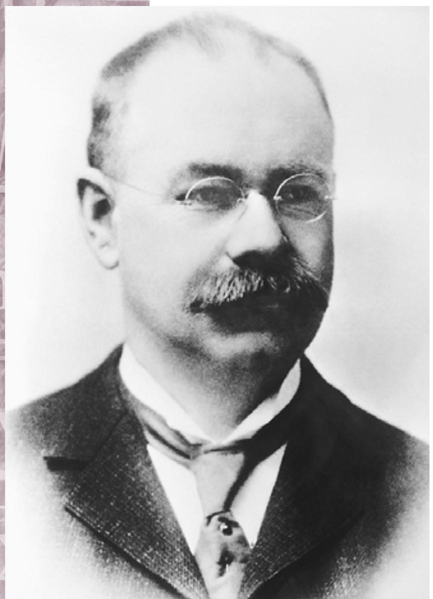
Hollerith, Herman

American Mathematician and Inventor
1860–1929

The presidential race between Texas Governor George W. Bush and U.S. Vice-President Al Gore in November 2000 illustrated the importance of accurately counting every vote. The hand recounts also revealed how collecting and interpreting vast amounts of data by hand can pose many difficulties,



Herman Hollerith invented the tabulating machine, and his company, through mergers, eventually became IBM.

binary existing in only two states, such as "off" or "on," "one" or "zero"

digital describes information technology that uses discrete values of a physical quantity to transmit information

including, but not limited to, error and bias. Herman Hollerith's invention of a tabulating machine was the first attempt to solve these problems.

During the 1880 U.S. census, the logistical problems of gathering and tabulating great amounts of data quickly enough for the data to remain useful was first recognized. In fact, the data from the 1880 census took nearly seven years to tabulate, far too long for the results to accurately reflect statistics that were needed for determining seats in the U.S. House of Representatives. The U.S. Census Bureau devised a competition in which a prize would be awarded for the best method of tabulating census data.

Herman Hollerith, a former statistician for the U.S. Census Bureau and professor of mechanical engineering at the Massachusetts Institute of Technology, won the competition. His work had led him to an interest in the mechanical manipulation of data. It seemed to lend itself perfectly to being automated. He edged out two competitors who invented "chip" and "slip" systems with his "card" system. His tabulating machine was put to use in the very next census in 1890.

Hollerith's "integrating machine" punched holes into stiff paper cards similar to the ones he had seen conductors using on trains. By punching holes next to the appropriate descriptors on the card, such as "large nose" or "dark hair," the conductor could create a profile of each passenger. Hollerith designed his cards to be the same size as dollar bills so that storage cabinets for currency could also be used for the "Hollerith cards."

Hollerith's system was essentially a **binary** system. "Punched" and "not punched" corresponded to the 1s and 0s we are familiar with in the twenty-first century's **digital** data storage systems. The cards were run under a group of brushes that completed electrical circuits when a "punch" was encountered. A corresponding mechanical counter then advanced for each punch, and in this way counted the total.

The Hollerith tabulating device allowed the 1890 census to be completed in only six weeks and at a savings of $5 million, an almost unbelievable improvement. In addition, data could be sorted and organized based on selected characteristics with little additional effort. More data than ever before could be collected and analyzed.

Fueled by the success of his machine, Herman formed the Herman Hollerith Tabulating Machine Company in 1896. However, his machine was so expensive the Census Bureau developed their own system for the 1910 census. Competition forced Hollerith to merge with another company, and the Computing-Tabulating-Recording Company was created in 1911. Thomas J. Watson later reorganized it into International Business Machines (IBM) Corporation. The success of Hollerith's machine was the basis of IBM's success and has led him to be remembered as a founder of information processing. SEE ALSO Mathematical Devices, Mechanical; Census.

Laura Snyder

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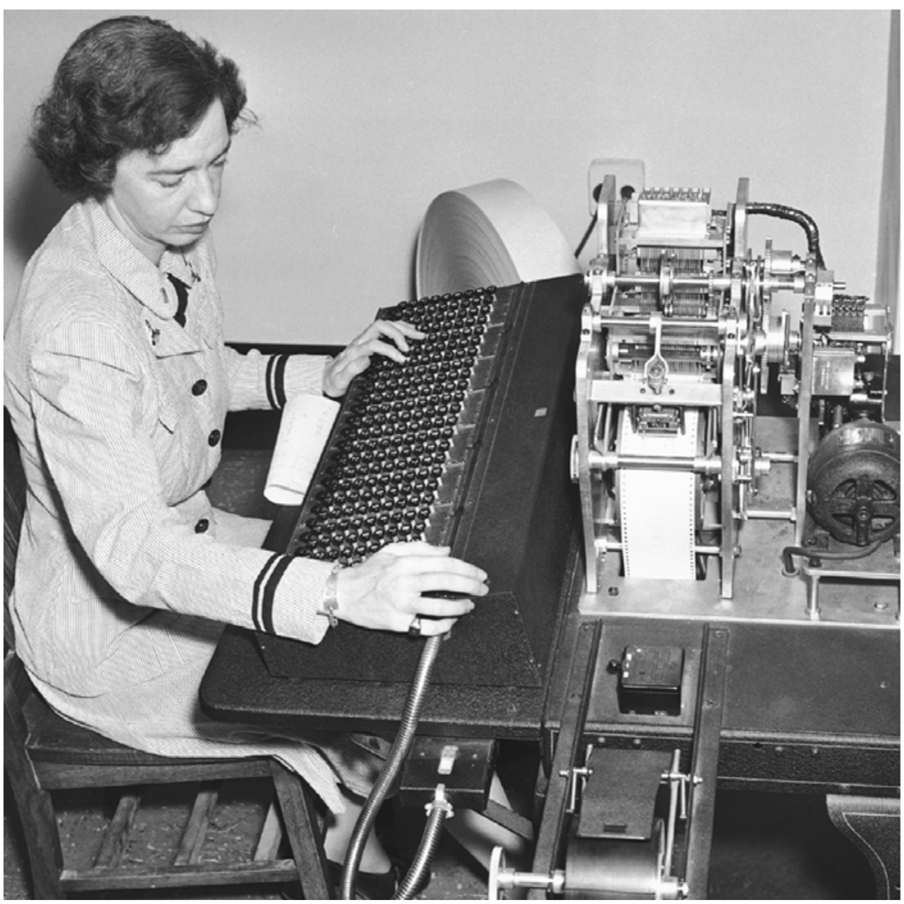
Hopper, Grace

American Mathematician and Computer Programming Pioneer
1906–1992

American mathematician and computer pioneer Grace Murray Hopper was called "Amazing Grace" by coworkers because of her determined ways. Born to parents who believed in quality education, Hopper was fascinated with gadgets, and she disassembled clocks and built vehicles with her "Structiron" kit. She was strongly influenced by her parents, and described her mother as having a "very great interest in mathematics" and her father as having "a house full of books, constant interest in reading, and insatiable curiosity."

Hopper entered Vassar in 1924 to study mathematics and physics and graduated with a bachelor's degree. She performed mathematics research at Yale and earned her master's degree in 1930 and her doctorate degree in 1934. During this era, these were rare achievements, especially for a woman.



Grace Hopper is best known for her contribution to the design and development of the COBOL programming language for business applications. She is shown here working on an early form of computer.

HOW DID COMPUTERS GET "BUGS?"

Grace Hopper was at Harvard University in 1945 when, while working on the Mark-II, she discovered a hardware failure caused by a trapped moth. She coined the now common term "bug" to refer to unexpected computer failures.

Hopper taught mathematics at Vassar and continued her tenure there as a professor until 1943. Her unusual methods applied mathematics to real life. She required her probability students to play dice and instructed students to plan a city by managing expenses.

Hopper's great-grandfather, a Navy rear-admiral, was her personal hero. When the United States entered World War II in 1941, she was crestfallen when the U.S. Navy would not accept women. By 1943 a shortage of men allowed women to enter the ranks. Hopper eagerly joined, but was rejected because she was too old, did not weigh enough, and was considered essential to the war effort as a civilian professor of mathematics. Undaunted, Hopper convinced the Navy to accept her, and in 1943, she started officer training, graduating at the top of her class. Assigned to Harvard's Computation Project, she worked with Howard Aiken on the Mark-I, which is considered one of the first programmable **digital** computers.

digital describes information technology that uses discrete values of a physical quantity to transmit information

Post-War Years

In 1946 Hopper ended her Navy duty but remained a reservist and was appointed a Harvard research fellow and continued work on the Mark-II and Mark-III computers. Even though colleagues said only scientists had enough knowledge to use computers, Hopper was undaunted and continued to write programs that made computers easily accessible.

In 1949 Hopper joined the Eckert-Mauchly Corporation as a senior mathematician where she worked on UNIVAC (UNIVersal Automatic Computer), the first computer to handle both numeric and textual information. Her original staff was comprised of four men and four women. Hopper liked hiring women, she said, because "Women turn out to be very good programmers for one very good reason. They tend to finish up things, and men don't very often finish."

During this time, Hopper designed an improved **compiler** that translated instructions from English commands to **machine language**. This reduced the need for writing tedious machine code. She finished the A-O compiler in 1952 using easy terms like SUB (for subtraction) and MPY (for multiplication).

compiler a computer program that translates symbolic instructions into machine code

machine language electronic code the computer can utilize

In 1957 Hopper developed FLOW-MATIC, the first commercial, data-processing compiler that allowed computers to be used for automated billing and payroll calculation. FLOW-MATIC became the foundation for Hopper's next development in 1959, the computer language COmmon Business Oriented Language, or COBOL. Whereas IBM's FORTRAN programming language used a highly condensed, mathematical code, COBOL used common English language words. COBOL was written for use on different computers and was intended to be independent of any one computer company. For her wide-reaching influence on COBOL's development, Hopper was deemed the "grandmother of COBOL."

Later Honors

Hopper retired from the Navy in 1966 but was recalled to help standardize computer languages. In 1969 she was named the first computer science "Man of the Year" by the Data Processing Management Association and was known among coworkers as "the little old lady who talks to computers."

Her office contained a skull-and-crossbones flag and a clock that ran backward to remind people to use flexible thinking.

Grace Hopper retired in 1986 as the oldest Naval officer on active duty. In 1991 President George Bush awarded Rear-Admiral Hopper the National Medal of Technology, saying she was the first woman to receive America's highest technology award as an individual, and recognizing her "as a computer pioneer, who spent a half century helping keep America on the leading edge of high technology." Throughout her life, Hopper held to her belief that "Most problems have more than one solution." SEE ALSO Computers, Evolution of Electronic.

William Arthur Atkins (with Philip Edward Koth)

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Human Body

Humans come in different shapes and sizes. There is no shape or size that is considered "right." Medical professionals have created healthy guidelines for people. When a baby is born, doctors track many variables to make sure it is developing correctly. These charts and guidelines are not perfect, but they form a framework from which to assess a person's health.

Tracking Growth

Over the years, physicians have developed various methods for tracking growth. Keeping a record of a child's growth patterns helps doctors determine if the child is developing properly. As a person ages, doctors continue to monitor growth, helping patients maintain a healthy weight. Physicians also review height to check for problems of the spine, bone, and other medical conditions.

Growth Charts. Pediatric growth charts have been used by medical professionals since 1977. The charts are used for many measurements of growth in a child. For example, when a child's height is measured, she is then placed into a percentile. If a child is in the seventy-fifth percentile in height, it means that she would be taller than 75 percent of all other children in her age group.

Originally, the sampling for growth charts was taken from a small portion of the population. These children were primarily Caucasian, formula-fed, and middle-class. This sample was problematic because it did not reflect the diversity of the United States. To help make charts that were representative of the whole, and in turn were more accurate, samples were taken

from a much larger portion of the population and from people with varying backgrounds.

scaling the process of reducing or increasing a drawing or some physical process so that proper proportions are retained between the parts

There are various mathematical concepts behind growth charts. **Scaling** is one important factor in charting growth in an infant. Though often associated with architecture, humans also undergo scaling. While an infant is much smaller than an adult, the support material remains the same: bone. Bone is a type of tissue that can only support a limited degree of force due to strain. This is evident in the number of broken bones humans can experience when faced with too much strain—for instance, as a result of activities such as sports. As humans grow, their bones also grow but only with a narrow tolerance for strain.

sampling selecting a subset of a group or population in such a way that valid conclusions can be made about the whole set or population

outliers extreme values in a data set

✶The tallest man in documented medical history was Robert Wadlow, who at 22 years old stood 8 feet, 11.1 inches tall, weighed 440 pounds, and wore a size 37AA shoe.

"Average" Height. The determination of "average" human dimensions is based on statistics and illustrates the statistical concepts of **sampling**, central tendency, and **outliers**. For example, rare genetic or medical conditions can cause a small percentage of the human population to be very tall or very short. Persons diagnosed with gigantism✶ may reach adult heights of more than 8 feet, whereas persons with dwarfism are commonly less than 4 feet, 10 inches tall.

Because the heights of "average" people are statistically determined by *excluding* individuals with these rare conditions, the physical dimensions of tall or short people are therefore deemed "disproportionate" or "outside the average." Yet if the sampling set included *only* individuals with these special conditions, and excluded everyone else, then their statures would be considered well within average. Hence, what is considered average depends on the sampling and calculation methods by which it is derived.

Many consumer products are built for "average" people—they are constructed for persons within an average height and weight range and who are not physically challenged. Such construction can cause difficulties for those who are not "average." As a result, some manufacturers offer products to help people make the necessary adjustments. For example, some people could not drive automobiles comfortably without enhancements like adjustable foot pedals, which bring the brake and accelerator pedals closer or farther away from drivers. This allows shorter and taller people to drive more easily, keeping a safe distance from the car's steering wheel airbag. Gadgets also exist to help disabled drivers, including hand controls that can be used to brake or accelerate.

Allometry. Another mathematical principle known as allometry (from the Greek word *alloios*, which means different) is especially important in the study of how humans grow. Biological allometry and human anatomy are concerned with the different growth rates that an organism experiences throughout its lifetime. A human infant does not have the same proportions of limb-to-head-to-body ratios as a human adult. The head of an infant compared to its body size is much different from an adult's. As a human grows, limbs and body size increase considerably while the head does not. This change in ratio occurs in all kinds of animals. A baby horse is born with extraordinarily long legs compared to its body. As it grows, the body catches up with the legs.

An entire branch of science deals with the allometric changes of organisms. One of the modern methods of observing allometric change is to

place imaginary points on the organism, in this case, a human infant's head. A computer scans for the imaginary points and constructs a grid system. As the infant grows, these points are continually tracked, and the points shift in position relative to one another. When the infant reaches adulthood, it is possible to see how the skull grew, where it grew the fastest and the slowest, and how the proportions have changed. The initial square grid looks wavy and distorted. Changes in body shape and physiology, such as in bone structure, breathing rate, and muscle strength can be expressed as variables and tracked. The basic allometric equations used to monitor these changes are as follows:

$$y = a \times x^b \text{ and } \log y = \log a + b \log x.$$

This equation is used by biologists to plot two **variables** on logarithmic coordinates (using logarithmic scales on the x and y axes of a Cartesian coordinate system). The result is a straight line. Many biological measurements that relate to change in body size use this general equation. The exponent b represents the **slope** of the line. This equation can help biologists track the changes in variables (the arbitrary points used to make an initial grid). Computers can track the three-dimensional (3-D) changes that occur with growth and produce allometric 3-D scaling grids.

There is no such thing as an ideal weight. Instead, every person has a range of what is healthy for her based on a number of variables, including age, height, sex, and muscle mass.

How Much Should a Person Weigh?

The new indicator used for weight measurements is the Body Mass Index (BMI). Formerly, weight-for-stature charts had been used when assessing weight. With this type of chart, a person's weight was evaluated only in relation to one variable, his or her height. When looking at a person's BMI, other factors, such as age, are considered.

A BMI is the fraction of a person's weight divided by his height squared. If a person is between the range of 20–25, then he is said to fall within a healthy weight range. Once a person goes above a BMI of 25, he may suffer from weight-related health problems. Additionally, people with a BMI of less than 20 may also suffer health problems from being underweight. The BMI is the preferred way to chart weight because it looks at more variables than how much a person weighs and what his height is. The body mass index can also help doctors be aware of people who may be more predisposed to being either overweight or underweight.

While all growth charts are helpful in determining development, they really serve only as guidelines. There is no one correct height or weight for a person, only a range of what is healthy. Growth charts do not serve the purpose of locating an "ideal." Instead, they help to indicate whether someone is healthy. SEE ALSO Dürer, Albrecht; Leonardo da Vinci; Logarithms; Ratio, Rate, and Proportion; Scale Drawings and Models.

Brook E. Hall

variable a quantity that may assume any one of a set of values known as the domain

slope the ratio of the vertical change to the corresponding horizontal change

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Hypatia

A Byzantine church historian wrote that Hypatia was so learned in literature and science that she exceeded all philosophers of her time.

Greek Philosopher and Mathematician
370–415 C.E.

Hypatia of Alexandria was a leading mathematician and philosopher of the ancient era. Her father, Theon, was the last known head of the Museum at Alexandria, Egypt, an ancient center of classical learning. He tutored Hypatia and passed on to her his love of mathematics. Eventually her reputation as a mathematician exceeded and outlasted his.

Only fragments of Hypatia's work and a list of several titles of her treatises on mathematics remain. She translated and popularized the works of Greek mathematicians, including Diophantus's (third century) *Arithmetica*, a book noted for **integral solutions** to **linear equations**, and Apollonius's *Conic Sections.* Hypatia also edited the *Almagest*, an important work by the Greek astronomer and mathematician Claudius Ptolemy. She taught and wrote about a number of math topics on which further progress was not made until centuries later.

One of Hypatia's most eminent students, Synesius, wrote her letters asking her advice on scientific matters, and these letters are one of the key sources about her work. He credits her with detailed knowledge of the **astrolabe** and the **hydroscope**, as well as other devices used for studies in astronomy and physics. Historians living in her time praised her learning, as well as her beauty and character.

integral solutions solutions to an equation or set of equations that are all integers

linear equation an equation in which all variables are raised to the first power

astrolabe a device used to measure the angle between an astronomical object and the horizon

hydroscope a device designed to allow a person to see below the surface of water

Hypatia's ties to a politician who disagreed with Alexandria's Christian bishop led to Hypatia's death in 415 C.E., when she was murdered by a mob of religious fanatics. Following Hypatia's murder, many of her students migrated to Athens, which by 420 C.E. acquired a considerable reputation in mathematics.

All of Hypatia's works are believed to have been lost in the seventh century, when the books of the library at Alexandria were burned by Arab invaders. SEE ALSO Apollonius of Perga.

Shirley B. Gray

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IMAX Technology

Ask anyone who has seen an IMAX film in an IMAX theater to describe their experience, and they will probably use words like "huge," "giant," and "awesome." This reaction is exactly what the designers of the IMAX film technology had in mind—a motion picture experience like no other. What exactly is it that makes IMAX films so different?

IMAX's History

In the late 1960s, filmmakers experimented with multiple-screen films. They used up to three cameras to film movies in a very wide format, which could then be played back to an audience using three projectors at three times the normal width. However, these systems had many problems. First, three cameras had to be used simultaneously to film the movie. Then the three projectors that were used to show the film had to be perfectly aligned and synchronized. Another problem was that the edges of each projector's image tended to vary in brightness.

Knowing that multiple-screen films were very popular with audiences, a group of Canadian filmmakers and business people decided to design a system that had the same large-scale effect as a multiple-screen film, but with a single camera and projector. After several years of work, the first IMAX film premiered at the 1970 World Expo in Osaka, Japan. The whole system was a hit, and since that time IMAX theaters have been built and are in use all over the world.

The Characteristics of IMAX

The magic of IMAX motion pictures stems from the combination of specialized equipment that is used to film and project IMAX movies. Every part of the system has been developed especially for IMAX films.

Screen Size. The first thing an IMAX theater patron is likely to notice is the screen size. The typical IMAX screen is about 50 feet tall and 70 feet wide. With this large screen, and the positioning of the seats in the theater, the screen fills even the **peripheral vision** of the audience. So when the camera moves, the audience experiences the sensation of movement. The sensation of motion can be made more spectacular when the film has been shot at high speeds, like from the cockpit of a jet fighter or from the front car on a roller coaster.

peripheral vision the outer area of the visual field

A typical 45-minute large-format IMAX movie uses 3 miles or 15,840 feet of film. Here a projectionist loads the film onto the large-scale reel.

Some IMAX flat screens are as large as 100 feet high. Even more spectacular are IMAX dome screens. IMAX dome screens can be almost 90 feet in diameter. In the dome, the audience sits at a reclined angle so that spectators look up at an angle toward the center of the domed screen. Even in flat-screened IMAX theaters, the audience area is designed so that every seat has a full, unobstructed view of the screen. In an IMAX theater, patrons feel like they are *in* the movie, rather than just watching it.

Sound System. Nearly everyone has experienced some sort of surround sound system in regular movie theaters, but IMAX theaters have sound systems that are loud enough and spectacular enough to complement their unique visual experience. IMAX theaters have six-channel wrap-around sound systems. That means up to six different audio tracks can be channeled anywhere in the theater to give the audience the perception that not only are the images in motion, but so are the sounds. With up to 13,000 watts of power driving dozens of speakers, the sound systems are loud enough to recreate the concussion that is felt during a space shuttle launch.

Film Size. Motion picture film is divided into frames. In the camera, the film is moved behind the lens and shutter and each frame is exposed for just a moment. When the film is developed and played back in a projector, the moving frames recreate the motion that was captured by the camera.

The size of the film frame affects the quality of the image that is produced. Standard motion picture film is 35 mm (millimeters) wide, and has a height of 4 perforations. The perforations are tiny holes on the edges of the film that the camera and projector use to move the film. High-quality motion picture film is 70 mm wide and has a height of 5 perforations. IMAX

film is even larger. It is 70 mm wide and has a height of 15 perforations. Additionally, IMAX film is run through cameras and projectors horizontally, so the perforations are at the top and bottom of the frame. Regular film is run through the cameras and projectors vertically, so the perforations are on the sides of the frame. Because of its size, IMAX film is often referred to as 15/70 film.

Special Cameras. IMAX movies are not just regular movies that have been printed on 15/70 film. The making of an IMAX movie begins with filming, during which special IMAX cameras use 15/70 film. The cameras weigh over 50 pounds apiece, and the film is loaded in cartridges that weigh about 5 pounds. Because of the speed of the IMAX film in the cameras, each film cartridge will provide only 5 to 7 minutes of footage. This makes for quite a bit of film reloading, or the use of multiple cameras.

Mega Projectors. The projectors used to produce the unparalleled IMAX movie experience have been compared in size to a compact car. To begin, the film is loaded into the projector on reels that are the size of truck wheels. The reels are mounted horizontally on either side of the projector because IMAX film moves horizontally instead of vertically. The film itself moves through the projector in a unique way that is called a "rolling loop." Air is used to hold each frame momentarily on the rear of the lens, producing a picture that is far more steady and clear than a normal motion picture projector.

To produce an image bright enough to light up the giant IMAX screens, the projectors use special xenon bulbs that consume up to 15,000 watts of power. This is equivalent to 150 100-watt light bulbs. With a bulb so bright, special measures must be taken to cool the projector. Both air, flowing at 800 cubic feet per minute, and water, flowing at 5 gallons per minute, cool the bulb, which can have a surface temperature of 1,300° F.

DOES FILM SIZE MATTER?

The area of an IMAX film frame—the area that is exposed—is over ten times larger than 35 mm (millimeter) film, and over three times larger than standard 70 mm film. This extra area makes the images sharper and more vibrant than other film formats.

IMAX Movies

Because of the expense and technical difficulties of filming IMAX movies, most of the 110 or so films that exist are short films, 40 minutes or less in length. Due to the spectacular visual quality of the IMAX format, many films are documentaries that are intended to show the audience images that are normally inaccessible. Consequently, the impact of the films about space exploration, wildlife, wilderness areas, or extreme environments is unforgettable.

The IMAX film system is a remarkable experience for audiences all over the world. Since everything about the system is like a regular movie multiplied by several factors, perhaps the adage "bigger is better" really applies to the IMAX system. SEE ALSO Ratio, Rate, and Proportion.

Max Brandenberger

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Induction

In mathematics, induction is a technique for proving certain types of mathematical statements. The induction principle can be illustrated by arranging a series of dominoes in a line. Suppose two facts are known about this line of dominoes.

1. The first domino is knocked over.
2. If one domino is knocked over, then the next domino is always knocked over.

What can be concluded from these statements? If the first domino is knocked over, then the second domino is knocked over, which knocks over the third, fourth, fifth, and so on, until eventually all of the dominoes fall.

Induction is a simple but powerful idea when applied to mathematical statements about positive integers. For example, consider the following statement: $n^2 \geq n$ for all positive integers, n. To prove that this statement is true using induction, it is necessary to prove two parts: first, that the statement is true for $n = 1$; and second, that if the statement is true for a positive integer $n = k$, then it must be true for $n = k + 1$. Demonstrating both of these parts proves that the mathematical statement has to be true for all positive integers.

Suppose using the induction principle it has been shown that $n^2 \geq n$. It is then instructive to see how the statement is true for all positive integers, n. The first part says that $n^2 \geq n$ is true for $n = 1$, which is, in effect, knocking over the first domino. According to the second part, $n^2 \geq n$ is also true for $n = k + 1$ when it is true for $n = k$, so it is true for $1 + 1 = 2$. This proves that the next domino is always knocked over. Now apply the second part again and take $k = 2$. Continuing this process proves that $n^2 \geq n$ is true for all positive integers.

Using the induction principle, it can also be shown that $2n$ is always an even number for all positive integers, n. Substitute 1, 2, 3, and 4 for n, and the results are 2, 4, 6, and 8, which are all even numbers. But how can it be certain that, without fail, every positive integer n will result in an even number for $2n$? It looks obvious, but often what looks obvious is not necessarily a valid proof. The induction principle, however, provides a valid proof.

The mathematical statement we want to prove is that $2n$ is an even number when n is a positive integer. To test the first part, we know that for $n = 1$, $2n$ is 2×1, or 2. The first even number is 2. So the statement is true for $n = 1$. To test the second part, suppose that $2n$ is an even number for some positive integer $n = k$. Therefore, $2k$ is even. Remember, adding 2 to any even number always produces an even number. So $2k + 2$ is also an even number, but $2k + 2 = 2(k + 1)$. Hence, $2(k + 1)$ is an even number. Assuming that the statement is true for $n = k$ leads to the fact that the statement is true for $n = k + 1$. Therefore, the induction principle proves that $2n$ is an even number for all positive integers, n. SEE ALSO Proof.

Rafiq Ladhani

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Inequalities

An inequality is a mathematical statement that looks exactly like an equation except that the equals sign is replaced with one of the five inequality symbols.

$\neq$	not equal to
$<$	less than
$\leq$	less than or equal to
$>$	greater than
$\geq$	greater than or equal to

Imagine two friends of ages 9 and 12. Each of the five inequality symbols can be used to write a statement about their ages, depending on which aspect about their ages is to be emphasized.

$9 \neq 12$ says that 9 is not equal to 12

$9 < 12$ says that 9 is less than 12

$9 \leq 12$ says that 9 is less than or equal to 12

$12 > 9$ says that 12 is greater than 9

$12 \geq 9$ says that 12 is greater than or equal to 9

Notice that $9 < 12$ and $12 > 9$ are two ways to represent the same relationship; when the 9 and 12 are reversed, the inequality is also reversed.

The difference in meaning of "greater than" and "greater than or equal to" is subtle. For example, let a represent the age of a person. How would an equality be written to show that the age limit for voting is 18?

The expression $a > 18$ means that the age has to be greater than 18. The expression $a \geq 18$ means that the age either can be greater than 18 or it can be equal to 18. Since a person who is exactly 18 years old is allowed to vote, the inequality $a \geq 18$ is the correct one to use for this situation.

Various limits in society can be expressed using inequalities. Just a few examples are speed limits, minimum and maximum bank withdrawals, minimum and maximum fluid levels, grade requirements for college admittance, and minimum sales to make a fundraising goal. Some common phrases that indicate inequality within word problems are "minimum," "maximum," "at most," "at least," and superlatives like "oldest" and "smallest."

Solving Inequalities

Every possible equation can be made into an inequality. Furthermore, inequalities are solved the same way as equations, with one exception; namely, when an inequality is multiplied or divided by a negative number on both sides of the inequality sign.

For example, start with the true statement $5 < 6$. When both sides are multiplied by -2, the statement becomes $-10 < -12$. This resulting

SUMMARIZING INEQUALITIES

Two points must be remembered when dealing with inequalities.

- If each side of an inequality is multiplied or divided by a positive number, then no changes are made to the inequality symbol.
- If each side of an inequality is multiplied or divided by a negative number, then the equality symbol must be reversed.

statement is false because -10 is greater than -12. Since multiplying (or dividing) an inequality by a negative number results in a false statement, the inequality symbol must be reversed to maintain a true statement. In this example, the correct answer is $-10 > -12$. As another example, to solve $-4x > 12$, divide both sides by -4, and then reverse the inequality to yield $x < -3$.

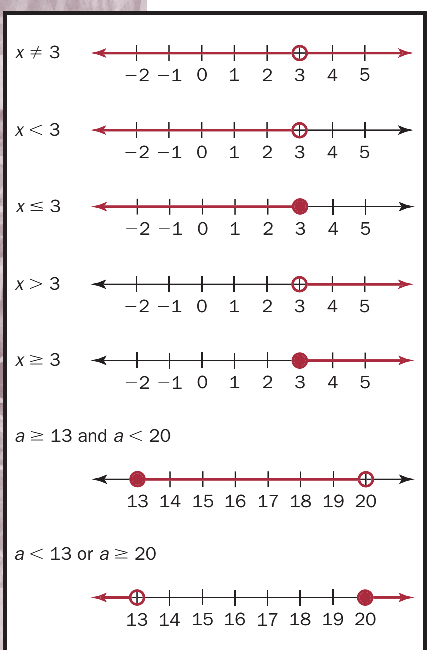
A Practical Example. Consider the following problem. A student has saved \$40, and wants to save a total of at least \$300 to buy a new bicycle and bicycle gear. His job pays \$7.25 an hour. Find the number of hours he needs to work in order to save at least \$300.

To solve this problem, let h be the number of hours he needs to work. The words "at least" indicate that the combination of savings and pay must be greater than or equal to \$300. The solution below shows that the student must work 36 or more hours to meet his goal of saving \$300.

$$\$40 + 7.25h \geq \$300$$

$$7.25h \geq 260 \qquad \textit{Subtract 40 from each side.}$$

$$h \geq 35.9 \qquad \textit{Divide each side by 7.25.}$$



Examples of graphs of inequalities in one variable.

Graphing Inequalities

The number lines in the top five rows of the boxed figure illustrate each case that may arise in graphing inequalities in one variable, x. A closed circle on a number indicates that it is included in the solution set, and an open circle indicates that it is not included in the solution set. Shading to the left with a darkened arrow on the end indicates all numbers of lesser value are included in the solution set; shading to the right with a darkened arrow on the end indicates all numbers of greater value are included in the solution set.

Graphing Compound Inequalities. Compound inequalities are two inequalities separated by the words "and" or "or." The solution set to a compound inequality that is separated by the word "and" is the region where the two graphs overlap. This is known as an intersection.

Consider how to write a compound inequality that represents the possible ages of a teenager, and then how to graph the solution set. To write this inequality, let a represent age. So $a \geq 13$ and $a < 20$. The graph of the intersection is shown in the sixth number line in the boxed figure.

Compound inequalities that represent solutions that fall between two numbers are frequently written in an abbreviated notation with the variable in the middle. Hence, the inequality described above can also be written as $13 \leq a < 20$.

The solution set to a compound inequality that is separated by the word "or" is the combination of all points on both graphs. This is known as a union.

Consider how to write a compound inequality to represent the possible ages of a sibling who is not a teenager, and then how to graph the solution set.

To write this inequality, let a represent age. So $a < 13$ or $a \geq 20$. The graph of the union is shown in the seventh number line in the boxed figure.

The corollary to compound inequalities in one variable is a system of inequalities in two variables. The solution to a system of inequalities is the intersection of the two graphs, just like the solution to a system of equations. However, the intersection of a system of inequalities usually consists of a whole region of points. This is especially useful since real-world problems often involve choosing from several possible solutions.

Inequalities in Two Variables

The graph below shows how to graph a linear inequality in two variables. Graphs in two variables are drawn using either a solid or dashed line. A solid line indicates that the points on the line are included in the solution set, and a dashed line indicates that the points on the line are not included in the solution set. For a linear graph, the solid or dashed line divides the plane into two regions, only one of which will be the solution set.

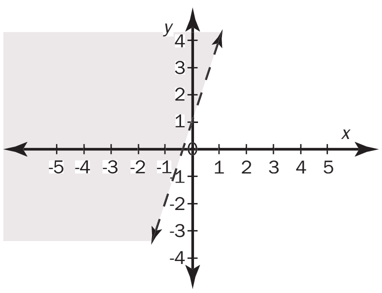
Consider how to solve and graph $y + 7 > 3x + 8$. First, subtract 7 from both sides to obtain $y > 3x + 1$. Graph the equation $y = 3x + 1$ using a dashed line. The dashed line divides the plane into two regions. Choose one point from each region that will be easy to substitute into the inequality. The points chosen below are (0, 0) and (0, 2). Only one of the two points will be a solution to the inequality, indicating which region includes the set of solutions.

First, substitute (0, 0).

$y > 3x + 1$

$0 > (3 \times 0) + 1$

$0 > 1$ (FALSE)



Next, substitute (0, 2).

$y > 3x + 1$

$2 > (3 \times 0) + 1$

$2 > 1$ (TRUE)

Hence, the point (0, 2) is a solution to the inequality. If more points in the same region are tested, they will also be solutions. Hence, the graph of the solution set includes points to the left of, but not including, the line $y = 3x + 1$, as shown above. SEE ALSO Functions and Equations.

Michelle R. Michael

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Infinity

Few concepts in mathematics are more fascinating or confounding than infinity. While mathematicians have a longstanding disagreement over its very definition, one can start with the notion that infinity (denoted by the symbol ∞) is an unbounded number greater than all real numbers.

Writing about infinity dates back to at least the Greek philosopher Aristotle (384 B.C.E.–322 B.C.E.). He stated that infinities come in two varieties; actual infinities (of which he could find no examples) and potential infinities, which he taught were legitimate only as thought. Indeed, the German Karl Gauss (1777–1855) once scolded a fellow mathematician for using the concept, stating that use of infinity "is never permitted in mathematics."

The French mathematician and philosopher René Descartes (1595–1650) proposed that because "finite humans" are incapable of producing the concept of infinity, it must come to us by way of an infinite being; that is, Descartes saw the existence of the idea of infinity as an argument for the existence of God. English mathematician John Wallis (1616–1703) suggested the use of ∞ as the symbol for infinity in 1655. Before that time, ∞ had sometimes been used in place of M (1000) in Roman numerals.

Defining Infinity

Although students are typically taught that "one cannot divide by 0," it can be argued that $\frac{1}{\infty} = 0$ (read as "one divided by infinity"). How is this possible? Observe the following progression.

$\frac{1}{1} = 1.0$
$\frac{1}{2} = 0.5$
$\frac{1}{4} = 0.25$
$\frac{1}{8} = 0.125$
$\frac{1}{16} = 0.0625$
$\frac{1}{32} = 0.03125$

Note that as the denominator, or the divisor, becomes larger, the value of the fraction (or the "quotient") becomes smaller. What happens if the denominators become very large?

$\frac{1}{512} = 0.00195$
$\frac{1}{4{,}096} = 0.00024$
$\frac{1}{10{,}000{,}000} = 0.0000001$

One can see that as the denominator becomes extremely large, the fraction values approach 0. Indeed, if one thinks of infinity as "ultimately large," one can see that the value of the fraction will likewise be "ultimately small," or 0. Hence, one informal (but useful) way to define infinity is "the number that 1 can be divided by to get 0." Actually, there is no need to use the number 1 as the numerator here; any number divided by infinity will produce 0.

Using algebra, one can come up with another definition of infinity. By transforming the following equation we see that infinity is what results if 1 is divided by 0.

If $\frac{1}{\infty} = 0$
Then $1 = \infty \times 0$
And $\infty = \frac{1}{0}$.

Notice that this approach to informally defining infinity produces an equation (the middle equation of the three above) in which something times 0

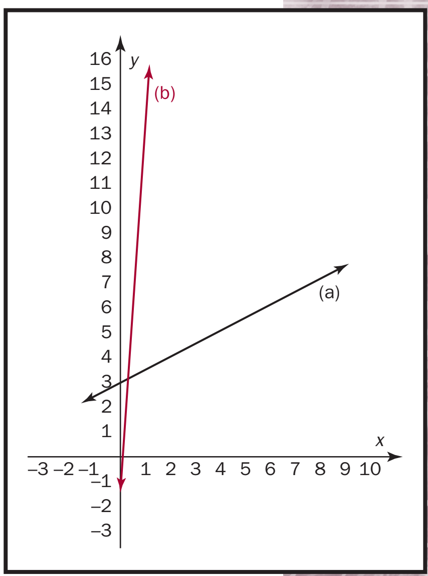
does not give 0! Because of this difficulty, and because the rules of algebra used to write and transform the equations apply to numbers, some mathematicians claim that division by 0 should not be allowed because ∞ may not be a defined number. They argue that dividing by 0 does *not* give infinity, but rather that infinity is undefined.

Another method of attempting to define infinity is to examine sets and their elements. If in counting the elements of a set one-by-one the counting never ends, the set can be said to be infinite.

Infinity as a Slope. Infinity is also sometimes defined as "the slope of a vertical line on the coordinate plane." In **coordinate geometry**, it is accepted that the slope of any straight line is defined as the change in vertical height divided by the change in horizontal distance between any two points on the line. The slope is often shown as a fraction in lowest terms, and sometimes called "rise over run."

coordinate geometry the concept and use of a coordinate system with respect to the study of geometry

In the figure, the slope of line (a) is $\frac{1}{2}$. If a line is very steep, the rise will be very large compared to the run, giving a very large numerical slope. The slope of line (b) is $\frac{13}{1}$. A much steeper line will result in a fraction such as $\frac{1{,}000{,}000{,}000}{1}$. Such a line would appear to be vertical, even though it would not be quite vertical if viewed in greater detail. Thus, the slope of extremely steep lines approaches infinity, and the slope of a "completely steep" line, that is, a vertical line, can be thought of as equal to infinity.



As a line becomes steeper (b), its slope will approach infinity. Yet the slope of a vertical line may be considered undefined rather than infinite because the calculation requires division by 0.

Yet on a "completely steep" or vertical line, any two points give a run of 0. This means that one could define the slope of the line as any number over 0. This again allows the conclusion that division by 0 results in infinity, unless one maintains that the slope of a vertical line is undefined.

The Nature of Infinity

Although several definitions of infinity were provided, note that none of them state that infinity is the highest possible number. Consider this: On a number line, how many points are between points 4 and 5? An infinite number, of course, because actual points have no dimension, even though their two-dimensional representations have a very small dimension on the paper, blackboard, or computer screen. But consider further: How many points are between points 4 and 6? Also an infinite number, certainly, but this set appears to be twice as large as the one between points 4 and 5. This use of **set theory** as an approach to understanding infinity forces one to look at several curious possibilities.

set theory the branch of mathematics that deals with the well-defined collections of objects known as sets

1. There are different sizes of infinity.
2. A set with an infinite number of elements is the same size as one of its "smaller" subsets.
3. Elements can be added to a set that already has an infinite number of elements.

Which of these possibly contradictory statements is true? It may be impossible to answer the question. Galileo (1564–1642) felt that the second statement was true. The great German mathematician and founder of set theory Georg Cantor (1845–1918) added to our understanding of infinity by choosing not to see the statements as contradictions at all, but to accept them as simultaneous truths. Cantor defined orders of infinity. An infinite

DEVELOPMENT OF SET THEORY

German mathematician Georg Cantor (1845–1918) was an active contributor to the development of set theory. He also became known for his definition of irrational numbers.

set that can be put into one-to-one correspondence with the counting numbers is the smallest infinite set, called aleph null. Other larger infinite sets are called aleph one, aleph two, and so on. One can see that working with infinity produces various counterintuitive and even paradoxical results; this is why it is such an interesting concept.

There are numerous examples of infinity in pre-college mathematics. One case: it is accepted that 0.999. . . is exactly equal to 1.0. Yet how can a number which has a 0 in the units place be exactly equal to a number with a one in that place? The idea that there are an infinite number of nines in the first number allows us to make sense of the proposition. The number 0.999. . . is said to "converge on 1," meaning that 0.999. . . becomes 1 when the infinite number of nines is considered.

Another example of how infinity comes into play in common mathematics is in the decimal representation of π (pi), or 3.14159. . . . The digits making up π go on forever without any pattern, even though the size of π never gets even as large as 3.15.

No one has ever come across an infinite number of real things. Infinity remains a concept, brought to life only by the imagination. SEE ALSO DESCARTES, AND HIS COORDINATE SYSTEM; DIVISION BY ZERO; LIMIT.

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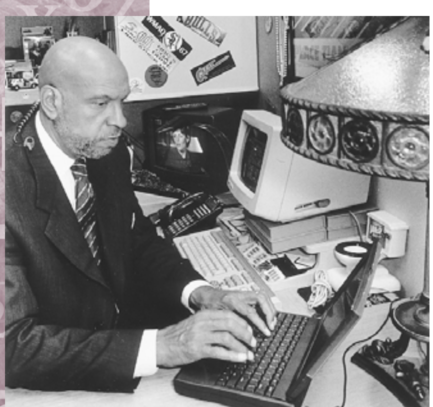
statistics the branch of mathematics that analyzes and interprets sets of numerical data

probability the likelihood an event will occur when compared to other possible outcomes

Insurance Agent



Insurance agents use computers and mathematics to assess risk, determine premiums, and calculate the amount of claims.

Insurance agents help people buy insurance plans to protect against illness, accidents, or emergencies. The three major categories of insurance are life, health, and property. Insurance agents may be self-employed or they may work for a particular insurance company. Insurance agents work for individuals or for companies to select an insurance provider, fill out complex application forms, and maintain records. In the event of a loss, agents help their clients settle insurance claims.

Most companies prefer to hire insurance agents who have college degrees. Insurance agents often have bachelor's degrees in business or economics. However, some companies will hire a person without a college degree if that person has proven sales ability or success in other types of work.

The mathematics of insurance can be very complicated. Computers help insurance companies use **statistics** to estimate the **probability** of an

event occurring. For example, the odds of an individual being involved in an automobile accident vary depending upon the driver's age, sex, driving history, location, and type or style of car. These statistics are used to determine appropriate rates for insurance plans. The riskier a person appears, based on the statistics, the higher the rate he or she will have to pay for insurance. Computers make it easier and faster to process insurance application forms and service requests.

Mathematics is used in the insurance business in other ways. For group coverage, insurance agents have to calculate the total cost, based on the number of persons being covered. Also, discounts may apply to the policy, thereby yielding some percentage off the normal cost. Most importantly to an insurance agent, the agent's pay, or commission, is often a percentage of the dollar amount of the policy sold. SEE ALSO Probability, Theoretical; Statistical Analysis.

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Integers

Integers are the set of numbers {. . . , −4, −3, −2, −1, 0, 1, 2, 3, 4, . . .} that encompass the counting numbers, {1, 2, 3, . . .}, the negative of the counting numbers, {. . . −3, −2, −1} and zero. Integers can be shown on a simple number line.

The integers on the left side of zero (0) are called negative numbers, and are represented by a negative sign (−) placed before them, as in −5, −10, and −15.* The integers on the right side of 0 are called positive numbers. Examples include 5, 10, and 15. The positive integers are known as counting numbers or natural numbers. The positive integers and 0 are called whole numbers. Zero is an integer but it is neither positive nor negative.

***Some historians believe the first evidence of the use of negative numbers was around 300 B.C.E. in China.**

Integers are used in everyday life. A debt or a loss is often expressed with a negative integer. A gain is usually expressed with a positive integer. When the temperature is warmer than the zero point of the temperature scale, it is represented with a positive sign; when it is colder than the zero point, it is represented with a negative sign. SEE ALSO Numbers, Real; Numbers, Whole; Zero.

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Integral *See Calculus.*

Interest

When you borrow money to buy a car or a house you are not only expected to pay back that money, but to pay interest on it, too. Interest is a fee paid by a borrower to the lender for the use of money. It is calculated as a percentage of the loan amount. When you deposit money into a savings account or certificate of deposit or buy a savings bond, you are loaning money and so you are paid interest. Interest plays an important role in economics because it serves as an incentive for those with available money to lend it to those needing it. There are many different ways that this fee is expressed and calculated.

Types of Interest

The term "simple interest" refers to a percentage of the loan that must be paid back in addition to the original loan. For example, if you borrow $1,000 at 10 percent simple interest and pay it back five years later, you will pay back the $1,000 plus 10 percent or $100 additional dollars for each year, a total of $500 in interest.

Few loans, however, are actually based on simple interest. Loans are usually based on "compound interest," where the total of the outstanding original money and the accumulated interest are calculated on a regular basis to compute the interest owed. In the preceding example, if interest were to be compounded annually (i.e., once each year), the interest for the first year would still be $100. But in the second year the borrower would owe interest not just on the original $1,000, but on the additional $100 of interest that was owed to the lender for the first year. The total interest for the second year would be $110. Similarly in the third year the interest would grow to $121. By the end of 5 years the total interest, compounded annually, would be $610.51, in contrast to the $500 in simple interest. This is illustrated in part (a) of the table.

Examples of Interest

(a) Compounded Annually

Year	Beginning value	Interest at 10%	Total at end of year
1	1,000.00	100	1,100.00
2	1,100.00	110	1,210.00
3	1,210.00	121	1,331.00
4	1,331.00	133.1	1,464.10
5	1,464.10	146.41	1,610.51

(b) Compounded Monthly

Month	Beginning value	Interest at 6%	Total at end of month
1	1,000.00	5.00	1,005.00
2	1,005.00	5.03	1,010.03
3	1,010.03	5.05	1,015.08
4	1,015.08	5.08	1,020.15
5	1,020.15	5.10	1,025.25
6	1,025.25	5.13	1,030.38
7	1,030.38	5.15	1,035.53
8	1,035.53	5.18	1,040.71
9	1,040.71	5.20	1,045.91
10	1,045.91	5.23	1,051.14
11	1,051.14	5.26	1,056.40
12	1,056.40	5.28	1,061.68

"Discount interest" is interest that is subtracted from the loan when it is first made. Following the above examples, if you borrowed $1,000 at a discount interest rate of 10 percent, you would only receive $900, but would be expected to pay back $1,000. Since in this case you have really only borrowed $900, which you end up paying $100 interest on, you are paying a higher rate of interest than with simple or compound interest. In this example, a 10 percent discount rate would be the same as an 11.11 percent compound rate.

Compound interest is not necessarily compounded on an annual basis. By compounding interest more often the lender is able to get interest on interest. This can have a significant affect on the total earnings made on a savings deposit or other loan. One thousand dollars in a savings account paying 6 percent interest compounded annually would earn $60 in interest in one year. What if that same account were compounded monthly, as shown in part (b) of the table? Six percent is not earned each month. Instead, the annual rate of 6 is divided by the number of months, 12, giving a half percent per month. So, using the table, you will note that after the first month, the account earns $5. But during second month there is now $1,005 in the account. The extra $5 also earns interest.

You can see from part (b) of the table that you earn an extra $1.68 by compounding the same savings monthly. In this case the **rate** is still 6 percent, but the yield is 6.168 percent. The rate is sometimes called APR and the yield APY, for annual percentage rate and annual percentage yield, respectively. Note that if a bank offers you a yield of 6 percent they are actually offering you a rate of less than 6 percent. When comparing interest rates, it is important that you do not confuse rates with yields.

rate the portion of the principal, usually expressed as a percentage, paid on a loan or investment during each time interval

So if you get a little more for computing interest monthly, what would happen if you computed it daily? The 6 percent rate would be divided by the number of days in the year (365) and the result would be the interest for one day. While this is only about $0.16, that extra $0.16 starts to earn interest too. By the end of the year, this results in $61.83 in interest being earned in the account or the yield for a 6 percent APR savings account compounded daily, day in to day out, which is 6.183 percent. Although the interest could be computed over smaller intervals, the interest calculation is a function that approaches a **limit**. The limit is so closely approached at the daily compounded value that there is little to be gained by compounding the interest more than daily.

limit a mathematical concept in which numerical values get closer and closer to a given value or approach that value

Interest and Loans

Loans such as automobile, credit cards, and home mortgage loans work in a similar fashion, with the bank serving as the lender. Automobile and mortgage loans are generally **fixed term**, meaning that they are expected to be paid off completely at a set time in the future. Automobile loans are usually a 3- to 5-year term, while home mortgage loans can be for terms as long as 30 years. These loans allow the consumer to buy these expensive items before they have all of the money for them, and pay for them while they use them. The cost of this convenience, however, is the interest. Home mortgage loans can also be either at a fixed rate, agreed to at the start of the loan, or can be variable-rate loans, where the interest rate can change. Although variable rate loans are generally offered at a lower percentage than fixed-rate loans, the borrower must face the risk that the interest rate could be adjusted and become higher than the rate that was offered for the fixed-rate loan.

fixed term for a definite length of time determined in advance

Credit card loans generally have a significantly higher interest rate and require a minimum payment each month. Borrowers frequently find that they make no progress in lowering their credit card debt because more credit is incurred as the outstanding balance is paid off.

Credit cards are an extremely convenient but expensive financial tool, and many borrowers get into serious financial trouble when they let their credit card debt grow. For example, if you have a credit bard balance of $1,000, the interest on this balance will be $15 per month if the annual interest rate of the card is 18 percent. The minimum payment that the card issuer charges will be somewhat higher than the monthly interest charge, but not by much. For this example, assume a minimum payment of $50. By paying only the minimum required payment each month, and adding interest, your balance is not significantly reduced from month to month. At this rate it could take years to pay off your balance.

When a borrower pays off a loan such as a house mortgage or a car loan, they make a payment at regular intervals, usually monthly. Some of the payment pays the interest on the loan for that month. The rest is applied to the current outstanding value of the loan and reduces it, so that the loan will be completely paid off at the agreed upon time. In the case of a 30-year home loan, a borrower will find that the largest part of the monthly payment is applied to interest payments at first and a small amount is applied to the balance of the loan, called the principal. The result is that after a few years you still owe nearly the full loan amount. Slowly the borrower makes progress against the principal, until finally after 30 years the loan is paid off.

For example, if a borrower borrows $100,000 on a 30-year mortgage at 8 percent fixed rate, after one year the borrower will have paid $8,805.17, not including taxes or insurance, but will have only reduced the principal by $835.36 (the remaining $7,969.81 goes towards interest). It is valuable to realize that if the loan has no per-payment penalty, you can significantly reduce the total cost of the loan by making additional payments in the early years, as these additional payments apply completely to the loan principal. For example, if after the first year the borrower was able to pay back an additional $904, the loan would be paid off a full year earlier, avoiding the last year of payments totaling $8,805.17. In this way a few extra payments each year or a slightly higher monthly payment can significantly reduce the term on the loan and result in significant savings.

By building a savings account that benefits from compound interest, a small investment will grow well over time. If a person who is 20 years old deposits $100 monthly into a savings account that pays 7 percent interest, they would have deposited $48,000 by age 60. But thanks to compound interest, there would be over $265,000 in the savings account. SEE ALSO Percent.

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Interior decorators and designers must use mathematics to achieve accurate measurements and make drawings to scale.

Interior Decorator

Although the terms "interior decorator" and "interior designer" refer to different professions, they are often used interchangeably. Technically, an interior decorator focuses on a room's surface—its color and decor and the artistic arrangement of the objects within it. An interior designer is more of an architect, concerned with the design and structure of the room. Nevertheless, the two fields have merged, and anyone interested in a career in interior decorating needs the same kind of training and experience. In the United States, this begins with a 4- or 5-year degree program from a school accredited by the Foundation for Interior Design Education and Research. These programs typically include course work in interior design, art, architecture, and technology. After graduation and 2 years of work experience, the aspiring interior decorator is qualified for the state licensing examination administered by the National Council for Interior Design Qualification.

As one prominent practitioner comments, being an interior decorator is "more than lingering over fabric swatches or doodling out designs." An interior decorator must know as much about business (including budgeting), engineering principles, materials science, drafting, and building safety codes as about color and arrangement. The interior designer has to be able to take accurate measurements of room areas, angles, elevations, and the like. A critical skill is the ability to envision and make drawings to scale to ensure that furnishings and other objects fit in the space being decorated. It is also a good idea for designers to develop strong computer skills, especially the ability to use CAD (computer-aided design) programs. SEE ALSO Architect.

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Internet

For many people, a good deal of the day is spent online. The ability to send e-mail messages and "surf" the World Wide Web has already become matter-of-fact. But an amazing amount of technology and mathematics must occur for e-mail and Internet access to be successful.

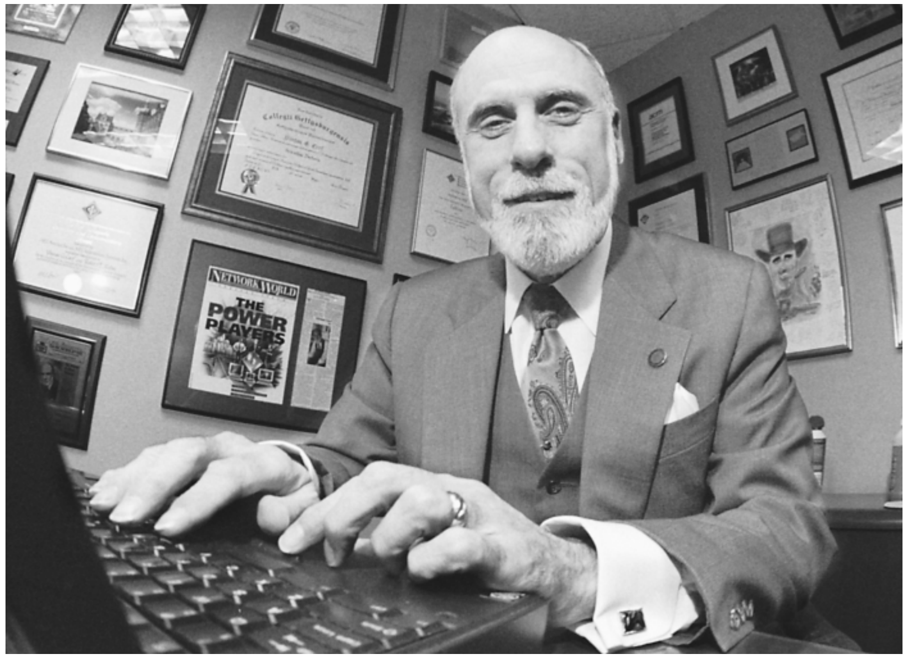
A Brief History of the Internet

The general consensus is that the conception of the Internet occurred in the early 1960s as part of the Department of Defense's Advanced Research Projects Agency (ARPA), which was conceived and headed by J. C. R. Licklider from the Massachusetts Institute of Technology. The intent was to share supercomputers among researchers in the United States.

Because computers in the 1960s were so large and expensive, it was important to find a way for many people, often at different locations, to be able to use the same computer. By the end of the decade, ARPANET was developed to solve this problem, and in 1969 four universities—Stanford, University of California–Los Angeles, University of California–Santa Barbara, and the University of Utah—were the first to be successfully connected.

The ARPANET was not available for commercial use until the late 1970s. By 1981 there were 213 different hosts (central computers) available on the ARPANET, although many were completely incompatible with one another because each "spoke" a different language. Things were somewhat disjointed until Bob Kahn and Vint Cerf created TCP/IP (Transfer Control Protocol/Internet Protocol), which became the common language for all Internet communication. This transformed the disparate collection known as ARPANET into one cohesive group, the Internet.

Even though the intent of the ARPANET and Internet was to allow researchers to share data and access remote computers, e-mail soon became the most popular application to communicate information. In the 30-plus



Vinton Cerf is the co-developer of the computer networking protocol, TCP/IP, which has become the common language for Internet communications.

years since then, not much has changed. In an average week, approximately 110 million people are online in the United States. If, on average, each of those people sends ten e-mails per week (a conservative estimate), then there are more than a billion e-mails sent every week.

Traveling on the Internet

Although e-mail is something that is often taken for granted, a great deal must happen for an e-mail message to go from one device to another. Depending on its destination, an e-mail message's travel path can be either very short or very long.

Sending e-mail is similar in some ways to sending a letter through regular mail: there is a message, an address, and a system of carriers that determines the best way to deliver the mail. The biggest differences between sending e-mail and regular mail are the first and last steps.

When an e-mail message is sent, it is first broken down into tiny chunks of data called "IP packets." This is accomplished by a mailing program (such as Outlook Express or Eudora) using the TCP Internet language. These packets are each "wrapped" in an electronic envelope containing web addresses for both the sender and recipient.

Next, the packets are sent independently through the Internet. It is possible that every single packet (and there can easily be hundreds of them) is sent on a different path. They may go through many levels of networks, computers, and communications lines before they reach their final destination.

The packets' journey begins within the Internet Service Provider (ISP) or network (AOL or MSN, for example), where the address on the envelopes is examined. Addresses are broken into two parts: the recipient name and the domain name. For example, in an e-mail message sent to

John_Doe@msn.com, "John_Doe" is the recipient name and "msn.com" is the domain name.

Based on the domain name, the router (a piece of equipment that determines the best path for the packets to take) will determine whether the packets remain on the network or need to be sent to a different router. If the former is the case, the packets are sent directly to the recipient's e-mail program and reassembled using TCP.

If the recipient is on a different network, things get more complex. The packets are sent through the Internet, where an Internet router determines both where they need to go and the best path to get there. Decisions like these are made by problem-solving programs called **algorithms**, which find the optimal path for sending the packets.

algorithm a rule or procedure used to solve a mathematical problem

Each packet is sent from one network to another until it reaches its final destination. Because they determine where the packets should go, routers can be likened to different transportation stations within a huge transportation system containing buses, trains, and airplanes. To get from one part of the world to another, a message may have to go through several stations and use multiple types of transportation.

For example, assume that two travelers are both starting in New York City and heading for Los Angeles. They get separated and end up taking different modes of transport yet still end up at the same point. This is what happens to the packets when they make the trip from the originating computer to their eventual destination; that is, they can get separated and sent on different paths to their final destination. Routers determine the optimal path for each packet, depending on data traffic and other factors.

The packets often arrive at the final destination at different times and in the wrong order. The recipient will not see an e-mail message until all of the packets arrive. They are then recombined in the correct order by the recipient's mail program, using TCP, into a message that the recipient can read.

Connection Speed

How quickly all of this occurs can be influenced by many factors, some within the control of the e-mail user and others beyond it. One factor that can be controlled is the way information is received and sent to and from the originating computer. Popular types of connections available in 2001 are telephone modems, DSL (Digital Subscriber Line), Cable, T1 and T3.

Telephone modems are the earliest and slowest of the possible types of connections. In relation to the transportation metaphor used previously, they would be the buses. Under optimal conditions, one can download or upload information at rates of between 14 and 56 kbps (kilobits per second) with a modem. (One kilobit equals one thousand bits.) A *bit* is what makes up the data that are sent.✶

✶Eight bits equals one byte, and one byte equals a single character (a letter or numeral).

Actual transmission speeds for modems tend to be much slower than the optimal speeds because there is a vast, constant stream of data being transferred back and forth. Compare this to driving on a highway. Even though the speed limit may be 65 miles per hour (mph), because of traffic and road conditions, one may need to drive less than 65 mph. On the Internet, it is almost always rush hour.

Under perfect conditions, the 56,000 characters of data per second—which comes out to over 3 million characters per minute—that can downloaded may sound like a lot of information, but it really is not. Most text messages (such as e-mail messages) are relatively small and will download quickly using a modem. Audio, video, or other multimedia files, however, cause more of a problem. These files can easily be upwards of 5 or 10 million bytes each, and thus use a much greater **bandwidth**.

bandwidth a range within a band of wavelengths or frequencies

Faster alternatives to modems are now widely available. The most common alternatives for home use are DSL and cable modems. DSL works through the phone line. Speeds for DSL tend to be in the range of 1.5 mbps (megabits per second). One megabit is equal to 1,000 kilobits.

Cable modems, unlike DSL, have nothing to do with phone lines. Cable modems transmit data using the cable that carries cable television signals. They offer fast speeds of up to 6 mbps. Even though this is a very good speed, an ISP may limit the available bandwidth, which restricts the size of files that can be uploaded or downloaded.

For large companies, universities, and the Internet Service Providers, speeds need to be high and bandwidths need to be enormous. T1 and T3 lines, which are dedicated digital communication links provided by the telephone company, are used for this purpose. They typically carry traffic to and from private business networks and ISPs, and are not used in homes.

Both T3 and T1 lines have their advantages in certain areas. With T3 connections one can potentially access speeds of nearly 45 mbps, or somewhere around one thousand times that of a modem. Transmission speeds for T1 lines are considerably slower, running at 1.5 mbps. The advantage of T1 is privacy. T1 connection lines are not shared with other users. In contrast, T3 connection lines (as well as modems, cable, and DSL) are shared.

Consider the highway metaphor once again. Having a T1 line is like maintaining a private two-lane highway on which only certain people are allowed to drive. Having a T3 line is more like driving on a 4-lane autobahn (the highway system in Germany, where there is no speed limit), with three of the lanes clogged up with slow-moving trucks. On the autobahn the potential exists to go very fast, but the traffic often prevents drivers from reaching high speeds. So whether the T1 or T3 is more desirable depends on which is more valued—speed or privacy.

When an e-mail message is sent, there is a very good possibility that the packets will encounter nearly all of these types of connections on their journeys—just like people can use planes, trains, and automobiles. The next time you hit the "send" button, think about all of the logical and mathematical operations that are about to happen. SEE ALSO Computers and the Binary System; Internet Data, Reliability of; Numbers, Massive.

Philip M. Goldfeder

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Internet Data, Reliability of

The Internet can be a wonderful resource for all types of information including **quantitative** data. The Internet is accessible, both for those who want to obtain information and for those who want to make data available. Since there is so much readily accessible information, Internet users must learn to filter the data they find on the **World Wide Web**, because the providers of information may misuse numbers and mathematics in order to appear convincing. Therefore, Internet users should learn to evaluate data carefully, critically, and even skeptically.

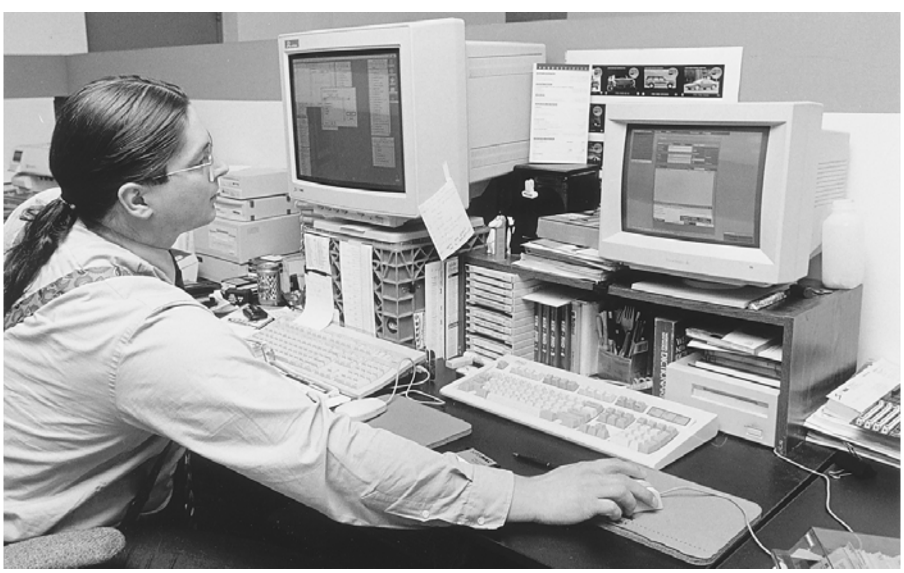
quantitative of, relating to, or expressible in terms of quantity

World Wide Web the part of the Internet allowing users to examine graphic "web" pages

To start, your local librarian could be of invaluable assistance in helping you discriminate between useful Internet sources of information and those web sites that would be a waste of time to visit. Remember that anyone can publish a web page on the Internet, and many web sites may be of little value in a search for specific quantitative information. Many librarians, particularly reference librarians, are specialists trained to effectively and efficiently search and obtain information for library patrons. The information specialist at your local library can save you significant time in your search for valid and reliable Internet data.

Filtering Information through the URL

Because the Internet is such a vast source of information of varying quality, web resources must be evaluated for authority, reliability, objectivity, accu-



Users of the Internet and its World Wide Web must think critically about the information they find. Statistics may appear authoritative, but their reliability can vary widely depending on their source.

racy, and currency. More traditional sources of information, such as an article in an encyclopedia, are screened with all those criteria in mind by authors, reviewers, editors, and publishers. That is not the case for most of the information on the Internet. No one has to approve the content of web sites, so it is your job to assess the appropriateness of the data you find on the Internet.

The domain name of the web page can reveal a great deal about the authority of the information on the web page. The domain name is the first part of the uniform resource locator (URL), and is the code that identifies the source of the web page. For example, of the address abc.com, the "abc" is the domain name.

Agencies of the U.S. government are assigned the .gov suffix, so you can trust that most of the information on those web sites has been screened for accuracy. The .edu suffix is usually assigned to 4-year degree-granting institutions. Much of the information posted on these sites has been prepared by scholars. Remember, however, that students also can post information on academic web pages, and sometimes faculty members post controversial and biased information to support their position. Thus, information on university web pages should still be treated with skepticism.

The .org domain suffix was originally meant for charities and non-profit organizations, a .com ending meant the site was for commercial businesses, and the .net suffix indicated the site is concerned with network issues. There are many sites with .com, .org, or .net suffixes that contain invaluable, reliable, and accurate information, but remember that the people creating those sites have particular economic, social, or political purposes in establishing a web presence. There is great potential that the data presented on their web pages may not be as complete, accurate, and unbiased as you require.

Web Pages with Authors

Beyond the domain name, the authority of the web page can be ascertained from the page itself. The name of the author(s) should be clearly visible in the header or footer of the web page, with contact information (e-mail, phone number, and address) provided. If the author is affiliated with an organization, this information should be clearly evident on the web page. You should scrutinize the author's education, experience, and reputation in the field you are investigating. That is easily accomplished by finding biographical information, often linked to the web page, and by looking for evidence of peer-reviewed, scholarly work by the author.

Closely connected to the authority of the web page is the reliability of the data found at that Internet site. Evidence that the information is reliable can be determined by observing if the information presented on the web site is of a reputable author or organization, if the data are taken from books or sources subject to quality control processes, or if the site itself is an online journal that is refereed by editors or other experts in the field.

Related to the reliability of the quantitative information obtained from a web page is the objectivity of the author. One of the purposes of a web site is for the author or organization to persuade viewers to adopt a particular point of view, and sometimes this is done in a subtle and surreptitious manner. The bias manifests itself in the presentation of incomplete or, worse,

inaccurate quantitative information slanted toward the perspective of the author. If the data look too good to be true, they probably are. Be wary of a hidden agenda, which can often be detected from the explicit or implicit purpose of the page. That purpose can only determined by careful and critical viewing of the site.

Looking at Information with a Skeptical Eye

Finally, you can assess the accuracy and currency of information found at an Internet site by observing if the information is timely and comprehensive. The document should be dated, particularly if the subject information is known to change rapidly. Sites that have many outdated links are not well-maintained and probably are not current. Web pages that acknowledge opposing points of view or are sponsored by non-commercial enterprises have a tendency to be more reliable.

While looking for data on the Internet, be aware that **search engines** do not evaluate web sites for the reliability or relevance of the information they contain. The various search engines have different **algorithms** for selecting sites, and those algorithms are generally not matched to your needs. For example, some search engines give priority to sites that are associated with sponsors of the search engine, and some give priority to the most popular web pages. Although search engines do not evaluate the quality of web pages, "evaluative" web sites examine other sites based on quality of content, authority, currency, and other useful criteria.

search engine software designed to search the Internet for occurrences of a word, phrase, or picture, usually provided at no cost to the user as an advertising vehicle

algorithm a rule or procedure used to solve a mathematical problem

Additionally, the lack of professionalism exhibited on the web page tells a great deal about the data contained on the page. The presence of spelling errors is an important indicator, as is the illogical organization of the site. Source documentation is extremely important, especially when it comes to the presentation of statistics and other quantitative information. The authors providing the Internet data should clearly identify the sources of the information they post, so that you can corroborate and confirm the validity of the data. If the source of the data is not clearly explained or specified, the information should be suspect. If there are links to the source data, this is a very good sign that the information may be accurate and reliable. Another positive indicator is the inclusion of a bibliography displaying related sources with proper attribution.

As a general rule, you should challenge all of the information you find on the Internet. Information coming from reliable sources, such as those sites containing a .gov domain name or a respected author, does not require as stiff a scrutiny as information posted on a personal web page. Validate the information by finding other Internet sources or printed sources that support the information found on an Internet site. You should never use Internet information you cannot verify. Information is power, and in learning to become skeptical and critical readers and viewers of Internet data, you will obtain the power that comes with having accurate information. SEE ALSO Internet.

Dick Jardine

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Inverses

The additive inverse of a number undoes the effect of adding that number. This means that, for example, the effect of adding a number by subtracting the same number can be undone. So, if 7 is added to 4, the result is 11. If 7 is subtracted from 11, the addition is undone, and the result is 4.

$$4 + 7 = 11 \text{ and } 11 - 7 = 4$$

Multiplication and division are related in the same way: They are inverse operations that undo each other. If 6 is multiplied by 3, the result is 18. Then 18 can be divided by 3 to undo the multiplication, and the result is 6.

$$6 \times 3 = 18 \text{ and } 18 \div 3 = 6$$

Most mathematical operations have an inverse operation that undoes or reverses its effect. The squaring of a number, for example, as in $7^2 = 49$, can be undone by taking the square root. The square root of 49 is 7. Thus, squaring and taking the square root are also inverse operations.

Inverse operations can also be used to find the additive inverse of a specific number. For example, -9 is the additive inverse of 9 since the sum of -9 and 9 is 0. Additive inverses come in pairs; 9 is the additive inverse of -9, just as -9 is the additive inverse of 9. Any two numbers are additive inverses if they add up to 0.

Visualize a pair of additive inverses on the number line. The number 9 and its additive inverse -9 are both nine units away from 0 but on opposite sides of 0. For this reason, -9 is called the opposite of 9, and 9 is the opposite of -9. The opposite of a number may be positive or negative. The opposite of -4 is 4, a positive number. The opposite of 8 is -8, a negative number. The number -2 can be read as "the opposite of 2" or as "negative 2."

A negative number is always to the left of 0 on a number line. Every number on the number line has an additive inverse. The additive inverse of 0 is 0 because $0 + 0 = 0$.

Multiplicative inverses come in pairs also. Any two numbers are multiplicative inverses if they multiply to 1. For example, because the product of 3 and $\frac{1}{3}$ is 1, 3 and $\frac{1}{3}$ are multiplicative inverses. In the same way, $\frac{2}{3}$ and $\frac{3}{2}$ are multiplicative inverses because their product is 1.

Decimal numbers also have inverses, of course. The decimal number 0.04 can be written as the fraction $\frac{4}{100}$, so the multiplicative inverse of 0.04 is $\frac{100}{4}$, which equals 25. This can be checked by multiplying 0.04 and 25 to

SUMMARIZING INVERSES

Inverses can be summarized in a few sentences.

Additive inverses add to 0. The additive inverse of a number is the opposite of the number. The additive inverse of a positive number is a negative number.

Multiplicative inverses multiply to 1. A number and its multiplicative inverse are called reciprocals. The reciprocal of a number is 1 divided by that number. The reciprocal of a fraction is the same as the fraction flipped.

verify that the product is indeed 1. The additive inverse of 0.04 is -0.04 because these two numbers add to 0.

A pair of numbers that multiply to 1, such as $\frac{1}{3}$ and 3 or 0.04 and 25, are also called reciprocals. To find the reciprocal of any number, write 1 over that number. Thus $\frac{1}{3}$ is the reciprocal of 3. One shortcut for finding the reciprocal of a fraction is to "flip" the fraction. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$; the reciprocal of $\frac{7}{10}$ is $\frac{10}{7}$. Pairs of reciprocals always multiply to give a product of 1.

Every number on the number line, except 0, has a multiplicative inverse. The multiplicative inverse of 0, or the reciprocal of 0, is undefined because division by 0 is undefined. SEE ALSO Integers; Mathematics, Definition of; Numbers, Real.

Lucia McKay

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K

Klein Bottle *See Mathematics, Impossible.*

Knuth, Donald

American Computer Scientist and Mathematician
1938–

Donald Ervin Knuth is considered one of the world's leading computer scientists and mathematicians. Yet in high school, Knuth found mathematics uninspiring. Although he achieved the highest grade-point average in the history of his high school, Knuth doubted his ability to succeed in college mathematics; and so did his advisor. So when Knuth graduated from high school in 1956, he entered the Case Institute of Technology (now Case Western Reserve) in Cleveland, Ohio, on a physics scholarship.

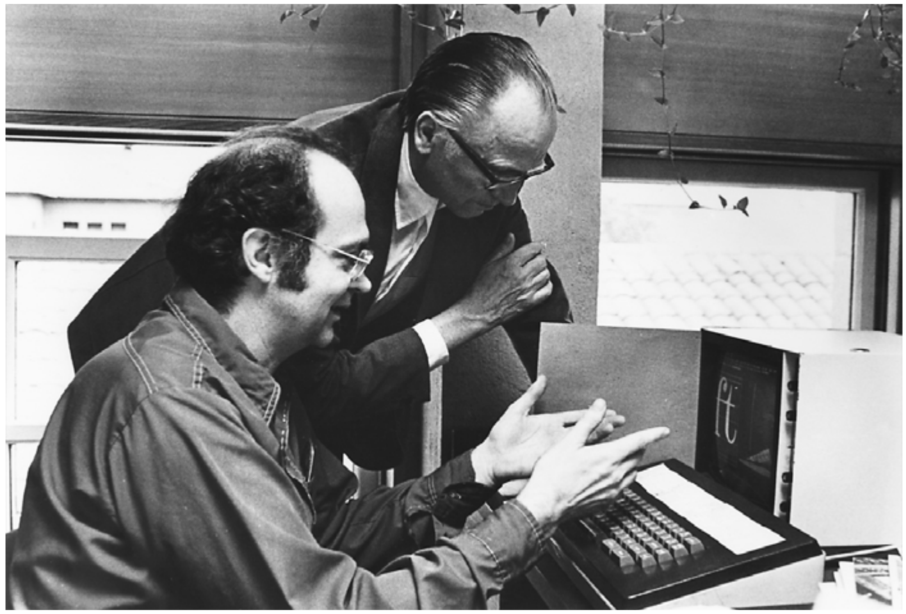
In his freshman year, Knuth encountered Paul Guenther, a mathematics professor who persuaded him to switch majors from physics to mathematics. Fearing he would fail, Knuth spent hours working out extra calculus problems, only to discover that his abilities far exceeded those of his classmates. One particularly difficult professor assigned a special problem and offered an immediate "A" in the course to any student who could solve it. Although Knuth initially considered the problem unsolvable, he did not give up. Making another attempt one day, Knuth solved the problem, earned his "A," and skipped class for the rest of the semester. The following year, Knuth earned an "A" in abstract mathematics and was given the job of grading papers for the very course he had skipped.

Knuth graduated *summa cum laude* from Case in 1960 with a Bachelor of Science (B.S.) and Master of Science (M.S.) in mathematics. He went on to earn his Doctor of Philosophy (Ph.D.) in mathematics from the California Institute of Technology in 1963, and joined the faculty as an assistant professor of mathematics. In 1968 Knuth joined the faculty at Stanford University, and served until his retirement in 1993, after which he was designated Professor Emeritus of the Art of Computer Programming.

Knuth's awards and honors include 17 books, more than 150 papers, 18 honorary doctorates, the 1974 Turing Award, Stanford University's first chair in computer science, the 1979 National Medal of Science (which was presented by then-President Jimmy Carter), and the 1996 Kyoto Prize for

NOT SO *MAD* AFTER ALL

Donald Knuth's prolific publishing career began at Milwaukee Lutheran High School where his science project won honorable mention in the Westinghouse Science Talent Search. His project, titled "Portzebie System of Weights and Measures," was defined as the thickness of *MAD Magazine* No. 26, and the basic unit of power was called the "whatmeworry." The editors of *MAD* recognized the importance of Donald's work. They purchased the piece for $25 and published it in their June 1957 edition.



Donald Knuth, (seated), shown here in 1980 with Herman Zaph at Stanford University, pioneered the TeX and METAFONT systems for digital typography. Knuth is perhaps most famous for his pioneering multivolume works, *The Art of Computer Programming* and *Computers and Typesetting*.

lifetime achievement in the arts and sciences (awarded by the Inawori Foundation).

Randy Lattimore

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Kovalevsky, Sofya

Russian Mathematician and Educator
1850–1891

Russian mathematician Sofya Kovalevsky was born in Moscow, Russia, the daughter of a minor nobleman. She became interested in mathematics at a very young age, when an uncle discussed mathematical concepts with her. Because of a wallpaper shortage, her nursery was papered with her father's lecture notes from a course in calculus, and at age 11 she studied the notes, recognizing principles her uncle had discussed. Under a tutor, she became so enamored with mathematics that she neglected her other studies.

When her father stopped her mathematics lessons, Kovalevsky borrowed an algebra book and read it while the family slept. At age 14 she read a physics textbook written by a neighbor and taught herself **trigonometry** so that she could understand the optics section. The neighbor recognized her ability and persuaded her father to send her to St. Petersburg, Russia, to continue her education.

trigonometry the branch of mathematics that studies triangles and trigonometric functions

Long, Hard Road to Success

The story of Kovalevsky's adult life was one of doors closing in her face because she was a woman. After finishing her secondary education, she arrived in Heidelberg, Germany, in 1869 to study mathematics and natural sciences, only to discover that the university did not admit women. Instead, she attended classes unofficially for three semesters. In 1870 she decided to try her fortunes at the University of Berlin. Again, the university did not admit women, but an eminent professor agreed to tutor her privately. By 1874 she had written papers on Abelian integrals and Saturn's rings. A third paper, on partial differential equations, was published in an influential mathematics journal. On the recommendation of Kovalevsky's tutor, Germany's University of Göttingen granted her a Ph.D. in 1874.

Sofya Kovalevsky was able to use her strength in mathematics to achieve positions that were highly uncommon for women of her time. She became the first female mathematician to hold a chair at a European university.

Unable to get a job teaching mathematics, Kovalevsky returned home, where shortly after her arrival, her father died. In her grief she neglected mathematics for the next 6 years. Instead, she wrote fiction, theater reviews, and science articles for a newspaper. Later in her life, Kovalevsky would go on to write plays.

In 1880 Kovalevsky resumed her study of mathematics. In 1882 she began work on the **refraction** of light and published three papers on the subject. Finally, in 1883 a door opened—she was granted a temporary appointment at the University of Stockholm in Sweden, where she taught courses in the latest mathematical topics. There she published a paper on crystals in 1885. She was appointed editor of a new journal, *Acta Mathematica*, and organized conferences with leading mathematicians.

refraction the change in direction of a wave as it passes from one medium to another

In 1888 Kovalevsky entered a paper titled "On the Rotation of a Solid Body about a Fixed Point" in a competition sponsored by the French Academy of Science. The committee thought so highly of the paper that it increased the prize money. In 1889 she won a prize from the Swedish Academy of Sciences for further work on the same topic and was elected as a corresponding member to the Imperial Academy of Sciences in Russia. Later that year the university granted her status as a professor.

Unfortunately, Kovalevsky's triumph did not last long. In 1891, at the summit of her career, she died of pneumonia in Stockholm. She was just 41 years of age.

Michael J. O'Neal

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Landscape Architect

L

According to the American Society of Landscape Architects, a landscape architect plans and designs the use, allocation, and arrangement of land and water resources through the creative application of biological, physical, mathematical, and social processes. While architects design buildings and structures, landscape architects are "architects of the land," designing parks, housing developments, zoos, waterfronts, and so on, as well as stormwater drainage systems, wetlands, and species habitats. Some of the specializations they pursue include regional landscape planning, urban planning, ecological planning and design, and historic preservation and reclamation. Forty-six states require landscape architects to be licensed by completing a bachelor's degree in the field and passing the Landscape Architect Registration Examination.

The work of a landscape architect is cross-disciplinary because it spans tasks ranging from urban planning to designing species habitats.

Among other skills, landscape architects must be sound mathematicians and be able to integrate mathematical models and planning methods into a technical design. To do this, it is necessary to take accurate measurements and compute areas, volumes, and the quantity of materials needed for each component of the job, all while staying within a budget. It is important for landscape architects to be competent with CAD (computer-aided design) software to help plan projects. Additionally, they need to understand the underlying mathematics principles in construction processes and support systems, as well as in methods of construction. For example, problems of drainage would require the landscape architect to manipulate contours and spot elevations, to calculate slopes, grades, and volumes of material, and to understand hydraulics. SEE ALSO Architect.

Michael J. O'Neal

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American Society of Landscape Architects. <http://www. asla.org>.

LEONARDO'S ROBOT

In 1495 Leonardo da Vinci designed a mechanical man capable of movement similar to humans'. Looking like a suit of armor, its inner workings are complete with pulleys, cables, and gears to make it move like the bones and muscles in the human body.

Using the principles behind Leonardo's mechanical man, modern-day engineers have manufactured a new type of robot—the anthrobot. More human-like than other types of mechanical robots, anthrobots have greater flexibility, dexterity, and motion. The human-like movements of the anthrobot have made it an ideal choice for NASA's space exploration program and the construction of a space station.

Leonardo da Vinci

Italian Painter, Scientist, and Mathematician
1452–1519

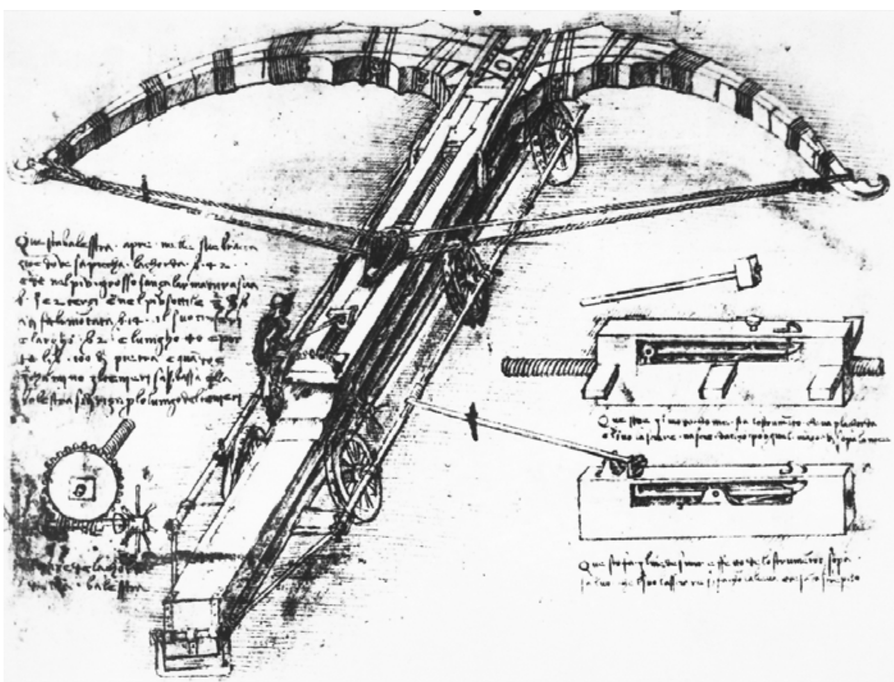
Leonardo da Vinci was born in the Italian town of Vinci. As a young boy, he showed a talent for painting. When he was 20 years old, he joined the painters' guild in Florence. Within a few years, Leonardo's talent was known all across Europe. Although he completed only thirty paintings, two of them—the *Mona Lisa* and *The Last Supper*—are among the most easily recognized paintings of all time.

Leonardo was more than a painter: He was a scientist and mathematician who explored botany, mechanics, astronomy, physics, biology, and optics. Leonardo developed prototypes of the modern helicopter, submarine, and parachute, and he attributed his scientific discoveries to mathematics. He wrote, "There is no certainty in science where mathematics cannot be applied."

Although Leonardo dabbled in different areas of mathematics, geometry was his chief focus. He discovered a proof of the Pythagorean theorem, dissected various geometric figures, and illustrated a book about geometry and art. At one point in Leonardo's life, a friend of his noted that "his mathematics experiments have distracted him so much from his painting that he can no longer stand his paint brush."

During the last three years of his life, Leonardo was a guest of Francois I, King of France. The king hoped Leonardo would produce some masterpieces for the royal court. He never did. Leonardo finished a few paintings he had already started and spent the rest of his time making scientific explorations. He died in Amboise, France.

Arthur V. Johnson II



Although known for his paintings, Leonardo da Vinci developed improvements to war implements, such as this mechanical crossbow.

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Light

Light is a form of electromagnetic radiation. Other types of electromagnetic radiation include radio waves, microwaves, infrared, ultraviolet, x-rays, and gamma rays. All electromagnetic waves possess energy. Moreover, electromagnetic waves (including light) are produced by accelerated electric charges (such as electrons). Light moves through space in a wave that has an electric part and a magnetic part. That is why it is called an electromagnetic wave.

Speed of Light

Light travels through empty space at a high speed, very close to 300,000 kilometers per second (km/s). This number is a universal constant: it never changes. Since all measurements of the speed of light in a vacuum always produce exactly the same answer, the distance light travels in a certain amount of time is now defined as the standard unit of length. For convenience, the speed of light is usually written as the symbol, c.

Characteristics of Waves

All waves, including light waves, share certain characteristics: They travel through space at a certain speed, they have frequency, and they have wavelength. The frequency of a wave is the number of waves that pass a point in one second. The wavelength of a wave is the distance between any two corresponding points on the wave. There is a simple mathematical relationship between these three quantities called the wave equation. If the frequency is denoted by the symbol f and the wavelength is denoted by the symbol λ, then the wave equation for electromagnetic waves is:

$$c = f\lambda.$$

Since c is a constant, this equation requires that a light wave with a shorter wavelength have a higher frequency.

Waves also have amplitude. Amplitude is the "height" of the wave, or how "big" the wave is. The amplitude of a light wave determines how bright the light is.

Wavelength and Color

There is a simple way to remember the order of wavelengths of light from longest to shortest: ROY G. BIV. The letters stand for red, orange, yellow, green, blue, indigo, and violet. (This violet is not the same as the crayon color called violet, which is a shade of purple.) The human eye perceives different wavelengths of light as different colors. Red is the color of the

longest wavelength the human eye can detect; violet is the shortest. Red light has a wavelength of around 700 nanometers (nm). (A nanometer is one-billionth of a meter.) Light with a wavelength longer than 700 nm is called infrared. ("Infra" means "below.") Violet light is around 400 nm. Electromagnetic radiation with a shorter wavelength is called ultraviolet. ("Ultra" means "beyond.") It is best not to use the terms "ultraviolet light" or "infrared light," for instance, because the word "light" should be applied only to wavelengths that the human eye can detect.

The refraction of light passing through water causes the spoon to look "disjointed" when viewed from the side.

refraction the change in direction of a wave as it passes from one medium to another

Refraction and Lenses

What happens when light encounters matter depends on the type of material. Glass, water, quartz, and other similar materials are transparent. Light passes through them. However, light slows down as it passes through a transparent material. This happens because the light is absorbed and reemitted by the atoms of the material. It takes a small amount of time for the atom to reemit the light, so the light slows down. In water, light travels around $0.75c$ or 225,000 km/s. In glass, the speed is even slower, $0.67c$. In diamond, light travels at less than half its speed in vacuum, $0.41c$.

When a beam of light passes from vacuum (or air) into glass, it slows down, but if the beam hits the glass at an angle, it does not all slow down at the same time. The edge of the beam that hits the glass first slows down first. This causes the beam to bend as it enters the glass. The change in direction of any wave as it passes from one material to another and speeds up or slows down is called **refraction**. Refraction causes water to appear to be shallower than it is in reality. Refraction causes a diamond to sparkle.

Refraction is also what creates a mirage. Sometimes the air a few centimeters above the ground is much warmer than air a few meters farther up. As light from the sky passes into this warmer air, it speeds up and bends away from the ground. An observer may see light from the sky and be fooled into thinking that it is a lake. Sometimes, even trees and houses can be seen in the mirage, but they will appear upside down.

Refraction of light allows a lens to perform its function. In a converging lens, the center of the beam reaches the lens first and slows down first. This causes the beam to be bent toward the center of the lens. A parallel beam of light passing through a good-quality lens will be bent so that all the light arrives at a single point called the focal point. The distance from the lens to the focal point is called the focal length, f. A diverging lens spreads the beam out so that it appears to be coming from the focal point.

In a slide projector, the lens projects an image of an object (the slide) onto a screen. The distance from the lens to the image and the distance of the lens to the object are related to the focal length by this strange-looking formula (d_i is the image distance and d_o is the object distance):

$$\frac{1}{f} = \frac{1}{d_i} + \frac{1}{d_o}.$$

Interpreting this formula is a little difficult. Remember that the focal length of the lens does not change, so $\frac{1}{f}$ is a constant. If the image distance (d_i) gets larger, the object distance (d_o) must get smaller to make the fractions add to the same constant value.

Reflection and Mirrors

When light hits a surface, it can also be **reflected**. Sometimes light is both refracted and reflected. If the object is opaque, however, the light will just be reflected. When light is reflected from a surface, it bounces off at the same angle to the surface. The angle of incidence is equal to the angle of reflection. SEE ALSO Vision, Measurement of.

reflected light or sound waves returned from a surface

Elliot Richmond

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Light Speed

Many science fiction writers feel that humans will only be able to feasibly explore the Milky Way galaxy (and beyond) when it is possible to travel at, or above, the speed of light. Unmanned probes have already been sent to explore the solar system and beyond. NASA's *Voyager I* and *Voyager II* blasted off from Earth in the late 1970s to explore the outer planets and are now far beyond them.

The Light-Year

The dimensions of the universe are so enormous that they overwhelm conventional units of distance (such as the meter, kilometer, or mile). Therefore, a much larger unit distance is needed—the light-year. A light-year is the amount of distance that light travels in vacuum in one Earth year. Astronomers have found light-years to be a convenient distance when measuring the distance between stars and other celestial bodies.

In one second, light travels approximately 186,000 miles. To extend this out one year, multiply 186,000 miles times the number of seconds in a minute (60), times the number of minutes in an hour (60), times the number of hours in a day (24), and times the number of days in a year (365). As a result, light travels approximately 5,865,696,000,000 (more than 5.8 trillion) miles in one year—the distance in one light-year.

To better grasp the distance involved in traveling in outer space, imagine flying NASA's space shuttle from Los Angeles to New York City. This journey would take approximately 20 minutes at a speed of about 17,000 mph (miles per hour). At that speed, a journey to the Sun would take about 228 days. Beyond the Sun, the next closest stars to Earth are those of the triple star system of Alpha Centauri A, Alpha Centauri B, and Proxima Centauri at 4.3 light-years away, more than 25 trillion miles distant. This flight on the shuttle would take about 170,000 years!

A voyage from Earth to the center of the galaxy, a distance of about 30,000 light-years, would take about 1.2 billion years. Even the two *Voyager* spacecraft, the fastest machines ever launched from Earth, are now traveling

at only 10 miles per second, not even one ten-thousandth the speed of light. These spacecraft would take 78,000 years to reach the Alpha Centauri star cluster.

As one can see, it would take extremely long periods of time to travel from Earth to other stars at conventional speeds, which is why the prospect of faster-than-light space travel has become so popular. It seems to be the only way to travel throughout the universe.

Is Traveling at the Speed of Light Possible?

Albert Einstein (1879–1955) developed his special theory of relativity in 1905. It declared that any material object can approach the speed of light, but it is impossible to go at or above this cosmic speed limit. The speed of light (denoted as c) in a vacuum—an approximation of what actually is found in interstellar space—is a fundamental constant of physics and nature. The speed of light is as basic as gravity, which Einstein tackled in his 1915 general theory of relativity.

According to Einstein, if one could travel at the speed of light, then time would stretch to infinity and distances would be abolished altogether. Yet one obstacle to traveling at the speed of light is that matter attempting to attain light speed requires more and more energy but with very little resulting additional speed. At a speed above the speed of light, an object theoretically would be going "backwards in time," an occurrence viewed by many scientists as impossible.

Interstellar space travel appears to be extremely, if not prohibitively, expensive, even if future technologies could make it possible. All the propulsion systems proposed so far for faster-than-light voyages, such as warp drives, would require huge amounts of energy—more energy that is even conceivable to produce. On the one hand, there are many trivial ways in which things, in a sense, can be going faster than light, and there may be other more genuine possibilities. On the other hand, there are also good reasons to believe that real faster-than-light travel and communication will always be unachievable.

Ways of Traveling Faster than the Speed of Light

One way to (apparently) travel faster than light is to make light travel slower. Light in a vacuum travels at a speed c, which is a universal constant, but in a dense medium such as water or glass, light slows down to $\frac{c}{n}$, where n is the refractive index of the medium (for instance, $n = 1.0003$ for air and $n = 1.4$ for water). It is possible for particles to travel through air or water at faster than the speed of light in the medium. Therefore, going faster than the speed of light really means exceeding the speed of light in vacuum c, not in a medium such as air or water.

Another way to (apparently) travel faster than light is to misrepresent speeds. If spaceship A is traveling away from a point at $0.7c$ in one direction, and another spaceship B is traveling away from the same point at $0.8c$ in the opposite direction, then the total distance between A and B may be thought to be increasing at $1.5c$ (derived from $0.7c + 0.8c$). However, this is not what is normally meant by relativistic speeds.

The true speed of spaceship A relative to spaceship B is the speed at which an observer in B observes the distance from A to be increasing. The

two speeds must be added using the relativistic formula for addition of velocities:

$$w = \frac{(u + v)}{(1 + uv/c^2)}$$

where

$v = 0.7c$ is the speed of spacecraft A,

$u = 0.8c$ is the speed of spacecraft B, and

c is the speed of light.

After inserting the appropriate values into the equation, the relative speed w is actually about $0.96c$ (that is, 0.96 times the speed of light) and, therefore, not faster than the speed of light.

Another way to (apparently) travel faster than light is to observe how fast a shadow can move. If you project a shadow of your finger using a nearby lamp onto a far away wall and then move your finger, the shadow will move much faster than your finger. It can actually move much faster than this if the wall is at some oblique angle. If the wall is very far away, the movement of the shadow will be delayed because of the time it takes light to get there, but its speed is still amplified by the same ratio. The speed of a shadow is therefore not restricted to being less than the speed of light.

These are all examples of things that can go faster than light, but they are not physical (material) objects. It is not possible to send information on a shadow, so faster-than-light communication is not possible in this way. Faster-than-light travel cannot logically be deduced just because some "things" go faster-than-light or appear to do so. This is not what is meant by faster-than-light travel, although it shows how difficult it is to define what is really meant by traveling faster than the speed of light.

Proposals for Faster-than-Light Travel

One proposal for traveling faster than the speed of light is to use *wormholes.* A wormhole is a four-dimensional shortcut through space-time in which two regions of the universe are connected by a narrow passageway. The wormhole would permit matter/energy to proceed from one spot in the universe to another in a shorter time than it would take light otherwise. Wormholes are a feature of general relativity, but to create them it is necessary to change the **topology** (or the physical features) of space-time.

topology the study of those properties of geometric figures that do not change under such nonlinear transformations as stretching or bending

The complete wormhole geometry consists of a black hole, a white hole, and two universe-regions connected at their horizons by a wormhole. According to Einstein's theory of gravitation, empty space is actually a tightly woven fabric of space and time. Massive objects warp the space-time fabric, just as a bedsheet would be pushed down into a deep valley if a whale were to lie on it. Anything that comes near the valley naturally rolls in, and that "falling" is the force perceived as gravity. If the whale twists around on the bed, its motion carries the bedsheet along.

If Einstein's theory is correct, space-time should likewise be dragged around massive objects. Black holes (like whales on a bedsheet) are objects that are so massive and dense that immense gravity warps space around the core, not allowing light or anything else to escape. A white hole, however, is a black hole running backward in time. Just as black holes (supposedly) pull things in, white holes (supposedly) push things out. This so-called

naturally made warping of space in the form of wormholes could provide a means of quickly traveling to distant regions of space.

A warp drive, sometimes called hyperspace drive, is a (theorized) mechanism for warping space-time in such a way that an object could move faster than light. The most famous spaceship to use warp drive (at least in science fiction) is Star Trek's *U.S.S. Enterprise*. Its concept expands space-time behind the spaceship and contracts space-time in front of the spaceship. The warp in space-time makes it possible for an object to go faster than light speed.

But the problem with developing a warp drive is the same problem with formulating large wormholes. To construct it, one would need a ring of exotic negative energy wrapped around the spaceship. Even if such exotic matter can exist, it is unclear how it could be deployed to make the warp drive work.

A more likely scenario for deep-space travel makes use of Einstein's special theory of relativity. For objects moving at relativistic speeds (speeds near but below the speed of light), there is an observable stretching out of time relative to an observer in a stationary reference frame. For example, say a spacecraft travels at a constant speed of nine-tenths the speed of light ($0.9c$) from Earth to Alpha Centauri, a distance of 4.3 light-years. According to **time dilation**, the time observed on Earth would be faster than the time onboard the spacecraft according to the equation

time dilation the principle of general relativity which predicts that to an outside observer, clocks would appear to run more slowly in a powerful gravitational field

$$t^* = t\{(1 - (\tfrac{v^2}{c^2}))\}^{1/2}$$

where

t^* is the time indicated by the spacecraft clock,

t is the time indicated by Earth's clock, and

c is the speed of light.

Therefore, according to Earth's clock, the spacecraft took 4.8 years to complete the trip (4.3 light-years divided by $0.9c$). However, onboard the spacecraft, the journey takes only $t^* = 4.8\{(1 - (\frac{(0.9c)^2}{c^2}))\}^{1/2}$, or 2.1 years.

A Future Reality

With present technology, it is possible to (theoretically) produce spaceships that could slowly accelerate to near the speed of light so that a trip to the nearer stars would (in Earth time) take perhaps a few hundred years. However, in the spaceship, the entire trip, due to time dilation, would perhaps take a few dozen years, depending on the time it took to accelerate up (and then accelerate down) to and from light speed and how closely to light speed the spacecraft traveled.

It is rather difficult to define exactly what is really meant by faster-than-light travel and communication. Many things such as shadows can go faster than light speed but not in a useful way that can carry information. There are several serious possibilities for real faster-than-light mechanisms that have been proposed in the scientific literature, but technical difficulties still exist.

Heisenberg uncertainty principle the principle in physics that asserts it is impossible to know simultaneously and with complete accuracy the values of certain pairs of physical quantities such as position and momentum

The **Heisenberg uncertainty principle** tends to stop the use of apparent faster-than-light quantum effects for sending information or matter. In general relativity there are potential means of faster-than-light travel, but they may be impossible to actually construct. It is highly unlikely that en-

gineers will be building spaceships with faster-than-light drives in the foreseeable future. It is curious, however, that theoretical physics, as presently understood, seems to leave the door open to the possibility.

Just because today something seems impossible, that does not mean that it will never become feasible. SEE ALSO Cosmos; Einstein, Albert; Space Exploration; Universe.

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Limit

The concept of limit is an essential component of **calculus**. Limits are typically the first idea of calculus that students study. Two fundamental concepts in calculus—the **derivative** and the **integral**—are based on the limit concept. Limits can be examined using three intuitive approaches: number sequences, functions, and geometric shapes.

calculus a method of dealing mathematically with variables that may be changing continuously with respect to each other

derivative the derivative of a function is the limit of the ratio of the change in the function; the change is produced by a small variation in the variable as the change in the variable is allowed to approach zero; an inverse operation to calculating an integral

integral a mathematical operation similar to summation; the area between the curve of a function, the *x*-axis, and two bounds such as $x = a$ and $x = b$; an inverse operation to finding the derivative

Number Sequences

One way to examine limits is through a sequence of numbers. The following example shows a sequence of numbers in which the limit is 0.

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$$

The second number in the sequence, $\frac{1}{2}$, is the result of dividing the first number in the sequence, 1, by 2. The third number in the sequence, $\frac{1}{4}$, is the result of dividing the second number in the sequence, $\frac{1}{2}$, by 2.

This process of dividing each number by 2 to acquire the next number in the sequence is continued in order to acquire each of the remaining values. The three dots indicate that the sequence does not end with the last number that appears in the list, but rather that the sequence continues infinitely.

If the sequence continues infinitely, the values in the sequence will get closer and closer to 0. The numbers in the sequence, however, will never actually take on the value of zero. The mathematical concept of approaching a value without reaching that value is referred to as the "limit concept." The value that is being approached is called the limit of the sequence. The limit of the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$ is 0.

The example below displays several sequences and their limits. In each case, the values in the sequence are getting closer to their limit.

Example 1: 0.9, 0.99, 0.999, 0.9999, 0.99999, 0.999999, 0.9999999, . . . Limit: 1

Example 2: 5.841, 5.8401, 5.84001, 5.840001, 5.8400001, 5.84000001, . . . Limit: 5.84

Example 3: $-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{5}, -\frac{1}{6}, -\frac{1}{7}, -\frac{1}{8}, -\frac{1}{9}, -\frac{1}{10}, \ldots$ Limit: 0

Not all sequences, however, have limits. The sequence 1, 2, 3, 4. . . increases and does not approach a single value. Another example of a sequence that has no limit is $-1.1, 2.2, -3.3, 4.4, -5.5, 6.6, \ldots$. Because there is no specific number that this sequence approaches, the sequence has no limit.

Functions

Limits can also be examined using functions. An example of a function is $f(x) = \frac{1}{x}$. One way to examine the limit of a function is to list a sample of the values that comprise the function. The left-hand portion of the table can be used to examine the limit of the function $f(x) = \frac{1}{x}$ as x increases.

As the values in the x column increase, the values in the $f(x)$ column get closer to 0. The limit of a function is equal to the value that the $f(x)$ column approaches. The limit of the function $f(x) = \frac{1}{x}$ as x approaches infinity is 0.

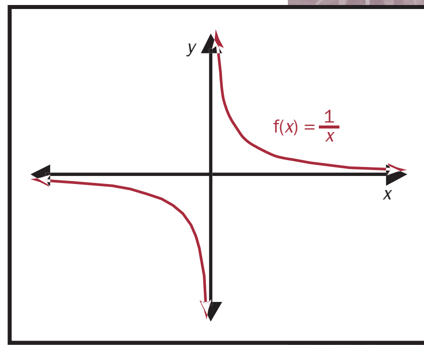
TABLE OF SAMPLE VALUES FOR THE FUNCTION $f(x)=\frac{1}{x}$

As x increases		As x approaches 0			
x	f(x)	x	f(x)	x	f(x)
1	1	1	1	–1	–1
2	$\frac{1}{2}$	$\frac{1}{2}$	2	$-\frac{1}{2}$	–2
3	$\frac{1}{3}$	$\frac{1}{4}$	4	$-\frac{1}{3}$	–3
4	$\frac{1}{4}$	$\frac{1}{5}$	5	$-\frac{1}{4}$	–4
5	$\frac{1}{5}$	$\frac{1}{6}$	6	$-\frac{1}{5}$	–5
6	$\frac{1}{6}$	$\frac{1}{7}$	7	$-\frac{1}{6}$	–6
7	$\frac{1}{7}$	$\frac{1}{8}$	8	$-\frac{1}{7}$	–7
.		.			.
.		.			.
.		.			.
100	$\frac{1}{100}$	$\frac{1}{100}$	100	$-\frac{1}{100}$	–100
.		.			.
.		.			.
.		.			.

Functions can also be plotted on a **Cartesian plane**. A graph of the function $f(x) = \frac{1}{x}$ is shown in the figure. The color curve represents the function. As the x values increase, the color curve or the $f(x)$ values get closer and closer to 0. Once again, the limit of the function $f(x) = \frac{1}{x}$ as x goes to infinity is 0.

It is important to consider what value x is approaching when determining the limit of $f(x)$. If x were approaching 0 in the preceding example, $f(x)$ would not have a limit. The reason for this can be understood using the middle and right-hand portions of the table.

The table suggests that the values for $f(x)$ continue to increase as x approaches 0 from values that are greater than 0. The table also suggests that the values for $f(x)$ continue to decrease as x approaches 0 from values that are less than 0. Because the $f(x)$ values do not approach a specific value, the function $f(x) = \frac{1}{x}$ does not have a limit as x approaches 0.

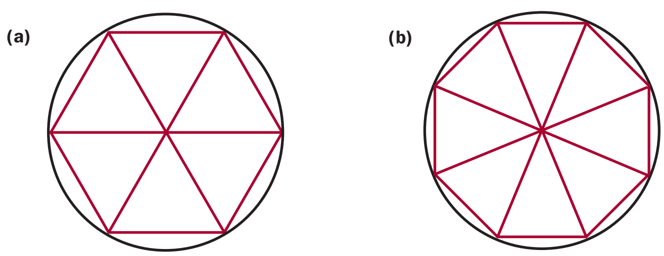


The limit of this function as x goes to infinity is 0. The function does not have a limit as x approaches 0.

Cartesian plane a mathematical plane defined by the x and y axes or the ordinate and abscissa in a Cartesian coordinate system

Geometric Shapes

A typical application of the limit concept is in finding area. For example, one method for estimating the area of a circle is to divide the circle into small triangles, as shown below, and summing the area of these triangles. The circle in (a) is divided into six triangles. If a better estimate of area is desired, the circle can be divided into smaller triangles as shown in (b).



If the exact area of the circle is needed, the number of triangles that divide the circle can be increased. The limit of sum of the area of these triangles, as the number of triangles approaches infinity, is equal to the standard formula for finding the area of a circle, $A = \pi r^2$, where A is the area of the circle and r is its radius.

In summary, limit refers to a mathematical concept in which numerical values get closer and closer to a given value or approaches that value. The value that is being approached is called the "limit." Limits can be used to understand the behavior of number sequences and functions. They can also be used to determine the area of geometric shapes. By extending the process that is used for finding the area of a geometric shape, the volume of geometric solids can also be found using the limit concept. SEE ALSO Calculus; Infinity.

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Lines, Parallel and Perpendicular

In mathematics, the term "straight line" is one of the few terms that is left undefined. However, most people are comfortable with this undefined concept, which can be modeled by a pencil, a stiff wire, the edge of a ruler, or even an uncooked piece of spaghetti. Mathematicians sometimes think of a line as a point moving forever through space. Lines can be curved or straight, but in this entry, only straight lines are considered.

A line, in the language of mathematics, has only one dimension—length—and has no end. It stretches on forever in both directions, so that its length cannot be measured. When a line is modeled with a piece of spaghetti, a line segment is actually being represented. The model of a line segment has thickness (or width), while the idea that it models—a mathematical line—does not. So a mathematical line is a notion in one's mind, rather than a real object one can touch and feel, just as the notion of "two" is an idea in one's mind—that quality and meaning that is shared by two apples, two trucks, and the symbols //, 2, ☺☺, and ii.

plane generally considered an undefinable term, a plane is a flat surface extending in all directions without end, and that has no thickness

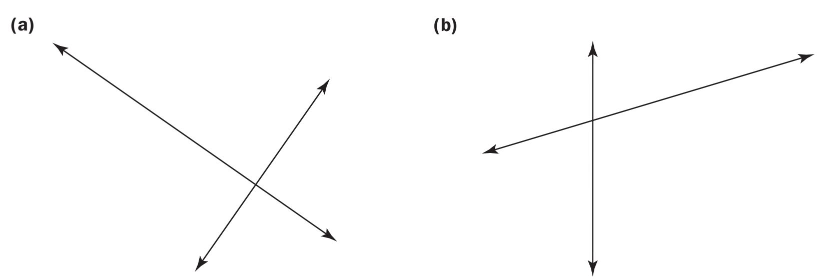
geometry the branch of mathematics that deals with the properties and relationships of points, lines, angles, surfaces, planes, and solids

Think of two straight lines in a **plane** (another undefined term in **geometry**). Someone can model this idea, imperfectly, by two pencils or two pieces of spaghetti lying on a desktop. Now, mentally or on a desktop, push these lines around, still keeping them on the plane, and see the different ways two lines can be arranged. If these two lines meet or cross, they have one point in common. In the language of mathematics, the two lines intersect at one point, their point of intersection. If two lines are moved so that they coincide, or become one line, then they have all of their points in common.

What other arrangements are possible for two lines in a plane? One can place them so that they do not coincide (that is, one can see that they are two separate lines), and yet they do not cross, and will never cross, no matter how far they are extended. Two lines in the same plane, which have no point in common and will never meet, are called parallel lines. If one draws a grid, or coordinate system, on the plane, she can see that two parallel lines have the same slope, or steepness. Are there any parallel lines in nature, or in the human-made world? There are many models of parallel lines in the world we build: railroad tracks, the opposite sides of a picture frame, the lines at the corners of a room, fence posts. In nature, parallel lines are not quite so common, and the models are only approximate: tracks of an animal in the snow, tree trunks in a forest, rays of sunlight.

perpendicular forming a right angle with a line or plane

The only other possible arrangement for two lines in the plane is also modeled by a picture frame, or a piece of poster board. Two sides of a rectangle that are not parallel are **perpendicular**. Perpendicular lines meet, or intersect, at right angles, that is, the four angles formed are all equal. The first pair of lines in part (a) of the figure below meet to form four equal angles; they are perpendicular. The second pair in part (b) forms two larger angles and two smaller ones; they are not perpendicular.



Perpendicular lines occur everywhere in buildings and in other constructions. Like parallel lines, they are less common in nature. On a coordinate system, two perpendicular lines (unless one of them is horizontal) have slopes that multiply to a product of –1; for example, if a line has a slope of 3, any line perpendicular to it will have a slope of $-\frac{1}{3}$. SEE ALSO Lines, Skew; Slope.

Lucia McKay

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Lines, Skew

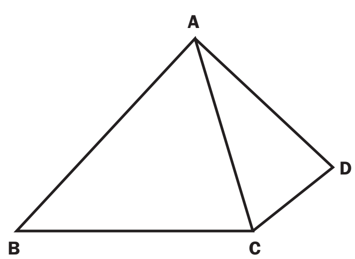
For geometric figures in a plane, two straight lines must either be parallel to one another or must intersect at one point. Skew lines are non-parallel and do not intersect. Skew lines must therefore lie in separate planes from one another. Since skew lines are defined in terms of distinct planes, discussing such lines leads directly to the branch of mathematics called **solid geometry**.

solid geometry the geometry of solid figures, spheres, and polyhedrons; the geometry of points, lines, surfaces and solids in three-dimensional space

Solid geometry is the branch of Euclidian geometry (named for Euclid, c. 325 B.C.E.–265 B.C.E.) that examines the relative positions, sizes, shapes, and other aspects of geometric figures that are not in a single plane. Whereas **plane geometry** is about two-dimensional space described by parameters such as length and width, solid geometry concerns itself with three-dimensional space.

plane geometry the study of geometric figures, points, lines, and angles and their relationships when confined to a single plane

One example of a three-dimensional object is a cube, which has height, length, and width. Another familiar example of a solid (three-dimensional) figure is the pyramid. Figures like these can be used to illustrate skew lines.



Examples of Skew Lines

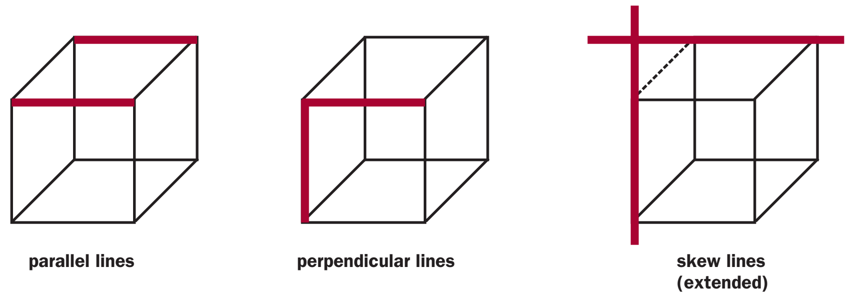
The edges of the pyramid above form skew lines. Each of the four faces of the pyramid (as well as its bottom) define a unique **plane**. Line segments *AB*, *AC*, and *BC*, for instance, define a unique plane, and each plane constitutes one of the four faces of the pyramid. None of the three line segments (*AB*, *AC*, *BC*) can be skew lines relative to one another because they all lie in the same plane. Recall that lines in the same plane either intersect (as do *AB*, *AC*, and *BC*) or are parallel to one another.

plane generally considered an undefinable term, a plane is a flat surface extending in all directions without end, and that has no thickness

SKEW LINES AND HIGHWAYS

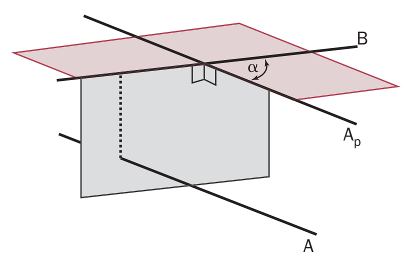
Several of the lines formed by elevated roadways at a highway interchange skew to one another; that is, they neither intersect nor are parallel. However, lampposts and overpass columns are vertical structures and, therefore, all of them must be parallel to one another.

There are, however, several pairs of line segments on the pyramid that form skew lines. Line segments *AB* and *CD* form a pair of such lines. These two segments are skew to one another because they are neither parallel nor intersecting. Even if the line segments are extended into infinite lines, they still remain skew. Though some of the line segments are hidden from view in this picture, one can envision several other pairs of skew lines formed by the edges of the pyramid.



A cube possesses many combinations of parallel, perpendicular, and skew line segments. The cubes above illustrate different line pairs: parallel, perpendicular, and skew. (The skew segments have been extended to indicate infinite lines.) The shortest distance between the two skew lines is the length of the dashed line segment, which is perpendicular to both of the indicated skew lines. Any other distance measured between the two skew lines will be longer than the dashed line segment.

The spatial orientation of any two skew lines can be described by two quantities: the closest or perpendicular distance between the two lines, and the angle between them. The illustration below shows these two quantities of distance and angle.



The two skew lines are *A* and *B*. Line *B* lies along the intersection of the two shaded planes. The dotted line segment joining lines *A* and *B* is perpendicular to both, and is the shortest distance between any two points on lines *A* and *B*.

To find the angle between *A* and *B*, a line, A_p, is constructed, which is parallel to *A* and which also intersects line *B*. The angle between lines A_P and *B* is designated by the Greek letter alpha (α). Because A_p and *A* are parallel lines, the angle α is also the angle between lines *A* and *B*. Therefore, by determining the distance (dotted line segment) and angle (α) between skew lines *A* and *B*, the relative position between the two is uniquely determined. SEE ALSO Lines, Parallel and Perpendicular.

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Locus

Sometimes it is useful in mathematics to describe the path that a point traces as it moves in a plane to meet certain conditions. For example, what is the path that a point on the end of the second hand of a clock traces in 60 seconds? This answer, of course, is a circle.

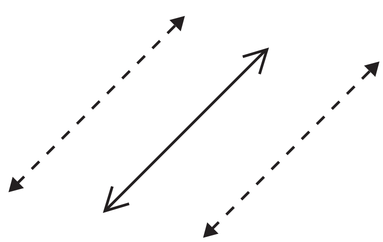
One way to define a circle is to say that a circle is the **locus** of all the points in a **plane** that are a given distance from a fixed point, called the center.

locus in geometry, the set of all points, lines or surfaces that satisfies a particular requirement

plane generally considered an undefinable term, a plane is a flat surface extending in all directions without end, and that has no thickness

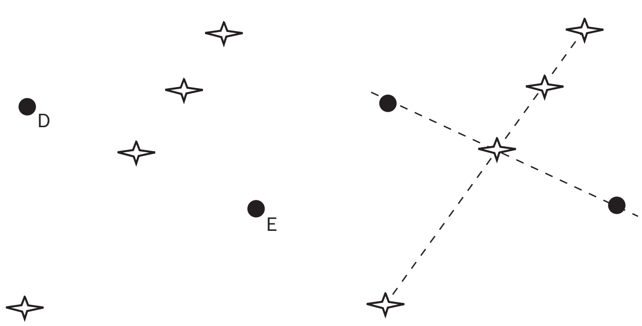
A locus in a plane can be thought of as all the possible locations or positions that a point can take as it moves to meet certain stated conditions. What is the locus of, or path traced by, a point in a plane that moves so that it is always three inches from point A? This locus will be a circle, with point A as the center and a radius of three inches.

What is the locus in a plane of all points that are 2 centimeters from a given line? This locus is made up of two lines, each parallel to the given line, one on each side, and at a distance of 2 centimeters from it, as illustrated in the figure below. The two dashed lines form the locus.



What is the locus of all points in a plane that are the same distance from point D and from point E? To answer this, one might draw some example points that are **equidistant** from D and E, such as the points marked with a star in the left-hand illustration of the figure below.

equidistant at the same distance



These example points indicate that the locus of all the points in a plane that are the same distance from D as they are from E is a line that is the

perpendicular forming a right angle with a line or plane

perpendicular bisector of the line segment that joins D and E, as shown in the right-hand illustration.

The idea of a locus can be used not just in a plane but also in three-dimensional space. For example, the preceding example, extended into space, becomes the locus of all points that are equidistant from points D and E. This locus will be the entire plane that is perpendicular to the plane containing DE and its perpendicular bisector and that contains the entire perpendicular bisector.

In space, the locus of all points at a given distance from a specific point is a sphere with a center at the point and a radius equal to the given distance. In space, the locus of all points at a given distance from a line segment is a cylinder with a hemisphere at each end.

parabola a conic section; the locus of all points such that the distance from a fixed point called the focus is equal to the perpendicular distance from a line

ellipse one of the conic sections, it is defined as the locus of all points such that the sum of the distances from two points called the foci is constant

hyperbola a conic section; the locus of all points such that the absolute value of the difference in distance from two points called foci is a constant

The idea of locus can also be used to define the conic sections. In a plane, a circle is the locus of all points at a given distance from a specific point; a **parabola** is the locus of all points such that each point on the curve is the same distance from a specific point as its distance from a specific line; an **ellipse** is the locus of all points such that, for each point on the curve, the sum of the distances from each of two separate specific points, called the foci, remains the same; and a **hyperbola** is the locus of all points such that, for each point on the curve, the absolute value of the difference of the distances to each of two separate specific points, called the foci, remains the same. SEE ALSO Conic Sections.

Lucia McKay

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Logarithms

The logarithm of a positive real number x to the base-a is the number y that satisfies the equation $a^y = x$. In symbols, the logarithm of x to the base-a is $\log_a x$, and, if $a^y = x$, then $y = \log_a x$.

Essentially, the logarithm to base-a is a function: To each positive real number x, the logarithm to base-a assigns x a number y such that $a^y = x$. For example, $10^2 = 100$; therefore, $\log_{10} 100 = 2$. The logarithm of 100 to **base-10** is 2, which is an elaborate name for the power of 10 that equals 100.

base-10 a number system in which each place represents a power of 10 larger than the place to its right

base-2 a binary number system in which each place represents a power of 2 larger than the place to its right

Any positive real number except 1 can be used as the base. However, the two most useful integer bases are 10 and 2. **Base-2**, also known as the binary system, is used in computer science because nearly all computers and calculators use base-2 for their internal calculations. Logarithms to the base-10 are called common logarithms. If the base is not specified, then base-10 is assumed, in which case the notation is simplified to $\log x$.

Some examples of logarithms follow.

$\log 1 = 0$ because $10^0 = 1$

$\log 10 = 1$ because $10^1 = 10$

$\log 100 = 2$ because $10^2 = 100$

$\log_2 8 = 3$ because $2^3 = 8$

$\log_2 2 = 1$ because $2^1 = 2$

$\log_5 25 = 2$ because $5^2 = 25$

$\log_3 (\frac{1}{9}) = -2$ because $3^{-2} = \frac{1}{9}$

The logarithm of multiples of 10 follows a simple pattern: logarithm of 1,000, 10,000, and so on to base-10 are 3, 4 and so on. Also, the logarithm of a number a to base-a is always 1; that is, $\log_a a = 1$ because $a^1 = a$.

Logarithms have some interesting and useful properties. Let x, y, and a be positive real numbers, with a not equal to 1. The following are five useful properties of logarithms.

1. $\log_a (xy) = \log_a x + \log_a y$, so $\log_{10} (15) = \log_{10} 3 + \log_{10} 5$
2. $\log_a (\frac{x}{y}) = \log_a x - \log_a y$, so $\log (\frac{2}{3}) = \log 2 - \log 3$
3. $\log_a x^r = r \log_a x$, where r is any real number, so $\log 3^5 = 5 \log 3$
4. $\log_a (\frac{1}{x}) = -\log_a x$, so $\log (\frac{1}{4}) = (-1) \log 4$ because $\frac{1}{4} = (4)^{-1}$
5. $\log_a a^r = r$, so $\log_{10} 10^3 = 3$

Logarithms are useful in simplifying tedious calculations because of these properties.

History of Logarithms

The beginning of logarithms is usually attributed to John Napier (1550–1617), a Scottish amateur mathematician. Napier's interest in astronomy required him to do tedious calculations. With the use of logarithms, he developed ideas that shortened the time to do long and complex calculations. However, his approach to logarithms was different from the form used today.

Fortunately, a London professor, Henry Briggs (1561–1630) became interested in the logarithm tables prepared by Napier. Briggs traveled to Scotland to visit Napier and discuss his approach. They worked together to make improvements such as introducing base-10 logarithms. Later, Briggs developed a table of logarithms that remained in common use until the advent of calculators and computers. Common logarithms are occasionally also called Briggsian logarithms. SEE ALSO Powers and Exponents.

Rafiq Ladhani

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Lotteries, State

State lotteries give participants the dream of winning huge jackpots while providing the state with funding for education, transportation, and other projects. State lotteries currently operate in about three-fourths of the states in the United States. These lotteries provide individuals with a relatively

inexpensive means of gambling on the possibility of winning a huge jackpot. The first legal state lottery was established in New Hampshire in 1964, culminating 10 years of legislative effort.

Probabilities of Winning

A wide range of games is offered by the numerous state lotteries across the country. The offerings of the Florida State Lottery in 2001, for example, include Florida Lotto, Mega Money, Fantasy 5, Play 4, and CASH 3, as well as numerous instant games played with scratch-off tickets. These games cater to the preferences of potential players by varying the size of the prizes awarded and the probabilities of winning.

Florida Lotto, the premier game in the Florida State Lottery, holds drawings twice a week. In this game, six balls are drawn from a container holding fifty-three balls numbered from 1 to 53. To win the jackpot, the six numbers chosen by a player must match the numbers on the six balls drawn. The order in which the numbers are selected does not matter. The Florida State Lottery claims that the odds of winning this jackpot are 1 in 22,957,480. Players can also win by matching five, four, or three numbers as well. The probability of matching five of six is 1 in 81,410, and continues to increase as the amount of matched numbers decreases.

The probability of winning the jackpot in this game can be verified by determining the amount of six-number combinations possible or by multiplying the probabilities of six consecutive successful selections. Calculating the number of combinations yields the following, which can be expressed using the notation of factorials, denoted as "!",

$$\frac{(53!)}{[(47!)(6!)]} = \frac{(53\times52\times51\times50\times49\times48)}{(6\times5\times4\times3\times2\times1)} = 22{,}957{,}480.$$

Of these combinations, only one corresponds to selecting all six numbers correctly. Alternatively, using probabilities, you could multiply the probabilities that the numbers on a ticket are selected on each of the six successive selections. The probability of the first number drawn matching one of the player's chosen numbers is 6 in 53. If this occurs, there are five favorable numbers in the remaining fifty-two, and so on. Therefore, the probability of six consecutive successful selections is $(\frac{6}{53})(\frac{5}{52})(\frac{4}{51})(\frac{3}{50})(\frac{2}{49})(\frac{1}{48})$ or 1 in 22,957,480.

Verifying the probability of correctly selecting five of the six numbers is only slightly more complex. Using the same logic as in the previous example, the probability of the first five matching numbers being selected on the first five draws is $(\frac{6}{53})(\frac{5}{52})(\frac{4}{51})(\frac{3}{50})(\frac{2}{49})$. If this occurs, there are forty-eight numbers remaining, and forty-seven are favorable to the result of correctly matching five of six numbers. Thus the probability of attaining this outcome would be $(\frac{6}{53})(\frac{5}{52})(\frac{4}{51})(\frac{3}{50})(\frac{2}{49})(\frac{47}{48})$. Though the fractions being multiplied would be different, the probability of the first number not matching and the next five matching would be $(\frac{47}{53})(\frac{6}{52})(\frac{5}{51})(\frac{4}{50})(\frac{3}{49})(\frac{2}{48})$ which is equal to the previous result. It is easily verifiable that the probability of the nonmatching number occurring on any of the six selections is equal. Thus the overall probability of correctly selecting five of six is $6(\frac{6}{53})(\frac{5}{52})(\frac{4}{51})(\frac{3}{50})(\frac{2}{49})(\frac{47}{48})$ or 1 in 81,409.50355, which rounds to 1 in 81,410.

Though people do win large lottery jackpots with regularity, the odds of an individual winning the lottery are very slim.

Fantasy 5 requires the player to match five numbers selected from 26 numbers. Prizes are also awarded to those who correctly match three or four numbers. The probability calculations for this game are analogous to those of Florida Lotto. In Fantasy 5, the probabilities of matching five, four, or three numbers are 1 in 65,780, 1 in 627, and 1 in 32, respectively.

At the other end of the spectrum is Florida's CASH 3 game. This game gives the player a probability of 1 in 1,000 of winning $500 on a $1 ticket or $250 on a 50-cent ticket. In order to win, the player must match a 3-digit number, each digit of which is a number from 0 to 9. In this game, unlike Lotto, each digit is randomly selected from its own set of numbers from 0 to 9 and the order of the result is important.

The Mega Money game adds a bit of a twist. In this game the player picks four numbers from the thirty-two numbers on the top of the ticket and one number from thirty-two on the lower half of the ticket. In order to win, all four numbers on the upper portion and the number on the lower portion of the ticket must be drawn. Finding the probability of matching the first four numbers can be found in the same way as determining the jackpot probability in Lotto. In Mega Money, the probability is $(\frac{4}{32})(\frac{3}{31})(\frac{2}{30})(\frac{1}{29})$ = 1 in 35,960. To determine the probability of winning the big prize in this game, you must multiply that result by the chances of correctly matching the one number on the lower part of the ticket, which is drawn from another bin of thirty-two balls. The probability of winning the big prize, which averages $200,000 in this game, is: $(\frac{1}{35{,}960})(\frac{1}{32})$ = 1 in 1,150,720.

Perspectives

When there are no winners in a large game, like Florida Lotto, new tickets are sold for the next drawing, increasing the value of the prize. Some feel that if the jackpot has not hit several times in a row, that it is "due to be hit." In fact, the probability of any particular combination winning does not change. However, the increase in interest and, correspondingly, in the number of tickets purchased makes it more likely that the jackpot will hit than if a smaller number of tickets were sold.

Consider a hypothetical game in which the probability of winning the jackpot is 1 in 1,000,000. If the jackpot is relatively low and only 100,000 tickets are sold, the probability of the jackpot being hit is at most 1 in 10. The probability is not necessarily equal to $\frac{1}{10}$ since almost certainly, more than one person will have selected the same combination. Now assume the jackpot has not hit for several drawings and a huge prize has accrued. If more than a million tickets have been sold covering 900,000 of the possible 1,000,000 combinations, the probability of the jackpot being hit will now be 90 percent. The chances of any individual ticket winning, however, remains 1 in 1,000,000. An interesting consequence here is that there is no guarantee that a winner will receive the entire jackpot. It is possible that two or more ticket holders will have chosen the correct combination of numbers and would then divide the jackpot equally.

Allocation of Revenue

The potential revenue generated by a lottery has provided a strong incentive for states to pass legislation to legalize them. While the exact figures vary from state to state, normally about thirty to thirty-eight percent of the intake goes toward funding state programs. A little more than half is returned in prize money and the remainder is applied to various expenses associated with operating the lottery, such as advertising and paying commissions to vendors. To put the monetary amounts from lotteries that go into state budgets into perspective, the New Hampshire state lottery provided the state department of education with over $65 million in one fiscal year, bringing the total amount of aid to education in that state to $665 million. California boasts of more than $12 billion being earmarked for its public schools since 1985, while the New York state lottery provided $1.35 billion to education in the 1999–2000 fiscal year.

Prize Payoffs

The allure of huge jackpots influences many individuals to purchase lottery tickets. Advertisements touting multi-million dollar jackpots are common. Information regarding the payoff procedures for jackpots is readily available from state lottery commissions as well as from other sources. Most lotteries allow the winner to choose between one immediate lump sum payment or yearly payments over a period of time.

A jackpot winner in the California Super Lotto Plus, for example, can opt for an immediate payment of roughly half of the jackpot amount or can take payment annually for 26 years. In the long-term plan, the first installment is 2.5 percent of the jackpot amount. Successive yearly payments in-

crease each year, with the final payment being about twice the initial. The sum of these twenty-six payments will be equal to the originally stated jackpot amount. Thus the winner of a $6 million jackpot could take approximately $3 million immediately or could get $150,000 as a first installment with successive yearly payments increasing each year and totaling $6 million after all the payments have been made.

Of course, lottery winners must take into consideration the taxes that must be paid on their winnings. Calculating taxes at one-third, a $6 million jackpot winner choosing the one-time payment option would get approximately $4 million. An individual may find that tax savings may be realized in the long term payoff; however, the large lump sum would not be available for investing purposes. SEE ALSO Probability, Experimental; Probability, Theoretical.

Robert J. Quinn

Lovelace, Ada Byron

English Mathematician and Scientist
1815–1852

Ada Lovelace was born Augusta Ada Byron, the daughter of the poet George Gordon (Lord Byron) and the mathematician and heiress Anne Isabella Milbanke. Although Lovelace inherited poetic inclinations from her father, her mother raised her to be a mathematician, and she subsequently contributed significantly to the earliest work on mechanical computing machines.

Ada Byron Lovelace is credited with writing the world's first computer program when she wrote instructions for Charles Babbage's Analytical Engine.

Lovelace received her early education at home and was assisted in her advanced studies by mathematician Augustus De Morgan and scientist Mary Somerville. Presented at court in 1833, she married William, eighth Lord of King, in 1835. He subsequently became Earl of Lovelace, and she became Countess of Lovelace.

Although involved in London society, Lovelace was interested in mathematics, particularly the calculating machines proposed by Charles Babbage, professor of mathematics at Cambridge. After Lovelace met Babbage, the pair became friends and coworkers.

Babbage proposed mechanical devices—the Difference Engine in 1833 and the more complex Analytical Engine in 1838—that would be able to make numerical calculations. Lovelace translated an Italian article describing Babbage's Analytical Engine and added commentary that was three times the length of the original article. Published in 1843, this article clearly shows that she was the first person to understand fully the significance of Babbage's inventions.

In the article, Lovelace described how the calculating machine could be programmed to compute Bernoulli numbers, foreshadowing modern computer programming. She also predicted the use of mechanical mathematical devices for such purposes as music composition and the production of graphics. Although thought to be whimsical at the time, her predictions have turned out to be quite accurate. SEE ALSO Babbage, Charles; Computers, Evolution of Electronic; Mathematical Devices, Mechanical.

J. William Moncrief

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Glossary

abscissa: the x-coordinate of a point in a Cartesian coordinate plane

absolute: standing alone, without reference to arbitrary standards of measurement

absolute dating: determining the date of an artifact by measuring some physical parameter independent of context

absolute value: the non-negative value of a number regardless of sign

absolute zero: the coldest possible temperature on any temperature scale; $-273°$ Celsius

abstract: having only intrinsic form

abstract algebra: the branch of algebra dealing with groups, rings, fields, Galois sets, and number theory

acceleration: the rate of change of an object's velocity

accelerometer: a device that measures acceleration

acute: sharp, pointed; in geometry, an angle whose measure is less than 90 degrees

additive inverse: any two numbers that add to equal 1

advection: a local change in a property of a system

aerial photography: photographs of the ground taken from an airplane or balloon; used in mapping and surveying

aerodynamics: the study of what makes things fly; the engineering discipline specializing in aircraft design

aesthetic: having to do with beauty or artistry

aesthetic value: the value associated with beauty or attractiveness; distinct from monetary value

algebra: the branch of mathematics that deals with variables or unknowns representing the arithmetic numbers

algorithm: a rule or procedure used to solve a mathematical problem

algorithmic: pertaining to an algorithm

ambiguity: the quality of doubtfulness or uncertainty

analog encoding: encoding information using continuous values of some physical quantity

analogy: comparing two things similar in some respects and inferring they are also similar in other respects

analytical geometry: describes the study of geometric properties by using algebraic operations

anergy: spent energy transferred to the environment

angle of elevation: the angle formed by a line of sight above the horizontal

angle of rotation: the angle measured from an initial position a rotating object has moved through

anti-aliasing: introducing shades of gray or other intermediate shades around an image to make the edge appear to be smoother

applications: collections of general-purpose software such as word processors and database programs used on modern personal computers

arc: a continuous portion of a circle; the portion of a circle between two line segments originating at the center of the circle

areagraph: a fine-scale rectangular grid used for determining the area of irregular plots

artifact: something made by a human and left in an archaeological context

artificial intelligence: the field of research attempting the duplication of the human thought process with digital computers or similar devices; also includes expert systems research

ASCII: an acronym that stands for American Standard Code for Information Interchange; assigns a unique 8-bit binary number to every letter of the alphabet, the digits, and most keyboard symbols

assets: real, tangible property held by a business corporation including collectible debts to the corporation

asteroid: a small object or "minor planet" orbiting the Sun, usually in the space between Mars and Jupiter

astigmatism: a defect of a lens, such as within an eye, that prevents focusing on sharply defined objects

astrolabe: a device used to measure the angle between an astronomical object and the horizon

astronomical unit (AU): the average distance of Earth from the Sun; the semi-major axis of Earth's orbit

asymptote: the line that a curve approaches but never reaches

asymptotic: pertaining to an asymptote

atmosphere (unit): a unit of pressure equal to 14.7 lbs/in^2, which is the air pressure at mean sea level

atomic weight: the relative mass of an atom based on a scale in which a specific carbon atom (carbon-12) is assigned a mass value of 12

autogiro: a rotating wing aircraft with a powered propellor to provide thrust and an unpowered rotor for lift; also spelled "autogyro"

avatar: representation of user in virtual space (after the Hindu idea of an incarnation of a deity in human form)

average rate of change: how one variable changes as the other variable increases by a single unit

axiom: a statement regarded as self-evident; accepted without proof

axiomatic system: a system of logic based on certain axioms and definitions that are accepted as true without proof

axis: an imaginary line about which an object rotates

axon: fiber of a nerve cell that carries action potentials (electrochemical impulses)

azimuth: the angle, measured along the horizon, between north and the position of an object or direction of movement

azimuthal projections: a projection of a curved surface onto a flat plane

bandwidth: a range within a band of wavelengths or frequencies

base-10: a number system in which each place represents a power of 10 larger than the place to its right

base-2: a binary number system in which each place represents a power of 2 larger than the place to its right

base-20: a number system in which each place represents a power of 20 larger than the place to the right

base-60: a number system used by ancient Mesopotamian cultures for some calculations in which each place represents a power of 60 larger than the place to its right

baseline: the distance between two points used in parallax measurements or other triangulation techniques

Bernoulli's Equation: a first order, nonlinear differential equation with many applications in fluid dynamics

biased sampling: obtaining a nonrandom sample; choosing a sample to represent a particular viewpoint instead of the whole population

bidirectional frame: in compressed video, a frame between two other frames; the information is based on what changed from the previous frame as well as what will change in the next frame

bifurcation value: the numerical value near which small changes in the initial value of a variable can cause a function to take on widely different values or even completely different behaviors after several iterations

Big Bang: the singular event thought by most cosmologists to represent the beginning of our universe; at the moment of the big bang, all matter, energy, space, and time were concentrated into a single point

binary: existing in only two states, such as "off" or "on," "one" or "zero"

binary arithmetic: the arithmetic of binary numbers; base two arithmetic; internal arithmetic of electronic digital logic

binary number: a base-2 number; a number that uses only the binary digits 1 and 0

binary signal: a form of signal with only two states, such as two different values of voltage, or "on" and "off" states

binary system: a system of two stars that orbit their common center of mass; any system of two things

binomial: an expression with two terms

binomial coefficients: coefficients in the expansion of ($x + y^n$, where n is a positive integer

binomial distribution: the distribution of a binomial random variable

binomial theorem: a theorem giving the procedure by which a binomial expression may be raised to any power without using successive multiplications

bioengineering: the study of biological systems such as the human body using principles of engineering

biomechanics: the study of biological systems using engineering principles

bioturbation: disturbance of the strata in an archaeological site by biological factors such as rodent burrows, root action, or human activity

bit: a single binary digit, 1 or 0

bitmap: representing a graphic image in the memory of a computer by storing information about the color and shade of each individual picture element (or pixel)

Boolean algebra: a logic system developed by George Boole that deals with the theorems of undefined symbols and axioms concerning those symbols

Boolean operators: the set of operators used to perform operations on sets; includes the logical operators AND, OR, NOT

byte: a group of eight binary digits; represents a single character of text

cadaver: a corpse intended for medical research or training

caisson: a large cylinder or box that allows workers to perform construction tasks below the water surface, may be open at the top or sealed and pressurized

calculus: a method of dealing mathematically with variables that may be changing continuously with respect to each other

calibrate: act of systematically adjusting, checking, or standardizing the graduation of a measuring instrument

carrying capacity: in an ecosystem, the number of individuals of a species that can remain in a stable, sustainable relationship with the available resources

Cartesian coordinate system: a way of measuring the positions of points in a plane using two perpendicular lines as axes

Cartesian plane: a mathematical plane defined by the x and y axes or the ordinate and abscissa in a Cartesian coordinate system

cartographers: persons who make maps

catenary curve: the curve approximated by a free-hanging chain supported at each end; the curve generated by a point on a parabola rolling along a line

causal relations: responses to input that do not depend on values of the input at later times

celestial: relating to the stars, planets, and other heavenly bodies

celestial body: any natural object in space, defined as above Earth's atmosphere; the Moon, the Sun, the planets, asteroids, stars, galaxies, nebulae

central processor: the part of a computer that performs computations and controls and coordinates other parts of the computer

centrifugal: the outwardly directed force a spinning object exerts on its restraint; also the perceived force felt by persons in a rotating frame of reference

cesium: a chemical element, symbol Cs, atomic number 55

Chandrasekhar limit: the 1.4 solar mass limit imposed on a white dwarf by quantum mechanics; a white dwarf with greater than 1.4 solar masses will collapse to a neutron star

chaos theory: the qualitative study of unstable aperiodic behavior in deterministic nonlinear dynamical systems

chaotic attractor: a set of points such that all nearby trajectories converge to it

chert: material consisting of amorphous or cryptocrystalline silicon dioxide; fine-grained chert is indistinguishable from flint

chi-square test: a generalization of a test for significant differences between a binomial population and a multinomial population

chlorofluorocarbons: compounds similar to hydrocarbons in which one or more of the hydrogen atoms has been replaced by a chlorine or fluorine atom

chord: a straight line connecting the end points of an arc of a circle

chromakey: photographing an object shot against a known color, which can be replaced with an arbitrary background (like the weather maps on television newscasts)

chromosphere: the transparent layer of gas that resides above the photosphere in the atmosphere of the Sun

chronometer: an extremely precise timepiece

ciphered: coded; encrypyted

circumference: the distance around a circle

circumnavigation: the act of sailing completely around the globe

circumscribed: bounded, as by a circle

circumspheres: spheres that touch all the "outside" faces of a regular polyhedron

client: an individual, business, or agency for whom services are provided by another individual, business, or industry; a patron or customer

clones: computers assembled of generic components designed to use a standard operation system

codomain: for a given function *f*, the set of all possible values of the function; the range is a subset of the codomain

cold dark matter: hypothetical form of matter proposed to explain the 90 percent of mass in most galaxies that cannot be detected because it does not emit or reflect radiation

coma: the cloud of gas that first surrounds the nucleus of a comet as it begins to warm up

combinations: a group of elements from a set in which order is not important

combustion: chemical reaction combining fuel with oxygen accompanied by the release of light and heat

comet: a lump of frozen gas and dust that approaches the Sun in a highly elliptical orbit forming a coma and one or two tails

command: a particular instruction given to a computer, usually as part of a list of instructions comprising a program

commodities: anything having economic value, such as agricultural products or valuable metals

compendium: a summary of a larger work or collection of works

compiler: a computer program that translates symbolic instructions into machine code

complex plane: the mathematical abstraction on which complex numbers can be graphed; the *x*-axis is the real component and the *y*-axis is the imaginary component

composite number: an integer that is not prime

compression: reducing the size of a computer file by replacing long strings of identical bits with short instructions about the number of bits; the information is restored before the file is used

compression algorithm: the procedure used, such as comparing one frame in a movie to the next, to compress and reduce the size of electronic files

concave: hollowed out or curved inward

concentric: sets of circles or other geometric objects sharing the same center

conductive: having the ability to conduct or transmit

confidence interval: a range of values having a predetermined probability that the value of some measurement of a population lies within it

congruent: exactly the same everywhere; having exactly the same size and shape

conic: of or relating to a cone, that surface generated by a straight line, passing through a fixed point, and moving along the intersection with a fixed curve

conic sections: the curves generated by an imaginary plane slicing through an imaginary cone

continuous quantities: amounts composed of continuous and undistinguishable parts

converge: come together; to approach the same numerical value

convex: curved outward, bulging

coordinate geometry: the concept and use of a coordinate system with respect to the study of geometry

coordinate plane: an imaginary two-dimensional plane defined as the plane containing the *x*- and *y*-axes; all points on the plane have coordinates that can be expressed as *x*, *y*

coordinates: the set of *n* numbers that uniquely identifies the location of a point in *n*-dimensional space

corona: the upper, very rarefied atmosphere of the Sun that becomes visible around the darkened Sun during a total solar eclipse

corpus: Latin for "body"; used to describe a collection of artifacts

correlate: to establish a mutual or reciprocal relation between two things or sets of things

correlation: the process of establishing a mutual or reciprocal relation between two things or sets of things

cosine: if a unit circle is drawn with its center at the origin and a line segment is drawn from the origin at angle theta so that the line segment intersects the circle at (x, y), then x is the cosine of theta

cosmological distance: the distance a galaxy would have to have in order for its red shift to be due to Hubble expansion of the universe

cosmology: the study of the origin and evolution of the universe

cosmonaut: the term used by the Soviet Union and now used by the Russian Federation to refer to persons trained to go into space; synonomous with astronaut

cotton gin: a machine that separates the seeds, hulls, and other undesired material from cotton

cowcatcher: a plow-shaped device attached to the front of a train to quickly remove obstacles on railroad tracks

cryptography: the science of encrypting information for secure transmission

cubit: an ancient unit of length equal to the distance from the elbow to the tip of the middle finger; usually about 18 inches

culling: removing inferior plants or animals while keeping the best; also known as "thinning"

curved space: the notion suggested by Albert Einstein to explain the properties of space near a massive object, space acts as if it were curved in four dimensions

deduction: a conclusion arrived at through reasoning, especially a conclusion about some particular instance derived from general principles

deductive reasoning: a type of reasoning in which a conclusion necessarily follows from a set of axioms; reasoning from the general to the particular

degree: 1/360 of a circle or complete rotation

degree of significance: a determination, usually in advance, of the importance of measured differences in statistical variables

demographics: statistical data about people—including age, income, and gender—that are often used in marketing

dendrite: branched and short fiber of a neuron that carries information to the neuron

dependent variable: in the equation $y = f(x)$, if the function f assigns a single value of y to each value of x, then y is the output variable (or the dependent variable)

depreciate: to lessen in value

deregulation: the process of removing legal restrictions on the behavior of individuals or corporations

derivative: the derivative of a function is the limit of the ratio of the change in the function; the change is produced by a small variation in the variable as the change in the variable is allowed to approach zero; an inverse operation to calculating an integral

determinant: a square matrix with a single numerical value determined by a unique set of mathematical operations performed on the entries

determinate algebra: the study and analysis of equations that have one or a few well-defined solutions

deterministic: mathematical or other problems that have a single, well-defined solution

diameter: the chord formed by an arc of one-half of a circle

differential: a mathematical quantity representing a small change in one variable as used in a differential equation

differential calculus: the branch of mathematics primarily dealing with the solution of differential equations to find lengths, areas, and volumes of functions

differential equation: an equation that expresses the relationship between two variables that change in respect to each other, expressed in terms of the rate of change

digit: one of the symbols used in a number system to represent the multiplier of each place

digital: describes information technology that uses discrete values of a physical quantity to transmit information

digital encoding: encoding information by using discrete values of some physical quantity

digital logic: rules of logic as applied to systems that can exist in only discrete states (usually two)

dihedral: a geometric figure formed by two half-planes that are bounded by the same straight line

Diophantine equation: polynomial equations of several variables, with integer coefficients, whose solutions are to be integers

diopter: a measure of the power of a lens or a prism, equal to the reciprocal of its focal length in meters

directed distance: the distance from the pole to a point in the polar coordinate plane

discrete: composed of distinct elements

discrete quantities: amounts composed of separate and distinct parts

distributive property: property such that the result of an operation on the various parts collected into a whole is the same as the operation performed separately on the parts before collection into the whole

diverge: to go in different directions from the same starting point

dividend: the number to be divided; the numerator in a fraction

divisor: the number by which a dividend is divided; the denominator of a fraction

DNA fingerprinting: the process of isolating and amplifying segments of DNA in order to uniquely identify the source of the DNA

domain: the set of all values of a variable used in a function

double star: a binary star; two stars orbiting a common center of gravity

duodecimal: a numbering system based on 12

dynamometer: a device that measures mechanical or electrical power

eccentric: having a center of motion different from the geometric center of a circle

eclipse: occurrence when an object passes in front of another and blocks the view of the second object; most often used to refer to the phenomenon

that occurs when the Moon passes in front of the Sun or when the Moon passes through Earth's shadow

ecliptic: the plane of the Earth's orbit around the Sun

eigenvalue: if there exists a vector space such that a linear transformation onto itself produces a new vector equal to a scalar times the original vector, then that scalar is called an eigenfunction

eigenvector: if there exists a vector space such that a linear transformation onto itself produces a new vector equal to a scalar times the original vector, then that vector is called an eigenvector

Einstein's General Theory of Relativity: Albert Einstein's generalization of relativity to include systems accelerated with respect to one another; a theory of gravity

electromagnetic radiation: the form of energy, including light, that transfers information through space

elements: the members of a set

ellipse: one of the conic sections, it is defined as the locus of all points such that the sum of the distances from two points called the foci is constant

elliptical: a closed geometric curve where the sum of the distances of a point on the curve to two fixed points (foci) is constant

elliptical orbit: a planet, comet, or satellite follows a curved path known as an ellipse when it is in the gravitational field of the Sun or another object; the Sun or other object is at one focus of the ellipse

empirical law: a mathematical summary of experimental results

empiricism: the view that the experience of the senses is the single source of knowledge

encoding tree: a collection of dots with edges connecting them that have no looping paths

endangered species: a species with a population too small to be viable

epicenter: the point on Earth's surface directly above the site of an earthquake

epicycle: the curved path followed by planets in Ptolemey's model of the solar system; planets moved along a circle called the epicycle, whose center moved along a circular orbit around the sun

epicylic: having the property of moving along an epicycle

equatorial bulge: the increase in diameter or circumference of an object when measured around its equator usually due to rotation, all planets and the sun have equatorial bulges

equidistant: at the same distance

equilateral: having the property that all sides are equal; a square is an equilateral rectangle

equilateral triangle: a triangle whose sides and angles are equal

equilibrium: a state of balance between opposing forces

equinox points: two points on the celestial sphere at which the ecliptic intersects the celestial equator

escape speed: the minimum speed an object must attain so that it will not fall back to the surface of a planet

Euclidean geometry: the geometry of points, lines, angles, polygons, and curves confined to a plane

exergy: the measure of the ability of a system to produce work; maximum potential work output of a system

exosphere: the outermost layer of the atmosphere extending from the ionosphere upward

exponent: the symbol written above and to the right of an expression indicating the power to which the expression is to be raised

exponential: an expression in which the variable appears as an exponent

exponential power series: the series by which *e* to the *x* power may be approximated; $e^x = 1 + x + x^{2/2!} + x^{3/3!} + \ldots$

exponents: symbols written above and to the right of expressions indicating the power to which an expression is to be raised or the number of times the expression is to be multiplied by itself

externality: a factor that is not part of a system but still affects it

extrapolate: to extend beyond the observations; to infer values of a variable outside the range of the observations

farsightedness: describes the inability to see close objects clearly

fiber-optic: a long, thin strand of glass fiber; internal reflections in the fiber assure that light entering one end is transmitted to the other end with only small losses in intensity; used widely in transmitting digital information

fibrillation: a potentially fatal malfunction of heart muscle where the muscle rapidly and ineffectually twitches instead of pulsing regularly

fidelity: in information theory a measure of how close the information received is to the information sent

finite: having definite and definable limits; countable

fire: the reaction of a neuron when excited by the reception of a neurotransmitter

fission: the splitting of the nucleus of a heavy atom, which releases kinetic energy that is carried away by the fission fragments and two or three neutrons

fixed term: for a definite length of time determined in advance

fixed-wing aircraft: an aircraft that obtains lift from the flow of air over a nonmovable wing

floating-point operations: arithmetic operations on a number with a decimal point

fluctuate: to vary irregularly

flue: a pipe designed to remove exhaust gases from a fireplace, stove, or burner

fluid dynamics: the science of fluids in motion

focal length: the distance from the focal point (the principle point of focus) to the surface of a lens or concave mirror

focus: one of the two points that define an ellipse; in a planetary orbit, the Sun is at one focus and nothing is at the other focus

formula analysis: a method of analysis of the Boolean formulas used in computer programming

Fourier series: an infinite series consisting of cosine and sine functions of integral multiples of the variable each multiplied by a constant; if the series is finite, the expression is known as a Fourier polynomial

fractal: a type of geometric figure possessing the properties of self-similarity (any part resembles a larger or smaller part at any scale) and a measure that increases without bound as the unit of measure approaches zero

fractal forgery: creating a natural landscape by using fractals to simulate trees, mountains, clouds, or other features

fractal geometry: the study of the geometric figures produced by infinite iterations

futures exchange: a type of exchange where contracts are negotiated to deliver commodites at some fixed price at some time in the future

***g*:** a common measure of acceleration; for example 1 *g* is the acceleration due to gravity at the Earth's surface, roughly 32 feet per second per second

game theory: a discipline that combines elements of mathematics, logic, social and behavioral sciences, and philosophy

gametes: mature male or female sexual reproductive cells

gaming: playing games or relating to the theory of game playing

gamma ray: a high-energy photon

general relativity: generalization of Albert Einstein's theory of relativity to include accelerated frames of reference; presents gravity as a curvature of four-dimensional space-time

generalized inverse: an extension of the concept of the inverse of a matrix to include matrices that are not square

generalizing: making a broad statement that includes many different special cases

genus: the taxonomic classification one step more general than species; the first name in the binomial nomenclature of all species

geoboard: a square board with pegs and holes for pegs used to create geometric figures

geocentric: Earth-centered

geodetic: of or relating to geodesy, which is the branch of applied mathematics dealing with the size and shape of the earth, including the precise location of points on its surface

geometer: a person who uses the principles of geometry to aid in making measurements

geometric: relating to the principles of geometry, a branch of mathematics related to the properties and relationships of points, lines, angles, surfaces, planes, and solids

geometric sequence: a sequence of numbers in which each number in the sequence is larger than the previous by some constant ratio

geometric series: a series in which each number is larger than the previous by some constant ratio; the sum of a geometric sequence

geometric solid: one of the solids whose faces are regular polygons

geometry: the branch of mathematics that deals with the properties and relationships of points, lines, angles, surfaces, planes, and solids

geostationary orbit: an Earth orbit made by an artificial satellite that has a period equal to the Earth's period of rotation on its axis (about 24 hours)

geysers: springs that occasionally spew streams of steam and hot water

glide reflection: a rigid motion of the plane that consists of a reflection followed by a translation parallel to the mirror axis

grade: the amount of increase in elevation per horizontal distance, usually expressed as a percent; the slope of a road

gradient: a unit used for measuring angles, in which the circle is divided into 400 equal units, called gradients

graphical user interface: a device designed to display information graphically on a screen; a modern computer interface system

Greenwich Mean Time: the time at Greenwich, England; used as the basis for universal time throughout the world

Gross Domestic Product: a measure in the change in the market value of goods, services, and structures produced in the economy

group theory: study of the properties of groups, the mathematical systems consisting of elements of a set and operations that can be performed on that set such that the results of the operations are always members of the same set

gyroscope: a device typically consisting of a spinning wheel or disk, whose spin-axis turns between two low-friction supports; it maintains its angular orientation with respect to inertial conditions when not subjected to external forces

Hagia Sophia: Instanbul's most famous landmark, built by the emperor Justinian I in 537 C.E. and converted to a mosque in 1453 C.E.

Hamming codes: a method of error correction in digital information

headwind: a wind blowing in the opposite direction as that of the course of a vehicle

Heisenberg Uncertainty Principle: the principle in physics that asserts it is impossible to know simultaneously and with complete accuracy the values of certain pairs of physical quantities such as position and momentum

heliocentric: Sun-centered

hemoglobin: the oxygen-bearing, iron-containing conjugated protein in vertebrate red blood cells

heuristics: a procedure that serves to guide investigation but that has not been proven

hominid: a member of family Hominidae; *Homo sapiens* are the only surviving species

Huffman encoding: a method of efficiently encoding digital information

hydrocarbon: a compound of carbon and hydrogen

hydrodynamics: the study of the behavior of moving fluids

hydrograph: a tabular or graphical display of stream flow or water runoff

hydroscope: a device designed to allow a person to see below the surface of water

hydrostatics: the study of the properties of fluids not in motion

hyperbola: a conic section; the locus of all points such that the absolute value of the difference in distance from two points called foci is a constant

hyperbolic: an open geometric curve where the difference of the distances of a point on the curve to two fixed points (foci) is constant

Hypertext Markup Language: the computer markup language used to create documents on the World Wide Web

hypertext: the text that contains hyperlinks, that is, links to other places in the same document or other documents or multimedia files

hypotenuse: the long side of a right triangle; the side opposite the right angle

hypothesis: a proposition that is assumed to be true for the purpose of proving other propositions

ice age: one of the broad spans of time when great sheets of ice covered the Northern parts of North America and Europe; the most recent ice age was about 16,000 years ago

identity: a mathematical statement much stronger than equality, which asserts that two expressions are the same for all values of the variables

implode: violently collapse; fall in

inclination: a slant or angle formed by a line or plane with the horizontal axis or plane

inclined: sloping, slanting, or leaning

incomplete interpretation: a statistical flaw

independent variable: in the equation $y = f(x)$, the input variable is x (or the independent variable)

indeterminate algebra: study and analysis of solution strategies for equations that do not have fixed or unique solutions

indeterminate equation: an equation in which more than one variable is unknown

index (number): a number that allows tracking of a quantity in economics by comparing it to a standard, the consumer price index is the best known example

inductive reasoning: drawing general conclusions based on specific instances or observations; for example, a theory might be based on the outcomes of several experiments

Industrial Revolution: beginning in Great Britain around 1730, a period in the eighteenth and nineteenth centuries when nations in Europe, Asia, and the Americas moved from agrarian-based to industry-based economies

inertia: tendency of a body that is at rest to remain at rest, or the tendency of a body that is in motion to remain in motion

inferences: the act or process of deriving a conclusion from given facts or premises

inferential statistics: analysis and interpretation of data in order to make predictions

infinite: having no limit; boundless, unlimited, endless; uncountable

infinitesimals: functions with values arbitrarily close to zero

infinity: the quality of unboundedness; a quantity beyond measure; an unbounded quantity

information database: an array of information related to a specific subject or group of subjects and arranged so that any individual bit of information can be easily found and recovered

information theory: the science that deals with how to separate information from noise in a signal or how to trace the flow of information through a complex system

infrastructure: the foundation or permanent installations necessary for a structure or system to operate

initial conditions: the values of variables at the beginning of an experiment or of a set at the beginning of a simulation; chaos theory reveals that small changes in initial conditions can produce widely divergent results

input: information provided to a computer or other computation system

inspheres: spheres that touch all the "inside" faces of a regular polyhedron; also called "enspheres"

integer: a positive whole number, its negative counterpart, or zero

integral: a mathematical operation similar to summation; the area between the curve of a function, the x-axis, and two bounds such as $x = a$ and $x = b$; an inverse operation to finding the derivative

integral calculus: the branch of mathematics dealing with the rate of change of functions with respect to their variables

integral number: integer; that is, a positive whole number, its negative counterpart, or zero

integral solutions: solutions to an equation or set of equations that are all integers

integrated circuit: a circuit with the transistors, resistors, and other circuit elements etched into the surface of a single chip of silicon

integration: solving a differential equation; determining the area under a curve between two boundaries

intensity: the brightness of radiation or energy contained in a wave

intergalactic: between galaxies; the space between the galaxies

interplanetary: between planets; the space between the planets

interpolation: filling in; estimating unknown values of a function between known values

intersection: a set containing all of the elements that are members of two other sets

interstellar: between stars; the space between stars

intraframe: the compression applied to still images, interframe compression compares one image to the next and only stores the elements that have changed

intrinsic: of itself; the essential nature of a thing; originating within the thing

inverse: opposite; the mathematical function that expresses the independent variable of another function in terms of the dependent variable

inverse operations: operations that undo each other, such as addition and subtraction

inverse square law: a given physical quality varies with the distance from the source inversely as the square of the distance

inverse tangent: the value of the argument of the tangent function that produces a given value of the function; the angle that produces a particular value of the tangent

invert: to turn upside down or to turn inside out; in mathematics, to rewrite as the inverse function

inverted: upside down; turned over

ionized: an atom that has lost one or more of its electrons and has become a charged particle

ionosphere: a layer in Earth's atmosphere above 80 kilometers characterized by the existence of ions and free electrons

irrational number: a real number that cannot be written as a fraction of the form a/b, where a and b are both integers and b is not zero; when expressed in decimal form, an irrational number is infinite and nonrepeating

isometry: equality of measure

isosceles triangle: a triangle with two sides and two angles equal

isotope: one of several species of an atom that has the same number of protons and the same chemical properties, but different numbers of neutrons

iteration: repetition; a repeated mathematical operation in which the output of one cycle becomes the input for the next cycle

iterative: relating to a computational procedure to produce a desired result by replication of a series of operations

iterator: the mathematical operation producing the result used in iteration

kinetic energy: the energy an object has as a consequence of its motion

kinetic theory of gases: the idea that all gases are composed of widely separated particles (atoms and molecules) that exert only small forces on each other and that are in constant motion

knot: nautical mile per hour

Lagrange points: two positions in which the motion of a body of negligible mass is stable under the gravitational influence of two much larger bodies (where one larger body is moving)

latitude: the number of degrees on Earth's surface north or south of the equator; the equator is latitude zero

law: a principle of science that is highly reliable, has great predictive power, and represents the mathematical summary of experimental results

law of cosines: for a triangle with angles A, B, C and sides a, b, c, $a^2 = b^2 + c^2 - 2bc \cos A$

law of sines: if a triangle has sides a, b, and c and opposite angles A, B, and C, then $\sin A/a = \sin B/ b = \sin C/c$

laws of probability: set of principles that govern the use of probability in determining the truth or falsehood of a hypothesis

light-year: the distance light travels within a vaccuum in one year

limit: a mathematical concept in which numerical values get closer and closer to a given value

linear algebra: the study of vector spaces and linear transformations

linear equation: an equation in which all variables are raised to the first power

linear function: a function whose graph on the x-y plane is a straight line or line segment

litmus test: a test that uses a single indicator to prompt a decision

locus (pl: loci): in geometry, the set of all points, lines, or surfaces that satisfies a particular requirement

logarithm: the power to which a certain number called the base is to be raised to produce a particular number

logarithmic coordinates: the x and y coordinates of a point on a cartesian plane using logarithmic scales on the x- and y-axes.

logarithmic scale: a scale in which the distances that numbers are positioned, from a reference point, are proportional to their logarithms

logic circuits: circuits used to perform logical operations and containing one or more logic elements: devices that maintain a state based on previous input to determine current and future output

logistic difference equation: the equation $x_{(n+1)} = r \times x_{n(1-xn)}$ is used to study variability in animal populations

longitude: one of the imaginary great circles beginning at the poles and extending around Earth; the geographic position east or west of the prime meridian

machine code: the set of instructions used to direct the internal operation of a computer or other information-processing system

machine language: electronic code the computer can utilize

magnetic trap: a magnetic field configured in such a way that an ion or other charged particle can be held in place for an extended period of time

magnetosphere: an asymmetric region surrounding the Earth in which charged particles are trapped, their behavior being dominated by Earth's magnetic field

magnitude: size; the measure or extent of a mathematical or physical quantity

mainframes: large computers used by businesses and government agencies to process massive amounts of data; generally faster and more powerful than desktops but usually requiring specialized software

malfunctioning: not functioning correctly; performing badly

malleability: the ability or capability of being shaped or formed

margin of error: the difference between the estimated maximum and minimum values a given measurement could have

mathematical probability: the mathematical computation of probabilities of outcomes based on rules of logic

matrix: a rectangular array of data in rows and columns

mean: the arithmetic average of a set of data

median: the middle of a set of data when values are sorted from smallest to largest (or largest to smallest)

megabyte: term used to refer to one million bytes of memory storage, where each byte consists of eight bits; the actual value is 1,048,576 (2^{20})

memory: a device in a computer designed to temporarily or permanently store information in the form of binomial states of certain circuit elements

meridian: a great circle passing through Earth's poles and a particular location

metallurgy: the study of the properties of metals; the chemistry of metals and alloys

meteorologist: a person who studies the atmosphere in order to understand weather and climate

methanol: an alcohol consisting of a single carbon bonded to three hydrogen atoms and an O–H group

microcomputers: an older term used to designate small computers designed to sit on a desktop and to be used by one person; replaced by the term personal computer

microgravity: the apparent weightless condition of objects in free fall

microkelvin: one-millionth of a kelvin

minicomputers: a computer midway in size between a desktop computer and a main frame computer; most modern desktops are much more powerful than the older minicomputers and they have been phased out

minimum viable population: the smallest number of individuals of a species in a particular area that can survive and maintain genetic diversity

mission specialist: an individual trained by NASA to perform a specific task or set of tasks onboard a spacecraft, whose duties do not include piloting the spacecraft

mnemonic: a device or process that aids one's memory

mode: a kind of average or measure of central tendency equal to the number that occurs most often in a set of data

monomial: an expression with one term

Morse code: a binary code designed to allow text information to be transmitted by telegraph consisting of "dots" and "dashes"

mouse: a handheld pointing device used to manipulate an indicator on a screen

moving average: a method of averaging recent trends in relation to long term averages, it uses recent data (for example, the last 10 days) to calculate an average that changes but still smooths out daily variations

multimodal input/output (I/O): multimedia control and display that uses various senses and interaction styles

multiprocessing: a computer that has two or more central processers which have common access to main storage

nanometers: billionths of a meter

nearsightedness: describes the inability to see distant objects clearly

negative exponential: an exponential function of the form $y = e^{-x}$

net force: the final, or resultant, influence on a body that causes it to accelerate

neuron: a nerve cell

neurotransmitters: the substance released by a neuron that diffuses across the synapse

neutron: an elementary particle with approximately the same mass as a proton and neutral charge

Newtonian: a person who, like Isaac Newton, thinks the universe can be understood in terms of numbers and mathematical operations

nominal scales: a method for sorting objects into categories according to some distinguishing characteristic, then attaching a label to each category

non-Euclidean geometry: a branch of geometry defined by posing an alternate to Euclid's fifth postulate

nonlinear transformation: a transformation of a function that changes the shape of a curve or geometric figure

nonlinear transformations: transformations of functions that change the shape of a curve or geometric figure

nuclear fission: a reaction in which an atomic nucleus splits into fragments

nuclear fusion: mechanism of energy formation in a star; lighter nuclei are combined into heavier nuclei, releasing energy in the process

nucleotides: the basic chemical unit in a molecule of nucleic acid

nucleus: the dense, positive core of an atom that contains protons and neutrons

null hypothesis: the theory that there is no validity to the specific claim that two variations of the same thing can be distinguished by a specific procedure

number theory: the study of the properties of the natural numbers, including prime numbers, the number theorem, and Fermat's Last Theorem

numerical differentiation: approximating the mathematical process of differentiation using a digital computer

nutrient: a food substance or mineral required for the completion of the life cycle of an organism

oblate spheroid: a spheroid that bulges at the equator; the surface created by rotating an ellipse 360 degrees around its minor axis

omnidirectional: a device that transmits or receives energy in all directions

Öort cloud: a cloud of millions of comets and other material forming a spherical shell around the solar system far beyond the orbit of Neptune

orbital period: the period required for a planet or any other orbiting object to complete one complete orbit

orbital velocity: the speed and direction necessary for a body to circle a celestial body, such as Earth, in a stable manner

ordinate: the *y*-coordinate of a point on a Cartesian plane

organic: having to do with life, growing naturally, or dealing with the chemical compounds found in or produced by living organisms

oscillating: moving back and forth

outliers: extreme values in a data set

output: information received from a computer or other computation system based on the information it has received

overdubs: adding voice tracks to an existing film or tape

oxidant: a chemical reagent that combines with oxygen

oxidizer: the chemical that combines with oxygen or is made into an oxide

pace: an ancient measure of length equal to normal stride length

parabola: a conic section; the locus of all points such that the distance from a fixed point called the focus is equal to the perpendicular distance from a line

parabolic: an open geometric curve where the distance of a point on the curve to a fixed point (focus) and a fixed line (directrix) is the same

paradigm: an example, pattern, or way of thinking

parallax: the apparent motion of a nearby object when viewed against the background of more distant objects due to a change in the observer's position

parallel operations: separating the parts of a problem and working on different parts at the same time

parallel processing: using at least two different computers or working at least two different central processing units in the same computer at the same time or "in parallel" to solve problems or to perform calculation

parallelogram: a quadrilateral with opposite sides equal and opposite angles equal

parameter: an independent variable, such as time, that can be used to rewrite an expression as two separate functions

parity bits: extra bits inserted into digital signals that can be used to determine if the signal was accurately received

partial sum: with respect to infinite series, the sum of its first n terms for some n

pattern recognition: a process used by some artificial-intelligence systems to identify a variety of patterns, including visual patterns, information patterns buried in a noisy signal, and word patterns imbedded in text

payload specialist: an individual selected by NASA, another government agency, another government, or a private business, and trained by NASA to operate a specific piece of equipment onboard a spacecraft

payloads: the passengers, crew, instruments, or equipment carried by an aircraft, spacecraft, or rocket

perceptual noise shaping: a process of improving signal-to-noise ratio by looking for the patterns made by the signal, such as speech

perimeter: the distance around an area; in fractal geometry, some figures have a finite area but infinite perimeter

peripheral vision: outer area of the visual field

permutation: any arrangement, or ordering, of items in a set

perpendicular: forming a right angle with a line or plane

perspective: the point of view; a drawing constructed in such a way that an appearance of three dimensionality is achieved

perturbations: small displacements in an orbit

phonograph: a device used to recover the information recorded in analog form as waves or wiggles in a spiral grove on a flat disc of vinyl, rubber, or some other substance

photosphere: the very bright portion of the Sun visible to the unaided eye; the portion around the Sun that marks the boundary between the dense interior gases and the more diffuse

photosynthesis: the chemical process used by plants and some other organisms to harvest light energy by converting carbon dioxide and water to carbohydrates and oxygen

pixel: a single picture element on a video screen; one of the individual dots making up a picture on a video screen or digital image

place value: in a number system, the power of the base assigned to each place; in base-10, the ones place, the tens place, the hundreds place, and so on

plane: generally considered an undefinable term, a plane is a flat surface extending in all directions without end, and that has no thickness

plane geometry: the study of geometric figures, points, lines, and angles and their relationships when confined to a single plane

planetary: having to do with one of the planets

planisphere: a projection of the celestial sphere onto a plane with adjustable circles to demonstrate celestial phenomena

plates: the crustal segments on Earth's surface, which are constantly moving and rotating with respect to each other

plumb-bob: a heavy, conical-shaped weight, supported point-down on its axis by a strong cord, used to determine verticality in construction or surveying

pneumatic drill: a drill operated by compressed air

pneumatic tire: air-filled tire, usually rubber or synthetic

polar axis: the axis from which angles are measured in a polar coordinate system

pole: the origin of a polar coordinate system

poll: a survey designed to gather information about a subject

pollen analysis: microscopic examination of pollen grains to determine the genus and species of the plant producing the pollen; also known as palynology

polyconic projections: a type of map projection of a globe onto a plane that produces a distorted image but preserves correct distances along each meridian

polygon: a geometric figure bounded by line segments

polyhedron: a solid formed with all plane faces

polynomial: an expression with more than one term

polynomial function: a functional expression written in terms of a polynomial

position tracking: sensing the location and/or orientation of an object

power: the number of times a number is to be multiplied by itself in an expression

precalculus: the set of subjects and mathematical skills generally necessary to understand calculus

predicted frame: in compressed video, the next frame in a sequence of images; the information is based on what changed from the previous frame

prime: relating to, or being, a prime number (that is, a number that has no factors other than itself and 1)

Prime Meridian: the meridian that passes through Greenwich, England

prime number: a number that has no factors other than itself and 1

privatization: the process of converting a service traditionally offered by a government or public agency into a service provided by a private corporation or other private entity

proactive: taking action based on prediction of future situations

probability: the likelihood an event will occur when compared to other possible outcomes

probability density function: a function used to estimate the likelihood of spotting an organism while walking a transect

probability theory: the branch of mathematics that deals with quantities having random distributions

processor: an electronic device used to process a signal or to process a flow of information

profit margin: the difference between the total cost of a good or service and the actual selling cost of that good or service, usually expressed as a percentage

program: a set of instructions given to a computer that allows it to perform tasks; software

programming language processor: a program designed to recognize and process other programs

proliferation: growing rapidly

proportion: the mathematical relation between one part and another part, or between a part and the whole; the equality of two ratios

proportionately: divided or distributed according to a proportion; proportional

protractor: a device used for measuring angles, usually consisting of a half circle marked in degrees

pseudorandom numbers: numbers generated by a process that does not guarantee randomness; numbers produced by a computer using some highly complex function that simulates true randomness

Ptolemaic theory: the theory that asserted Earth was a spherical object at the center of the universe surrounded by other spheres carrying the various celestial objects

Pythagorean Theorem: a mathematical statement relating the sides of right triangles; the square of the hypotenuse is equal to the sums of the squares of the other two sides

Pythagorean triples: any set of three numbers obeying the Pythogorean relation such that the square of one is equal to the sum of the squares of the other two

quadrant: one-fourth of a circle; also a device used to measure angles above the horizon

quadratic: involving at least one term raised to the second power

quadratic equation: an equation in which the variable is raised to the second power in at least one term when the equation is written in its simplest form

quadratic form: the form of a function written so that the independent variable is raised to the second power

quantitative: of, relating to, or expressible in terms of quantity

quantum: a small packet of energy (matter and energy are equivalent)

quantum mechanics: the study of the interactions of matter with radiation on an atomic or smaller scale, whereby the granularity of energy and radiation becomes apparent

quantum theory: the study of the interactions of matter with radiation on an atomic or smaller scale, whereby the granularity of energy and radiation becomes apparent

quaternion: a form of complex number consisting of a real scalar and an imaginary vector component with three dimensions

quipus: knotted cords used by the Incas and other Andean cultures to encode numeric and other information

radian: an angle measure approximately equal to 57.3 degrees, it is the angle that subtends an arc of a circle equal to one radius

radicand: the quantity under the radical sign; the argument of the square root function

radius: the line segment originating at the center of a circle or sphere and terminating on the circle or sphere; also the measure of that line segment

radius vector: a line segment with both magnitude and direction that begins at the center of a circle or sphere and runs to a point on the circle or sphere

random: without order

random walks: a mathematical process in a plane of moving a random distance in a random direction then turning through a random angle and repeating the process indefinitely

range: the set of all values of a variable in a function mapped to the values in the domain of the independent variable; also called range set

rate (interest): the portion of the principal, usually expressed as a percentage, paid on a loan or investment during each time interval

ratio of similitude: the ratio of the corresponding sides of similar figures

rational number: a number that can be written in the form a/b, where a and b are intergers and b is not equal to zero

rations: the portion of feed that is given to a particular animal

ray: half line; line segment that originates at a point and extends without bound

real number: a number that has no imaginary part; a set composed of all the rational and irrational numbers

real number set: the combined set of all rational and irrational numbers, the set of numbers representing all points on the number line

realtime: occuring immediately, allowing interaction without significant delay

reapportionment: the process of redistributing the seats of the U. S. House of Representatives, based on each state's proportion of the national population

recalibration: process of resetting a measuring instrument so as to provide more accurate measurements

reciprocal: one of a pair of numbers that multiply to equal 1; a number's reciprocal is 1 divided by the number

red shift: motion-induced change in the frequency of light emitted by a source moving away from the observer

reflected: light or soundwaves returned from a surface

reflection: a rigid motion of the plane that fixes one line (the mirror axis) and moves every other point to its mirror image on the opposite side of the line

reflexive: directed back or turning back on itself

refraction: the change in direction of a wave as it passes from one medium to another

refrigerants: fluid circulating in a refrigerator that is successively compressed, cooled, allowed to expand, and warmed in the refrigeration cycle

regular hexagon: a hexagon whose sides are all equal and whose angles are all equal

relative: defined in terms of or in relation to other quantities

relative dating: determining the date of an archaeological artifact based on its position in the archaeological context relative to other artifacts

relativity: the assertion that measurements of certain physical quantities such as mass, length, and time depend on the relative motion of the object and observer

remediate: to provide a remedy; to heal or to correct a wrong or a deficiency

retrograde: apparent motion of a planet from east to west, the reverse of normal motion; for the outer planets, due to the more rapid motion of Earth as it overtakes an outer planet

revenue: the income produced by a source such as an investment or some other activity; the income produced by taxes and other sources and collected by a governmental unit

rhomboid: a parallelogram whose sides are equal

right angle: the angle formed by perpendicular lines; it measures 90 degrees

RNA: ribonucleic acid

robot arm: a sophisticated device that is standard equipment on space shuttles and on the International Space Station; used to deploy and retrieve satellites or perform other functions

Roche limit: an imaginary surface around a star in a binary system; outside the Roche limit, the gravitational attraction of the companion will pull matter away from a star

root: a number that when multiplied by itself a certain number of times forms a product equal to a specified number

rotary-wing design: an aircraft design that uses a rotating wing to produce lift; helicopter or autogiro (also spelled autogyro)

rotation: a rigid motion of the plane that fixes one point (the center of rotation) and moves every other point around a circle centered at that point

rotational: having to do with rotation

round: also to round off, the systematic process of reducing the number of decimal places for a given number

rounding: process of giving an approximate number

sample: a randomly selected subset of a larger population used to represent the larger population in statistical analysis

sampling: selecting a subset of a group or population in such a way that valid conclusions can be made about the whole set or population

scale (map): the numerical ratio between the dimensions of an object and the dimensions of the two or three dimensional representation of that object

scale drawing: a drawing in which all of the dimensions are reduced by some constant factor so that the proportions are preserved

scaling: the process of reducing or increasing a drawing or some physical process so that proper proportions are retained between the parts

schematic diagram: a diagram that uses symbols for elements and arranges these elements in a logical pattern rather than a practical physical arrangement

schematic diagrams: wiring diagrams that use symbols for circuit elements and arranges these elements in a logical pattern rather than a practical physical arrangement

search engine: software designed to search the Internet for occurences of a word, phrase, or picture, usually provided at no cost to the user as an advertising vehicle

secant: the ratio of the side adjacent to an acute angle in a right triangle to the side opposite; given a unit circle, the ratio of the x coordinate to the y coordinate of any point on the circle

seismic: subjected to, or caused by an earthquake or earth tremor

self-similarity: the term used to describe fractals where a part of the geometric figure resembles a larger or smaller part at any scale chosen

semantic: the study of how words acquire meaning and how those meanings change over time

semi-major axis: one-half of the long axis of an ellipse; also equal to the average distance of a planet or any satellite from the object it is orbiting

semiconductor: one of the elements with characteristics intermediate between the metals and nonmetals

set: a collection of objects defined by a rule such that it is possible to determine exactly which objects are members of the set

set dancing: a form of dance in which dancers are guided through a series of moves by a caller

set theory: the branch of mathematics that deals with the well-defined collections of objects known as sets

sextant: a device for measuring altitudes of celestial objects

signal processor: a device designed to convert information from one form to another so that it can be sent or received

significant difference: to distinguish greatly between two parameters

significant digits: the digits reported in a measure that accurately reflect the precision of the measurement

silicon: element number 14, it belongs in the category of elements known as metalloids or semiconductors

similar: in mathematics, having sides or parts in constant proportion; two items that resemble each other but are not identical

sine: if a unit circle is drawn with its center at the origin and a line segment is drawn from the origin at angle theta so that the line segment intersects the circle at (x, y), then y is the sine of theta

skepticism: a tendency towards doubt

skew: to cause lack of symmetry in the shape of a frequency distribution

slope: the ratio of the vertical change to the corresponding horizontal change

software: the set of instructions given to a computer that allows it to perform tasks

solar masses: dimensionless units in which mass, radius, luminosity, and other physical properties of stars can be expressed in terms of the Sun's characteristics

solar wind: a stream of particles and radiation constantly pouring out of the Sun at high velocities; partially responsible for the formation of the tails of comets

solid geometry: the geometry of solid figures, spheres, and polyhedrons; the geometry of points, lines, surfaces, and solids in three-dimensional space

spatial sound: audio channels endowed with directional and positional attributes (like azimuth, elevation, and range) and room effects (like echoes and reverberation)

spectra: the ranges of frequencies of light emitted or absorbed by objects

spectrum: the range of frequencies of light emitted or absorbed by an object

sphere: the locus of points in three-dimensional space that are all equidistant from a single point called the center

spin: to rotate on an axis or turn around

square: a quadrilateral with four equal sides and four right angles

square root: with respect to real or complex numbers s, the number t for which $t^2 = s$

stade: an ancient Greek measurement of length, one stade is approximately 559 feet (about 170 meters)

standard deviation: a measure of the average amount by which individual items of data might be expected to vary from the arithmetic mean of all data

static: without movement; stationary

statistical analysis: a set of methods for analyzing numerical data

statistics: the branch of mathematics that analyzes and interprets sets of numerical data

stellar: having to do with stars

sterographics: presenting slightly different views to left and right eyes, so that graphic scenes acquire depth

stochastic: random, or relating to a variable at each moment

Stonehenge: a large circle of standing stones on the Salisbury plain in England, thought by some to be an astronomical or calendrical marker

storm surge: the front of a hurricane, which bulges because of strong winds; can be the most damaging part of a hurricane

stratopause: the boundary in the atmosphere between the stratosphere and the mesosphere usually around 55 kilometers in altitude

stratosphere: the layer of Earth's atmosphere from 15 kilometers to about 50 kilometers, usually unaffected by weather and lacking clouds or moisture

sublimate: change of phase from a solid to a gas

sublunary: "below the moon"; term used by Aristotle and others to describe things that were nearer to Earth than the Moon and so not necessarily heavenly in origin or composition

subtend: to extend past and mark off a chord or arc

sunspot activity: one of the powerful magnetic storms on the surface of the Sun, which causes it to appear to have dark spots; sunspot activity varies on an 11-year cycle

superconduction: the flow of electric current without resistance in certain metals and alloys while at temperatures near absolute zero

superposition: the placing of one thing on top of another

suspension bridge: a bridge held up by a system of cables or cables and rods in tension; usually having two or more tall towers with heavy cables anchored at the ends and strung between the towers and lighter vertical cables extending downward to support the roadway

symmetric: to have balanced proportions; in bilateral symmetry, opposite sides are mirror images of each other

symmetry: a correspondence or equivalence between or among constituents of a system

synapse: the narrow gap between the terminal of one neuron and the dendrites of the next

tactile: relating to the sense of touch

tailwind: a wind blowing in the same direction of that of the course of a vehicle

tangent: a line that intersects a curve at one and only one point in a local region

tectonic plates: large segments of Earth's crust that move in relation to one another

telecommuting: working from home or another offsite location

tenable: defensible, reasonable

terrestrial refraction: the apparent raising or lowering of a distant object on Earth's surface due to variations in atmospheric temperature

tessellation: a mosaic of tiles or other objects composed of identical repeated elements with no gaps

tesseract: a four-dimensional cube, formed by connecting all of the vertices of two three-dimensional cubes separated by the length of one side in four-dimensional space

theodolite: a surveying instrument designed to measure both horizontal and vertical angles

theorem: a statement in mathematics that can be demonstrated to be true given that certain assumptions and definitions (called axioms) are accepted as true

threatened species: a species whose population is viable but diminishing or has limited habitat

time dilation: the principle of general relativity which predicts that to an outside observer, clocks would appear to run more slowly in a powerful gravitational field

topology: the study of those properties of geometric figures that do not change under such nonlinear transformations as stretching or bending

topspin: spin placed on a baseball, tennis ball, bowling ball, or other object so that the axis of rotation is horizontal and perpendicular to the line of flight and the top of the object is rotating in the same direction as the motion of the object

trajectory: the path followed by a projectile; in chaotic systems, the trajectory is ordered and unpredictable

transcendental: a real number that cannot be the root of a polynomial with rational coefficients

transect: to divide by cutting transversly

transfinite: surpassing the finite

transformation: changing one mathematical expression into another by translation, mapping, or rotation according to some mathematical rule

transistor: an electronic device consisting of two different kinds of semiconductor material, which can be used as a switch or amplifier

transit: a surveyor's instrument with a rotating telescope that is used to measure angles and elevations

transitive: having the mathematical property that if the first expression in a series is equal to the second and the second is equal to the third, then the first is equal to the third

translate: to move from one place to another without rotation

translation: a rigid motion of the plane that moves each point in the same direction and by the same distance

tree: a collection of dots with edges connecting them that have no looping paths

triangulation: the process of determining the distance to an object by measuring the length of the base and two angles of a triangle

trigonometric ratio: a ratio formed from the lengths of the sides of right triangles

trigonometry: the branch of mathematics that studies triangles and trigonometric functions

tropopause: the boundry in Earth's atmosphere between the troposphere and the stratosphere at an altitude of 14 to 15 kilometers

troposphere: the lowest layer of Earth's atmosphere extending from the surface up to about 15 kilometers; the layer where most weather phenomena occur

ultra-violet radiation: electromagnetic radiation with wavelength shorter than visible light, in the range of 1 nanometer to about 400 nanometer

unbiased sample: a random sample selected from a larger population in such a way that each member of the larger population has an equal chance of being in the sample

underspin: spin placed on a baseball, tennis ball, bowling ball, or other object so that the axis of rotation is horizontal and perpendicular to the line of flight and the top of the object is rotating in the opposite direction from the motion of the object

Unicode: a newer system than ASCII for assigning binary numbers to keyboard symbols that includes most other alphabets; uses 16-bit symbol sets

union: a set containing all of the members of two other sets

upper bound: the maximum value of a function

vaccuum: theoretically, a space in which there is no matter

variable: a symbol, such as letters, that may assume any one of a set of values known as the domain

variable star: a star whose brightness noticeably varies over time

vector: a quantity which has both magnitude and direction

velocity: distance traveled per unit of time in a specific direction

verify: confirm; establish the truth of a statement or proposition

vernal equinox: the moment when the Sun crosses the celestial equator marking the first day of spring; occurs around March 22 for the northern hemisphere and September 21 for the southern hemisphere

vertex: a point of a graph; a node; the point on a triangle or polygon where two sides come together; the point at which a conic section intersects its axis of symmetry

viable: capable of living, growing, and developing

wavelengths: the distance in a periodic wave between two points of corresponding phase in consecutive cycles

whole numbers: the positive integers and zero

World Wide Web: the part of the Internet allowing users to examine graphic "web" pages

yield (interest): the actual amount of interest earned, which may be different than the rate

zenith: the point on the celestial sphere vertically above a given position

zenith angle: from an observer's viewpoint, the angle between the line of sight to a celestial body (such as the Sun) and the line from the observer to the zenith point

zero pair: one positive integer and one negative integer

ziggurat: a tower built in ancient Babylonia with a pyramidal shape and stepped sides

Topic Outline

GRAPHICAL REPRESENTATIONS

IDEAS AND CONCEPTS

MEASUREMENT

NUMBER ANALYSIS

NUMBER SETS

SCIENCE APPLICATIONS

SPATIAL MATHEMATICS

SYSTEMS

TECHNOLOGY

mathematics

mathematics

VOLUME 3
Ma-Ro

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Mandelbrot, Benoit B.

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American Geometer
1924–

Benoit B. Mandelbrot, called the father of **fractal geometry**, was born November 20, 1924, in Warsaw, Poland, into a well-educated Jewish family. In 1936 the family moved to France where Benoit spent time with his uncle, Szolem Mandelbrojt, who was a professor of mathematics at the prestigious Collège de France in Paris, and who took an interest in Benoit's education.

Szolem Mandelbrojt recommended that Benoit study the work of Gaston Julia, whose 1918 paper was considered a mathematical masterpiece and a source of good problems. At the time, Benoit expressed little interest for the kind of mathematics that he found in Julia's paper, much to the dismay of his uncle, but instead showed an interest in geometry. In 1944 Benoit was accepted into the Ecole Polytechnique and studied under the direction of Paul Lévy, who embraced Mandelbrot's interest in geometry.

Mandelbrot's untraditional education, which included self-instruction, gave him the ability to study mathematical issues in a unique way. He is shown here in front of a fractal (self-similar) image that bears his name.

In 1952 Mandelbrot received his Ph.D. (Docteur ès Sciences Mathématiques) from the University of Paris. After completing his degree, Mandelbrot went to the United States, where he held a position at the School of Mathematics at the Institute for the Advanced Studies (under J. von Neumann) at Princeton University. He returned to France in 1955, at which time he married Ailette Kagan.

Discontented with the style of mathematics work in France at the time, Mandelbrot returned to the United States in 1958 to accept a position as a fellow and professor in the research department at the world-famous laboratories of International Business Machines (IBM) in New York. IBM was beginning to lead the computer industry, and the company provided Mandelbrot with the freedom and resources to pursue his research interests.

Chaos and Fractals

fractal geometry the study of the geometric figures produced by infinite iterations

During the 1970s, Mandelbrot's research examined unusual or chaotic patterns of behavior in geometric shapes. In 1975 Mandelbrot coined the term "fractal," from the Latin *fractus* (meaning fragmented, irregular), as a way to describe the self-similar geometric patterns he had discovered. In addition, Mandelbrot revisited Julia's earlier work and, with the aid of computer

graphics, illustrated Julia's work as a source of some of the most beautiful fractal images known today. His work was first published in English in his book *Fractals: Form, Chance, and Dimension* (1977).

geometry the branch of mathematics that deals with the properties and relationships of points, lines, angles, surfaces, planes, and solids

The connection between chaos and **geometry** was further established with Mandelbrot's discovery in 1980 of what we have come to call the Mandelbrot Set. Named in his honor, it is certainly the most popular fractal and is often noted as the most popular object of contemporary mathematics. In addition, Mandelbrot had discovered fractal geometry and chaotic behavior in many aspects of nature. In his most recognized book, *The Fractal Geometry of Nature* (1982), Mandelbrot demonstrated that mathematical fractals have many features in common with shapes found in nature, such as snowflakes, mountains, ferns, and coastlines.

Honors and Achievements

Throughout his career, Mandelbrot has held many academic positions. In addition to being a Fellow of the IBM Thomas J. Watson Research Center and of the American Academy of Arts and Sciences, he held appointments as professor of the practice of mathematics and economics at Harvard University; professor of engineering at Yale University; and professor of physiology at the Einstein College of Medicine. He is recognized for his many remarkable achievements, prizes, and honors in the fields of mathematics, physics, engineering, and medicine, including the Barnard Medal for Meritorious Service to Science (1985), the Franklin Medal (1986), the Alexander von Humboldt Prize (1987), the Steinmetz Medal (1988), the Nevada Medal (1991), and the Wolf prize for physics (1993). On June 23, 1999, Mandelbrot received the Honorary Degree of Doctor of Science from the University of St Andrews. SEE ALSO Chaos; Fractals.

Gay A. Ragan and Óscar Chávez

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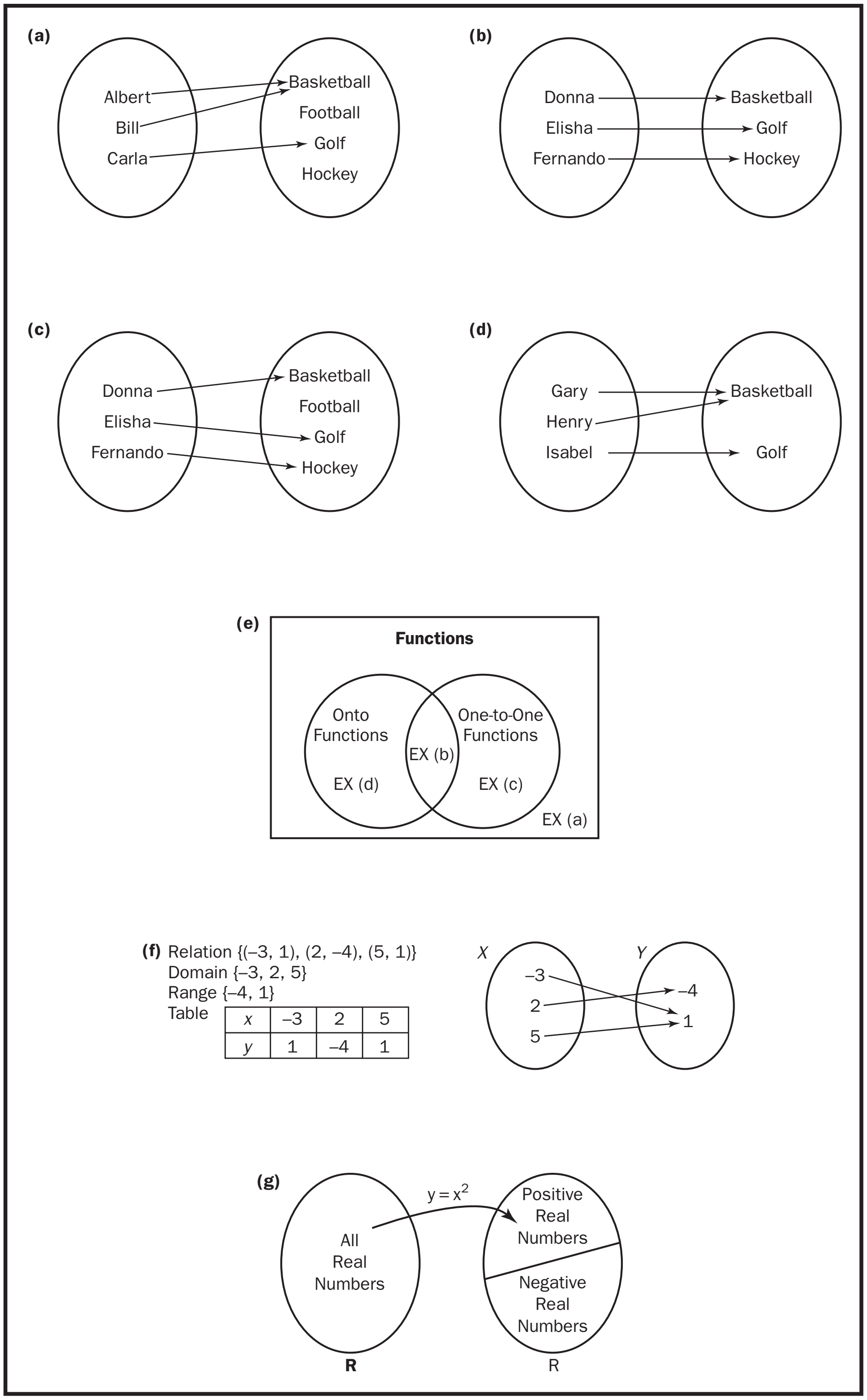
Mapping, Mathematical

A mapping is a function that is represented by two sets of objects with arrows drawn between them to show the relationships between the objects.



Examples of mathematical mapping.

In all mappings, the oval on the left holds values for the **domain**, and the oval on the right holds values for the **codomain**.

domain the set of all values of a variable used in a function

codomain for a given function *f*, the set of all possible values of the function; the range is a subset of the codomain

The figure shows a mapping of people and the sports they play. The mapping in (a) shows that Albert and Bill play basketball, Carla plays golf, and nobody plays football or hockey. The domain is {Albert, Bill, Carla}, and the codomain is {Basketball, Football, Golf, Hockey}. The sports that

range the set of all values of a variable in a function mapped to the values in the domain of the independent variable

set notation a standard way of using symbols to indicate sets and mathematical operations on them

they do play {Basketball, Golf} comprise the range. The domain, codomain, and **range** are always listed using **set notation**.

In (b) of the figure, the domain is {Donna, Elisha, Fernando}, the codomain is {Basketball, Golf, Hockey}, and the range is {Basketball, Golf, Hockey}. In this example, the codomain and the range are the same.

Formally defined, the domain is the set of values that can be assumed by the independent variable, and the range is the set of values that can be assumed by the dependent variable. In (a) and (b), the sport played depends on the person, so sports are the dependent variable and people are the independent variable.

One-to-One and Onto Functions

Some special types of mappings are the mappings of one-to-one and onto functions. A one-to-one correspondence is a type of function in which every object in the range is paired with, at most, one object from the domain. In other words, an object in the codomain can have no more than one arrow pointing to it. Example (b) is one-to-one, but example (a) is not one-to-one because both Albert and Bill play basketball; that is, basketball has two arrows pointing to it. Example (c) is the same as (b) except it also has football in the codomain; hence, it is a one-to-one correspondence because no element of the range has more than one arrow pointing to it.

An onto function has a relationship in which every object in the codomain is paired with at least one object in the domain. This means that one or more arrows must be pointing to every value in the codomain. In other words, the codomain and the range contain the same set of objects. Example (d) is an onto function. Example (b) is also an onto function: it is a special case because it is both one-to-one and onto. The diagram in (e) summarizes how examples (a) through (d) can be classified.

Mappings for Ordered Pairs and Beyond

Mappings often are used to describe sets of ordered pairs in which the domain is the set of all x-coordinates and the range is the set of all y-coordinates. A common term used to describe mappings is image. An image of an object in the domain is the y-value that is paired with it. In (f) of the figure, the image of -3 is 1, the image of 2 is -4, and the image of 5 is 1.

Note that (f) is onto because every element of the range has arrows pointing to it, but it is not one-to-one because one element of the range has two arrows pointing to it.

Mappings for equations often represent infinite sets of ordered pairs, and therefore must be drawn a little differently. The domain represents the set of input values (x-coordinates), and the range represents the set of output values (y-coordinates). The codomain may or may not be a larger set of numbers than the range.

Example (g) of the figure illustrates how the function $y = x^2$ is used to map from the domain, $\boldsymbol{R}$ to the codomain, R. Note that $\boldsymbol{R}$ is a symbol used to represent the set of all real numbers. Any real number can be input into the equation $y = x^2$; however, the output is always positive. So although the codomain is the set of all real numbers, the range is the set of positive real numbers.

Mappings can also be used to represent multidimensional problems such as mapping the ordered pairs of the vertices of a **polygon** to the corresponding ordered pairs for its reflection. They can also represent assigning a unique number to the coordinates of a three-dimensional object. Finally, mappings can represent n-dimensional spaces. SEE ALSO Functions and Equations.

polygon geometric figure bounded by line segments

Michelle R. Michael

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Maps and Mapmaking

Maps have been made for thousands of years as a means to convey information about the surface of the Earth. Some maps are highly complex, organized data sets that have been assembled by **cartographers**, mathematicians, or geographers who wish to illustrate the lay of the land. Maps are able to provide tremendous amounts of information on a two-dimensional surface.

cartographer a person who makes maps

The History of Mapmaking

The first attempt to establish a grid system on Earth was in 150 C.E. by the Greek astronomer Ptolemy. He established the concept of imaginary lines that intersected at regular intervals, making it possible to locate a particular position from reference points on a grid.

Using information from his astronomical predecessors, Ptolemy established **latitude** as imaginary parallel lines, equally spaced, that circled the world in a horizontal plane. The zero degree parallel was the equator. He chose this parallel as zero because it has the largest diameter of any latitude.

Ptolemy proposed the concept of **longitude**, equally spaced imaginary lines that run north and south, as a way of dividing Earth into a set of lines that are parallel to one another at the equator. The zero **meridian** was originally chosen arbitrarily by Ptolemy. Throughout the centuries, the **Prime Meridian** has been changed as a result of various political interests. Through an international consensus, it was agreed that the prime meridian would pass through Greenwich, England, going from the North to the South Poles.

latitude the number of degrees on Earth's surface north or south of the equator; the equator is latitude zero

longitude one of the imaginary great circles beginning at the poles and extending around Earth; the geographic position east or west of the prime meridian

meridian a great circle passing through Earth's poles and a particular location

Prime Meridian the meridian that passes through Greenwich, England

Mathematics in Maps

One of the first mathematical calculations to be determined for a map is the scale. Because it is usually infeasible to make a life-sized map, mapmakers must reduce the size of real objects and distances proportionally. The smaller a map scale, the more detail the maker can include.

Projections are one of the most difficult tasks of mapmaking. A projection is an attempt to draw the gridlines of a spherical object on a flat surface. In reality, it is impossible to draw the true gridlines of Earth on a flat surface without losing some of the mathematical accuracy of distance or

area. For example, on a flat map, Greenland appears much larger in proportion to areas that are closer to the equator than it is in reality.

Most projections are derived from geometric figures. To make geometric maps, **trigonometry** is used to solve the problems of distortion.

trigonometry the branch of mathematics that studies triangles and trigonometric functions

Gerardus Mercator was one of the first people to solve the problem of distortion in maps. In 1569 he drew his map so that the meridians remained vertical and equally spaced while the parallels increased in spacing as they approached the equator. Mercator determined the correct degree of increase between parallels from the **secant** of the latitude.

secant the ratio of the side adjacent to an acute angle in a right triangle to the side opposite; given a unit circle, the ratio of the *x* coordinate to the *y* coordinate of any point on the circle

Cylindrical maps represent Earth as a cylinder on which the parallels are horizontal lines and the meridians are perfectly vertical. These maps may seem impractical because of the tremendous distortion in the polar latitudes, but they are excellent for use with compass bearing, which can be plotted as straight lines.

Conic projections are maps drawn as a projection from a point above the North or South Poles. The map intersects Earth at a point **tangent** to a specific point on the sphere, usually a pre-selected parallel. **Polyconic projections** are used in large-scale map series. Each conic section is made to correspond to a particular latitude.

tangent a line that intersects a curve at one and only one point in a local region

polyconic projection a type of map projection of a globe onto a plane that produces a distorted image but preserves correct distances along each meridians

Azimuthal projections are maps made from viewing Earth from a particular perspective; either outer space, the interior of Earth, or different

Geographers and cartographers continue to improve the accuracy of maps. Mapmaking has evolved from the beautifully handpainted paper maps of ancient times to the sophisticated Global Positioning System (GPS) of the twenty-first century, which provides accurate geographic details via satellites.

hemispheres of Earth. The map itself is a flattened disk with its center at a point tangent to a reference point. The map represents a view from one of the specific points mentioned. These types of maps are most often seen as polar projections in which the polar land and sea are shown together as a circle.

Digital Maps

Maps are often updated using aerial photographs. These photographs are known as digital orthophoto quadrangles (DOQ). DOQs are altered so that they have the geometric properties of a map. There are four elements necessary in an aerial photograph: three identifiable ground positions, camera **calibration** specifications, a digital elevation model of the area in the photograph, and a high-resolution digital image of the photograph. The picture is then processed **pixel** by pixel to create an image with true geographic positions.

calibration act of systematically adjusting, checking, or standardizing the graduations of a measuring instrument

pixel a single picture element on a video screen; one of the individual dots making up a picture on a video screen or digital image

Geographers, mathematicians, and computer analysts, among others, continue to improve the quality and reliability of maps. Modern digital technology has helped enhance our current understanding of the surface of Earth. Mapmaking will continue to become more refined as technology continues to improve. SEE ALSO Cartographer; Geography; Global Positioning System.

Brook E. Hall

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Marketer

Why do consumers stay loyal to certain brand names, buy items they do not need, or upgrade belongings that function perfectly well? Marketers hope to answer these questions and more as they constantly strive to better understand consumer behavior.

A marketer is involved in one of the activities that takes a product or service from the producer to the consumer. Marketers necessarily focus on the buyer, relying on numerical analysis of consumer interests and satisfaction. **Demographics** and the condition of the economy must also be mathematically interpreted. In fact, marketing can only be deemed successful when a response to the product can be numerically measured.

demographics statistical data about people—including age, income, gender—that are often used in marketing

Effective marketers help businesses improve or **remediate** the item they are selling and get the word to consumers. Financial, marketing, and stakeholder scorecards are developed to allow marketers to do this. These score-

remediate to provide a remedy; to heal or to correct a wrong or a deficiency

Marketers pitched the singing California Raisins to the California Raisin Advisory Board in 1986. After the television advertisements aired, sales soared. The Raisins eventually appeared on tote bags, bedding, videos, and bookmarks and had their own television special.

cards are created using mathematical techniques of data collection and interpretation, flow-charting, regression analysis, measurement, and decision-making theory and **statistics**. Computer modeling makes many sophisticated techniques available to marketers, who should have a good understanding of mathematics.

statistics the branch of mathematics that analyzes and interprets sets of numerical data

deregulation the process of removing legal restrictions on the behavior of individuals or corporations

privatization the process of converting a service traditionally offered by a government or public agency into a service provided by a private corporation or other private entity

As the economy adjusts itself to the effects of globalization and technology, marketing techniques must be more innovative than ever for businesses to succeed in a highly competitive market. The forces of **deregulation** and **privatization** make the marketplace even more cutthroat. Products and services must give the company an edge over others in some way: higher quality or better service, lower prices, improvement or innovation, higher market share, customization or entrance into high growth markets. A marketer who aggressively performs marketing research and adjusts strategies accordingly will have success at predicting consumer behavior. SEE ALSO Consumer Data.

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Mass Media, Mathematics and the

Mathematics as a tool of the media can influence the values, beliefs, and ideas of its readers and listeners. As a result, students and consumers of information must learn how to recognize sound and usable data in a wilderness of numbers.

Thinking Critically about Numbers

The use of critical thinking skills can enable information consumers to analyze and critique the mathematics, data, and statistics that are reported in the media. Statistical methods and statistical terms are frequently used in articles dealing with social and economic trends, business conditions, opinion polls, and the census. But writers often do not use these terms accurately, and results are usually presented within a limited context. Consequently, the jargon can seem like **semantic** nonsense for readers, listeners, and viewers. In addition, results may be misleading, depending on the argument of the writers.

semantic the study of how words acquire meaning and how those meanings change over time

Critical thinking ability is radically different from using numbers just to add, subtract, multiply, and divide accurately. Our technological society requires everyone to know how and why crucial issues are put in mathematical form by the media. For instance, college admission requirements that use a combination of test scores and high school rank can be used by the media to report either a positive or a negative message.

Radio, television, newspapers, and the Internet are changing the way mathematics is viewed and, in turn, increasing the importance of mathematical modeling in the media. Students and consumers must be prepared to ask and answer different questions, such as:

- Is this the best mathematical model for the information being presented? What methods of mathematical analysis will best support the position?
- What variables should be included in the analysis to strengthen the position?
- Can mathematical models minimize the appearance of important data?
- Will percentages or fractions make a more striking impression?

Thinking Critically about Statistics

Students and consumers must understand when information is distorted or misrepresented because of a misuse of statistics. For example, the word "average" is frequently used in the media to convey information consisely. Yet mathematically, the average value can have different interpretations, and therefore may yield a biased picture of reality.

When the word "average" is encountered, several things must be considered. For instance, what form of average is being used? Statisticians use three types of averages, or measures of central tendency: mean, median, and mode. Data can be made to appear more favorable depending on which measure is chosen.

To calculate the mean, all of the values in a data set are summed up then divided by the total number of values. Hence, very high or low values can influence the average. For example, using the mean to determine the average salary of workers in a company will give the appearance of higher pay if the owner's much higher pay is included. Depending on the research question, the average calculated in this manner yields a misleadingly high figure.

SHARK ATTACKS: ON THE RISE?

Headlines may state that certain harmful events are increasing in frequency, duration, or severity. But just because a calculated number may be slightly higher than before does not necessarily mean that an event is worsening. In fact, a small increase could, in reality, be a decrease depending on the margin of error, which is partly determined by how the data were collected and analyzed.

Moreover, repeated news coverage can give the impression that numbers of events are increasing. For example, heavy coverage of shark attacks in 2001 gave the impression that attacks were increasing, even though 2001 statistically was considered a typical year.

With a median, which is the value that falls in the middle of the ranked distribution of data points, the data show that half the employees make more than that value, and half make less. The mode, the most frequently occurring value in a data set, reveals the most common pay. In this example, both the median and mode will be considerably less than the mean because they are less influenced by the owner's higher pay.

degree of significance a determination, usually in advance, of the importance of measured differences in statistical variables

ordinate the *y*-coordinate of a point in a Cartesian plane

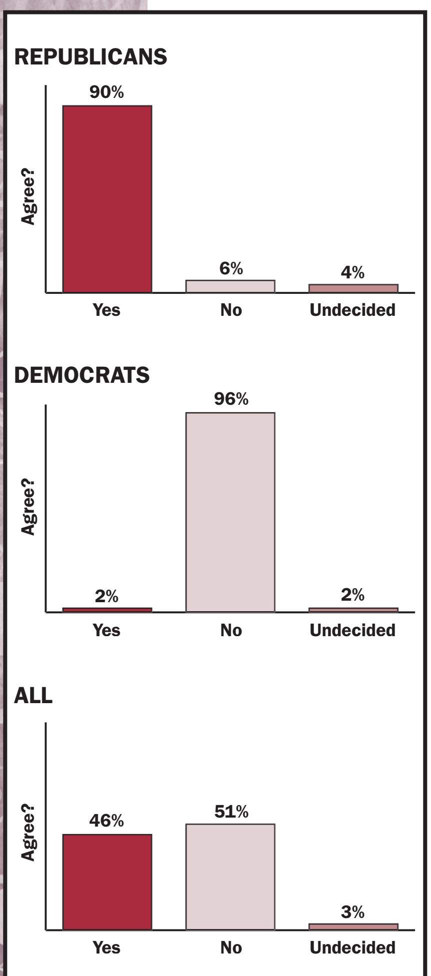
abscissa the *x*-coordinate of a point in a Cartesian coordinate plane

Another question to ask is "Who is included in the average?" If only tall people were included in a calculation of "average" height, the resulting mean would not represent the population as a whole.

Yet another question is "How large is the sample?" A sample is a portion of the population that is evaluated to gain information with the intention of generalizing to the whole. An inadequate statistical sample size will not produce conclusive results. For example, results cited from a small "independent" laboratory may be relying on an experimental study of six cases, hardly enough to determine any **degree of significance** in the results. (One could also ask: Which laboratory? How small?)

Consider what happens to percentages when a coin is tossed 10 times versus 1000 times. Ten tosses may give the result of 8 heads and 2 tails for an 80 percent result of heads. Yet 80 percent would be misleading, because one thousand tosses will bring the result closer to the actual probability of 50 percent.

Finally, the degree of significance also reveals the accuracy of the data. For most purposes, 5 percent is thought to be significant. This level of significance indicates that the probability that the results were generated randomly is less than 5 percent.



Responses to a poll can show very different results depending on who is polled.

Thinking Critically about Graphs

Technological advancements have made graphics a vital mathematical feature in the media. Although pictorial graphs are commonly used because they are appealing to the eye, they have a high potential for misuse—and they can further compound misused statistics.

Consider a survey in which American citizens were asked whether they agreed with a decision made by the president of the United States. The percentage of "yes" and "no" responses would predictably follow a pattern based on partisanship, as shown in the figure. But if responses from both parties are combined, the percentages of "yes" and "no" responses are fairly even.

One media report could show only the top graph, giving the indication that Americans strongly favored the president's decision. Another report could show only the middle graph, indicating strong opposition. Both graphs are accurate, but each one by itself shows only part of the picture. Hence, a conscientious media reporter must clearly indicate who was polled.

Another misuse of graphs involves proportions. For example, by simply changing the proportion on the **ordinate** and **abscissa**, or by completely eliminating the scale, a person preparing a graph can make it appear to rise or fall more quickly and thus give the impression of a drastic increase or decrease. (Darrell Huff, the author of *How to Lie with Statistics*, calls these "gee-whiz" graphs.) Furthermore, the use of proportions for objects can be misleading when a one-dimensional picture is represented in two dimen-

sions. The height of a moneybag may represent the comparison of two salaries where one salary is double the other. If the pricier money bag is made twice as tall as the other, yet the same proportion is maintained, the numbers still say two-to-one, but the visual impression—which is the dominating one most of the time—says the ratio is four-to-one. SEE ALSO Central Tendency, Measures of; Data Collection and Interpretation; Graphs; Polls and Polling; Probability, Experimental; Statistical Analysis.

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Mathematical Devices, Early

Early humans counted and performed simple calculations using tools such as their fingers, notches in sticks, knotted strings, and pebbles. Most early cultures evolved some form of a counting board or abacus to perform calculations. Pencil and paper eventually replaced these early counting boards, but a modern form of the abacus may still be seen in use in parts of Russia and Asia in the twenty-first century.

Counting Boards

Ancient cultures such as the Greeks, Babylonians, and Romans marked parallel lines on a table and placed pebbles on the lines for counting. In the Western hemisphere, the Mayans, Aztecs, and Incas used kernels of grain as counters. The parallel lines represented numbers, and pebbles or other counters placed on the lines denoted multiples of that number. Since the value assigned to a counter depended on the line on which it was placed, these early counting devices used a place value system. Some of the cultures that used these place value devices for computations then recorded the results of these calculations using a number system that did not use place value, such as Roman numerals.

The pebbles the ancient Romans used for their counting boards were called *calculii.* Our modern words "calculate" and "calculus" come from this root word.

Very few counting tables still exist. However, they must have been common because they are often mentioned in wills and inventories. In the fifth century B.C.E., the Greek historian Herodotus (c. 485 B.C.E.–425 B.C.E.) described counting tables that used pebbles and wrote examples of the calculations for which they could be used. One such calculation was computing the interest due on a loan.

One of the few counting tables still in existence was found on the Greek island of Salamis. It is now in two pieces, but it was once a very large marble slab, approximately 5 feet by 2 1/2 feet. The table is marked with 11 vertical lines, a blank space between them, and horizontal lines crossing the

vertical ones. Greek symbols appear along the top and bottom of the tablet. No one is certain what it was used for, but it could have been used for addition, subtraction, multiplication, and division.

To add numbers using a counting board or table, counters would be placed on the appropriate lines to denote the first number to be added. Additional counters were placed on appropriate lines to make up subsequent numbers to be added. If there were numbers to be carried, counters were removed from one line and an additional counter was placed on the line to represent the next higher number. At the end of the operations, the total value of the counters on the table indicated the sum. For subtraction, counters would be taken away, with any borrowing done manually. Since negative numbers were not used at the time, smaller numbers would be subtracted from larger ones.

By the thirteenth century a standard form of the counting table was prevalent in Europe. It was a table upon which lines were drawn to represent the place value of the counters to be put on the lines. The bottom line was the units place and each subsequent line represented ten times the value of the line below it. These lines formed a **base-10 system**. Each space between two lines represented numbers having five times the value of the line below the space. As soon as five counters appeared on a line or space, they were removed and replaced by one counter on the next higher space or line.

base-10 system a number system in which each place represents a power of 10 larger than the place to its right

Early counters were usually pebbles, but by the thirteenth century in Europe, counters resembled coins. These later counters came to be called *jetons* from the French verb *jeter*, meaning "to throw." They were quite common and at one time manufacturing jetons was a major industry in Europe.

The Abacus

The more modern wire and bead abacus began in the Middle East during the early Middle Ages, c. 500 C.E.–1000 C.E. The abacus is believed to have spread from Europe along trade routes to the east. It was first adopted by the merchants in each society because they had to perform many calculations in their daily business activities.

The Greeks used the word "abax" to denote the surface on which they placed their counting lines. This may have come from the Semitic word *abaq*, meaning dust. This term spread to Rome where counting boards were called *abaci*. The abacus was called a *choreb* by the Turks and a *stchoty* by the Russians. As the abacus was used in more societies, its form changed, but the principles of computation remained the same.

By 1300 a device resembling the modern abacus was in common use in China. It consisted of a rectangular wooden frame with a bar running down its length dividing the abacus into two parts. The upper part, smaller than the lower, was sometimes referred to as "heaven," and the lower part as "Earth." Dowels were placed through the dividing bar, **perpendicular** to it.

perpendicular forming a right angle with a line or plane

The "heaven" part of each dowel was strung with 2 beads, each representing 5 times the place value of the number corresponding to the dowel. The "Earth" part of each dowel contained 5 beads, each representing the place value of the corresponding number. The initial position of each bead was touching either the outer frame or a bead touching the outer frame. The beads were used for counting or computing by touching them either

to the bar or to a bead touching the bar. Any number from 0 to 15 could be represented by the beads on one dowel, although numbers greater than 9 would be carried to the next higher dowel. The Chinese called this device *suan pan*, or counting table.

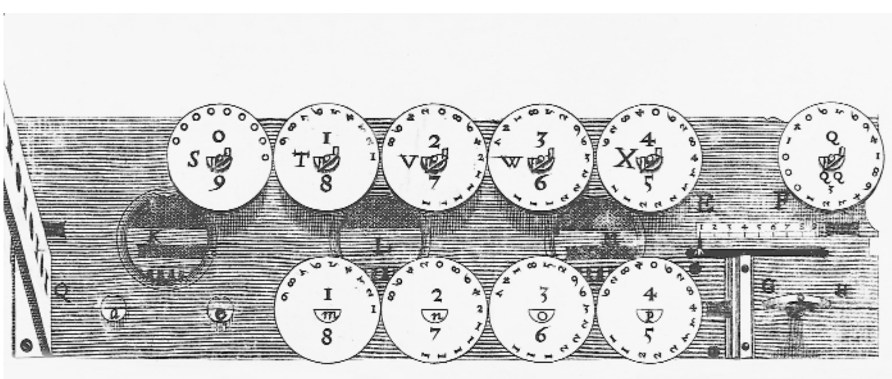
Around 1500, the wire and bead abacus spread from China to Japan, where it was called the *soroban.* The modern soroban has only one bead in heaven and four in Earth, so 9 is the highest number that can be represented on a dowel.

The abacus does not multiply and divide as efficiently as it adds and subtracts. Multiplying when one of the factors is a small number can be done by repeated addition. It is a little trickier to multiply by larger numbers. For example, to multiply 141 by 36, first multiply 141 by 3 by adding 141 three times. Then multiply that result by 10. In a base-10 system, this would involve shifting each digit to the left to add a zero to the end. Then multiply 141 by 6, by adding 141 six times, and then add the two results together to get the product of 141 and 36. The world would have to wait for a further mathematical development in order to have tools that could multiply, divide, raise numbers to powers, and extract square roots.

Napier's Bones

The Scottish mathematician John Napier (1550 C.E.–1617 C.E.) wanted to simplify the work involved in calculations. He accomplished this by inventing a calculating device, "Napier's bones," so-called because the better quality instruments were made of bone or ivory. Napier's bones consisted of flat rods with a number 0 through 9 at the top of each. Underneath the top number on each bone are nine squares, each divided in half by a diagonal from upper right to lower left. The first square contains the product of the number and 1, the second the product of the number and 2, and so on. The tens place is in the upper half, the units place in the lower half.

To multiply a multi-digit number by a single digit, the rods corresponding to the larger number are placed side by side. The solution is found in the row corresponding to the multiplier. The rightmost digit of the product is in the lower half square of the rightmost rod. The next digit is the sum of the number in the upper half of the rightmost rod and the lower half of the rod to its left, and so on. If a sum is more than 9, it is carried to the next higher digit; hence, the person using the rods must keep track of the numbers to be carried.



In 1672 Samuel Morland used Napier's logarithms to build an updated disc version of Napier's bones, which had consisted of flat rods.

Napier published a description of his invention in 1617, the year of his death. The bones became used widely in Europe and spread to China. Several improvements were made to Napier's bones over the years. One was the Genaille-Lucas ruler, which was similar to Napier's bones but designed to eliminate the need to carry from one digit to another. Napier's bones, and other related devices, could also be used for division and extracting square and cube roots. Napier's bones were used to build the first workable mechanical adding machine in 1623.

Logarithms and the Slide Rule

Another of Napier's inventions—logarithms—had a more lasting effect on simplifying calculations than his mechanical multiplier.✶ Logarithms are exponents, the power to which a number, such as 10 (called the base), is raised to yield a given number. Since exponents are added when two powers are multiplied together, the logarithm of a product is the sum of the logarithms of the factors. Likewise, when one power is divided by another, the exponent of the divisor is subtracted from the exponent of the dividend.

✶John Napier was the first to use and then popularize the decimal point to separate the whole number part from the fractional part of a number.

Calculations using logarithms involve adding and subtracting instead of multiplying and dividing. Logarithms can also be used to raise numbers to powers or extract roots by multiplying and dividing. Using logarithms replaces more complicated computations with simpler ones.

After Napier devised his system of logarithms, English mathematician Henry Briggs (1561–1631) developed extensive tables of logarithms. Within a few decades scientists and mathematicians throughout the world were using logarithms for their calculations. Anyone using logarithms for computations had to use the tables to look up the logarithm of each number in the calculation. This could be a tedious task (though not as tedious as calculations without logarithms), and the tables contained errors.

But logarithms had yet another contribution to make. An English astronomy and mathematics teacher named Edmund Gunter (1581–1626) plotted logarithms of numbers on a line (called Gunter's Line of Numbers) and multiplied and divided numbers by adding and subtracting lengths on the line.

An English clergyman named William Oughtred (1574–1660) refined Gunter's line by using two pieces of wood that slid against each other. Each piece of wood contained a scale in which the distance of a number from the end of its line is proportional to its logarithm. To multiply two numbers, one of the numbers is lined up with 1 and the product appears opposite the other number. Division reverses the process. Unfortunately, the slide rule was not accurate to many decimal places. It also required the user to keep track of where the decimal point belonged.

Despite its drawbacks, the slide rule was enormously successful. It eliminated the need for using tables of logarithms. The slide rule was used by scientists and mathematicians, as well as students, for over 300 years until it was replaced by the electronic hand-held calculator. Human computing ability has come a long way from sticks and pebbles. SEE ALSO Abacus; Bases; Logarithms; Mathematical Devices, Mechanical; Slide Rule.

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Mathematical Devices, Mechanical

The earliest mechanical mathematical devices were crude, slow, and subject to errors. Users had to adjust the device manually for some functions, such as carrying in addition, just as did the counting table and abacus. The ability to build faster and more accurate devices was constrained by the limited technology of the time.

Schickards's Mechanical Calculator

For many years the French mathematician and philosopher Blaise Pascal (1623–1662) was believed to have built the first mechanical calculator. However, research in the 1950s and 1960s revealed that an earlier mechanical calculator was built by Wilhelm Schickard (1592–1635), a German professor and minister.

Schickard also had an interest in mathematics, and was a friend of Johann Kepler (1571–1630), the prominent astronomer. By 1623 Schickard had produced the first workable mechanical adding machine capable of carrying and borrowing from one column to the next. The machine consisted of a set of Napier's bones drawn on cylinders so that any bone could be selected by turning a dial.

Horizontal slides exposed different sections of the bones to show single-digit multiples of the selected number. The result would be stored in a set of wheels called the accumulator. Whenever one of the wheels made a complete rotation by passing from 9 to 0, a tooth would increase the next highest digit in the accumulator by one.

In earlier computing devices, such as the abacus, digits were added separately and the user manually carried from one digit to the next higher digit. Carrying from one place to the next without human intervention was the greatest problem facing designers of mechanical digital calculators, but it could be accomplished by having two gears with a gear ratio of 10:1. When the first counter added 9 + 1, the second counter went from 0 to 1, and the first counter went back to 0. The number 10 was the result.

The accumulator in Schickard's machine had only six digits because of difficulties in propagating "carries" to several places, as in adding 99,999 + 1 to produce 100,000. Each subsequent carry put force on each previous gear so that when several carries were propagated, the force on the units gear could damage it.

Schickard wrote letters to Kepler telling him that he had built two copies of the machine, one for himself and one for Kepler. He included detailed drawings of his machine in these letters. The machine Schickard made for Kepler was destroyed by fire. Schickard and his entire family died in a plague and no trace of his machine has ever been found. However, Schickard's machine was reconstructed in 1960 based on the drawings Schickard had sent to Kepler.

Pascal's Calculating Machine

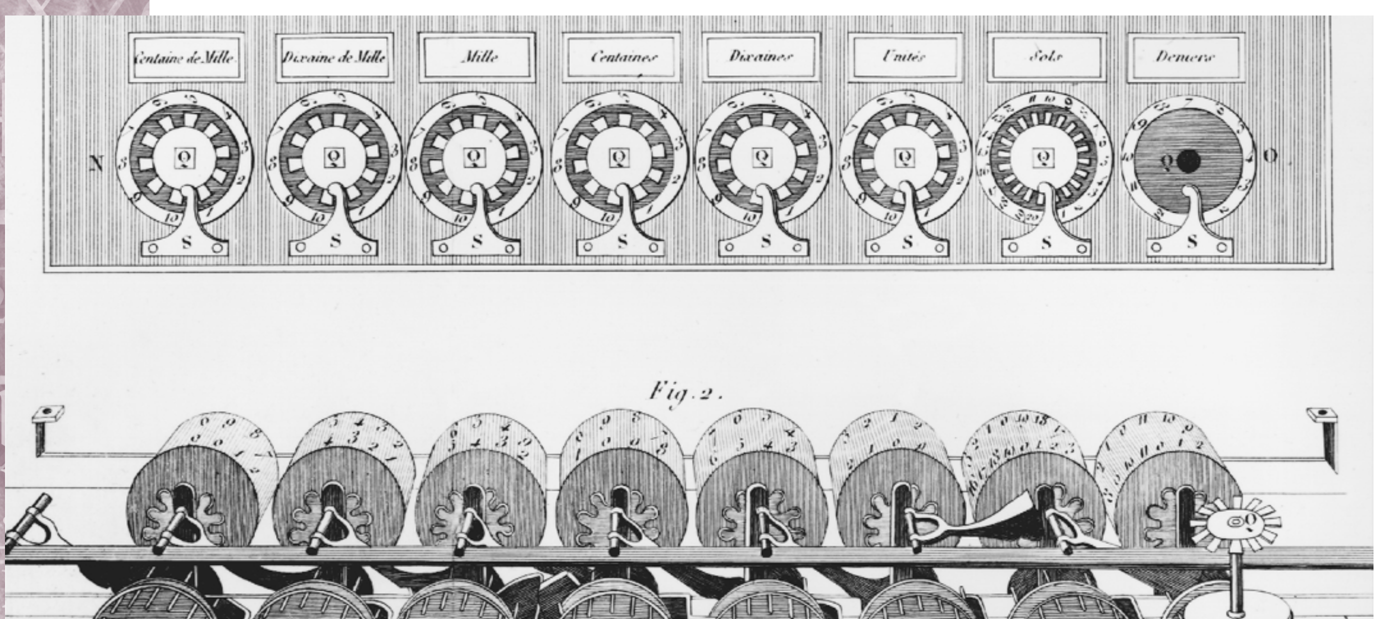
Pascal (1623–1662) produced the next important calculating machine. The son of a tax collector, Pascal invented the machine to help his father with the large number of calculations his work required. Pascal's machine, called the Pascaline, had a carry mechanism completely different from Schickard's.

In the Pascaline, weights were placed between each two accumulator wheels. When one accumulator wheel moved from 9 to 0, a weight would drop and move the accumulator wheel for the next higher digit by one place. This worked better than Schickard's carry mechanism, but it was very sensitive to external movements and would sometimes generate extra carries when the device was bumped.

The wheels in the Pascaline turned in only one direction and therefore could not subtract directly. In order to subtract, the nines complement of a number was added. For example, to subtract 461 from 1,090, first subtract 461 from 9,999. The result is 9,538. Then add 9,538 and 1,090 giving 10,628. Take the leading 1 and add it to the units place, giving 629. This is the result of the original subtraction.

This formula works because first 9,999 is added, then 10,000 is subtracted, then 1 is added, giving a net change of zero. The disadvantage of this process is that the nines complement had to be computed by the person operating the machine, which was slow and subject to human error. Few people of that time were well educated and hence most machine operators were unable to perform such calculations.

Prior to completing the finished product, Blaise Pascal went through over fifty designs for his calculating machine.



Although the Pascaline was not commercially successful, it was influential in the designs of susequent calculating machines. Pascal produced about fifty different models of his mechanical calculating machines during his life, all based on the same design. Many of them have survived to the twenty-first century. However, they are delicate and produce unreliable results.

Leibniz's Mechanical Multiplier

The Pascaline could multiply only by repeated addition and divide by repeated subtraction. This could be time-consuming if the multiplier and divisor were large numbers. Multiplying numbers directly became possible in about 1671 when the German mathematician (and co-developer of **calculus**) Gottfried Leibniz (1646–1716) invented a mechanical multiplier.

calculus a method of dealing mathematically with variables that may be changing continuously with respect to each other

To multiply a number by 3, for example, in Leibniz's multiplier, the number to be multiplied was entered into the machine by setting pointers that controlled the gears. Then a crank was turned three times. The machine had two layers so a multiplier could have two digits. To multiply a number by 73, the crank would be turned three times, then the top layer of the machine was shifted one decimal place to the left and the crank was turned seven more times.

Leibniz's machine included one of his inventions, the stepped drum. This drum had nine cogs on its exterior, each of a different length. The longest cog signified the number 9, and the shortest signified 1. A gear above the drum could change positions along the drum's length and engage a different number of cogs depending on where the drum was located. The gear could represent any number from 1 to 9 on the drum simply by moving.

Leibniz's machine, though ingenious, could not carry from one digit to the next higher digit without the help of the human operator. A pointer would indicate when a carry needed to be made, and the operator of the machine would push the pointer down, which would cause a carry to the next higher digit. If this carry caused another carry, another pointer would go up and the process would be repeated.

Leibniz had high hopes for his machine. He sent a copy of it to Peter the Great of Russia, asking the czar to send it to the emperor of China to encourage the emperor to increase trade with European nations.

Despite its limitations, the stepped-drum design endured for many years. Leibniz's basic idea had been sound, but the engineering and manufacturing technology of his time did not allow him to build accurate, reliable machines. In the years after Leibniz's death, engineering and design practice improved.

Later Calculating Devices

The first commercially successful calculating machine was built around 1820. This machine, called the "arithmometer," was developed by Charles Xavier Thomas of France. Thomas was able to build on Leibniz's innovations and develop an efficient, accurate carry mechanism for his arithmometer. Machines having the same general design as the arithmometer were sold for about 90 years.

The varying lengths of the cogs allowed calculating devices based on the stepped drum to represent any number with a fixed number of cogs

(nine). In 1885 an American, Frank Baldwin, and W. T. Odhner, a Swede, simultaneously but independently developed a gear with a variable number of teeth. The gears in this machine consisted of flat disks with movable pins that could be retracted into the disk or made to protrude.

Machines made of variable-toothed gears were reliable, but their main benefit was that the disks were very thin and so they were much smaller than the stepped-drum machines. A machine using stepped drums might be as large as an entire desk surface, whereas the variable-toothed gear machines would fit on the corner of a desk. These smaller machines were much more practical and more widely used.

The machines with variable-toothed gears, however, retained one drawback of the earlier machines. Performing calculations was a two-step process, which made them time-consuming. The operator first had to set levers or gears to enter a number and then pull a lever or crank to make the machine perform the calculation.

Because of the time required to enter and operate on numbers, the calculating machines produced up to this point were better suited for the work of scientists than that of office workers. Scientific calculations often required many operations on a few numbers, whereas business applications often required adding long lists of numbers.

Between 1850 and 1885 a number of patents were granted for devices that greatly simplified the work required to operate them; all were either never constructed or of too limited a capacity to be useful. Then, the year after Baldwin and Odhner built their variable-toothed machines, an American machinist named Dorr E. Felt produced a working calculating machine that had a range of uses.

Felt's machine combined the operations of entering a number and performing the calculation using that number into a stroke of a key. This machine was called a comptometer and required that the operator only push a key corresponding to numbers in order to add them. For example, to add 367, the operator would push the 3 in the hundreds column, the 6 in the tens column, and the 7 in the units column.

Desktop key-driven mechanical calculators remained in common use until electronic hand-held calculators replaced them in the 1970s. Though many improvements were made that would make them cheaper, smaller, and more reliable, the basic design of Felt's machine remained the same.

Babbage's Machine

One of the most influential mechanical mathematical machines was never built. In the late-eighteenth century, mathematical tables (such as trigonometric and logarithmic) were major tools of mathematicians and scientists. These tables contained many errors in calculation, copying, and printing. In 1827 an eccentric British genius named Charles Babbage (1791–1871) published the most accurate set of mathematical tables produced up to that time.

Years earlier, while a student of mathematics at Cambridge, Babbage had dreamed of building a machine that would quickly and accurately compute and print mathematical tables. Babbage was able to obtain funding from the British government for his machine, which he called the Difference En-

THE SCHEUTZ MACHINE

In 1853, a Swedish father-and-son team, Pehr and Edvard Scheutz, developed a tabulating machine based on Charles Babbage's designs. The Scheutz machine used the method of differences to produce several hundred mathematical tables.

gine. This machine would operate on the principle of differences. For example, consider the simple polynomial x^2. The first difference is the difference between the square of a number and the square of the previous number. The second difference is the difference between the first difference of a number and the first difference of the previous number.

x	x^2	First difference	Second difference
1	1		
2	4	3	
3	9	5	2
4	16	7	2
5	25	9	2

Note that the second differences of x^2 are all the same. Any second-degree polynomial has constant second differences, third-degree polynomials have constant third differences, and so on. This makes it possible to compute polynomial functions of large numbers simply by knowing the value for smaller numbers and adding to them.

This method of differences was useful for constructing mathematical tables because many functions, such as logarithms and trigonometric functions, do not have constant differences but can be approximated by polynomials that do. Babbage then proposed to build a machine that would use this method to construct accurate mathematical tables.

But financial and design problems plagued Babbage's Difference Engine and it was never built—if it had been constructed, it would have weighed about 2 tons.✶

✶**In 1991 researchers in London's Science Museum completed a working model of Babbage's Difference Engine.**

While working on designs for the Difference Engine, Babbage devised the idea of an improved machine, the Analytical Engine, which could compute any mathematical function, even those not based on common differences. This engine was designed with a memory and would have been controlled by a program that was punched into cards. The Analytical Engine was never built, but its concept was the forerunner of twentieth-century computers. SEE ALSO Babbage, Charles; Computers, Evolution of Electronic; Mathematical Devices, Early; Pascal, Blaise.

Loretta Anne Kelley

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Mathematics, Definition of

Over the centuries, people have thought of mathematics, and have defined it, in many different ways. Mathematics is constantly developing, and yet

the mathematics of 2,000 years ago in Greece and of 4,000 years ago in Babylonia would look familiar to a student of the twenty-first century.

Mathematics, says the mathematician Asgar Aaboe, is characterized by its permanence and its universality and by its independence of time and cultural setting. Try to think, for a moment, of another field of knowledge that is thus characterized. "In most sciences one generation tears down what another has built and what one has established another undoes. In Mathematics alone each generation builds a new story to the old structure," noted Hermann Henkel in 1884.

From Truth to Application

The mathematician and philosopher Bertrand Russell said that math is "the subject in which we never know what we are talking about nor whether what we are saying is true." Mathematics, in its purest form, is a system that is complete in itself, without worrying about whether it is useful or true.

Mathematical truth is not based on experience but on inner consistency within the system. Yet, at the same time, mathematics has many important practical applications in every facet of life, including computers, space exploration, engineering, physics, and economics and commerce.

In fact, mathematics and its applications have, throughout history, been inextricably intertwined. For example, mathematicians knew about **binary arithmetic**, using only the digits 0 and 1, for years before this knowledge became practical in computers to describe switches that are either off (0) or on (1). Gamblers playing games of chance led to the development of the **laws of probability**. This knowledge in turn led to our ability to predict behaviors of large populations by **sampling**.

binary arithmetic the arithmetic of binary numbers; base two arithmetic; internal arithmetic of electronic digital logic

laws of probability set of principles that govern the use of probability in determining the truth or falsehood of a hypothesis

sampling selecting a subset of a group or population in such a way that valid conclusions can be made about the whole set or population

chaos theory the qualitative study of unstable aperiodic behavior in deterministic nonlinear dynamical systems

The desire to explain the patterns in 100 years of weather data led, in part, to the development of mathematical **chaos theory**. Therefore, mathematics develops as it is needed as a language to describe the real world, and the language of mathematics in turn leads to practical developments in the real world.

The Rules of the Game

Another way to think of mathematics is as a game. When players decide to join in a game—say a game of cards, a board game, or a baseball game—they agree to play by the rules. It may not be "fair" or "true" in the real world that a player is "out" if someone touches the player with a ball before the player's foot touches the base, but within the game of baseball, that is the rule, and everyone agrees to abide by it.

One of the rules of the game of mathematics is that a particular problem must have the same answer every time. So, if Bill says that 3 divided by 2 is $1\frac{1}{2}$, and Maria says that 3 divided by 2 is 1.5, then mathematics asks if these two different-looking answers really represent the same number (as they do). The form of the answers may differ, but the value of the two answers must be identical if both answers are correct. Another rule of the game of mathematics is consistency. If a new rule is introduced, it must not contradict or lead to different results from any of the rules that went before.

These rules of the game explain why division by 0 must be undefined. For example, when checking division by multiplication it is clear that 10 di-

vided by 2 is 5 because 2×5 is 10. Suppose 10/0 is defined as 0. Then 0×0 must be 10, and that contradicts the rule that 0 times anything is 0. One may believe that 0 divided by 0 is 5 because 0×5 is 0, but then 0 divided by 0 is 4, because 0×4 is also 0. There is another rule in the game of mathematics that says if 0 divided by 0 is 5 and 0 divided by 0 is 4, then 5 must be equal to 4—and that is a contradiction that no mathematician or student will accept.

Mathematics depends on its own internal rules to test whether something is valid. This means that validity in mathematics does not depend on authority or opinion. A third-grade student and a college professor can disagree about an answer, and they can appeal to the rules of the game to decide who is correct. Whoever can prove the point, using the rules of the game, must be correct, regardless of age, experience, or authority.

Learning the Language

Mathematics is often called a language. Numbers and symbols are understood without the barrier of translation, and mathematics can be used to describe many aspects of today's world, from airline reservation systems to theories about the shape of space.

Yet learning the vocabulary of mathematics is often a challenge and can be confusing. For example, mathematicians speak of the "bottom" of a fraction as the "denominator," which is a pretty frightening word to a beginner. But, like any language, mathematics vocabulary can be learned, just as Spanish speakers learn to say *anaranjado*, and English speakers learn to say "orange" for the same color.

In *Islands of Truth* (1990), the mathematician Ivars Peterson says that "the understanding of mathematics requires hard, concentrated work. It combines the learning of a new language and the rigor of logical thinking, with little room for error." He goes on to say "I've also learned that mystery is an inescapable ingredient of mathematics. Mathematics is full of unanswered questions, which far outnumber known theorems and results. It's the nature of mathematics to pose more problems than it can solve." SEE ALSO Mathematics, New Trends in.

Lucia McKay

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Mathematics, Impossible

Geometric objects that cannot be constructed in three-dimensional space are considered "impossible." What makes them intriguing is that despite this difficulty, some representation of them is possible. Even though they

can never be built or held, it is possible to imagine what they would look like and to learn about how these impossible objects behave.

Local versus Global

In the study of mathematics, distinctions are routinely made between the local properties of an object—what small pieces of the object look like—and the properties of the object as a whole. Impossible objects are all "well-behaved" locally; it is only when we consider them globally that contradictions arise.

Consider two lithographs by Dutch artist M. C. Escher. (Although not shown in this entry, these artworks are easily viewed on various Internet web sites and on books of Escher's art.) The first is "Belvedere," created in 1958. At first glance, this seems to be a straightforward depiction of an open-air building with pillars and archways, with people scattered around at various points. But it does not take long to notice that this picture is filled with incompatibilities; many of the pillars do not start or end where they should, causing the building to twist around itself in unrealistic ways.

Similar trickery occurs in another Escher work, the 1961 lithograph entitled "Waterfall." This work depicts the peculiar phenomenon of water falling off a ledge and then proceeding to flow around a right-angled path until it reaches the top again. What makes this picture difficult to reconcile with reality is not that the water is fighting gravity but rather that the water is *not* fighting gravity. And once again, a close look at the picture uncovers all sorts of physical impossibilities.

In both examples, any small portion of the scene is a perfectly reasonable depiction of something that could exist in our three-dimensional world. It is only the way in which these small pieces are glued together that causes problems. This difference between local and global behavior leads us to the subject of **topology**, in which (among many other things) surfaces with similar behavior are studied.

topology the study of those properties of geometric figures that do not change under such nonlinear transformations as stretching or bending

Mathematical Surfaces

In mathematical terminology, a surface is an object that is "locally two-dimensional." An example is a hollow sphere. Although the entire sphere is three-dimensional and cannot be squashed into a plane without radically altering it, any small patch on the sphere looks like a slightly curved piece of infinitesimally thin paper and consequently is considered to be two dimensional. The formal way to say this is that any small patch of the sphere is "topologically equivalent" to a small patch of a plane. Two objects are said to be topologically equivalent if one can be stretched or compressed to form the other. (Imagine that the objects are made of infinitely stretchable and infinitely thin rubber.)

Moreover, the sphere is an orientable surface. When considering the surface of the sphere itself, it has no boundary. If you were traveling "in" the surface rather than on it, you would never fall off the edge but instead would return to your starting point.

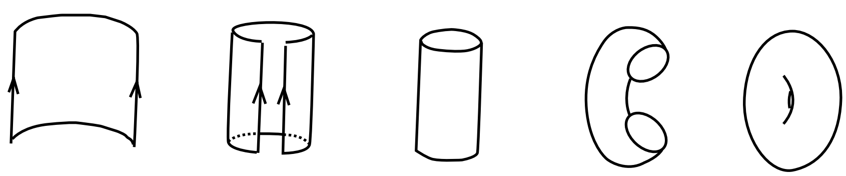
Building a Sphere. A number of surfaces can be built by gluing together regions in the plane. The sphere is an example of such a surface, since each

hemisphere is (topologically) planar. Another way to view the sphere as a planar object with some gluing is as follows.

First, draw a circle in the plane. Draw two dots on it, one at either end of the "equator"— that is, the horizontal diameter. The sphere will be formed by taking the disc (the interior of the circle) and gluing its boundary (the circle) together in the following way: each point on the top half of the circle (above the two dots) is glued to the point directly below on the lower half of the circle. It may take a little practice to be able to visualize this without actually doing the cutting and pasting, but the result is a shape that fully encloses a hollow center—in other words, a shape that, topologically speaking, is a sphere.

Building Nonorientable Surfaces. Using the technique of creating surfaces from two-dimensional regions, one can build objects more complicated than a sphere and whose final shapes are not as easy (or not even possible) to visualize. Two of these objects—a torus and a Klein bottle—can be formed by gluing together the edges of a square.

When the left and right edges of a square (a plane) are glued to each other, every point on an edge is glued to the point directly across from it. Once glued in this way, the square has become rolled into a cylinder, and looks like the cardboard tube in a roll of paper towels. The top and bottom edges of the square, which previously were line segments, have now become the circles at the top and bottom of the cylinder. These two circles are then glued to each other by bending the cylinder into a circular shape. The end product is a hollow doughnut-shaped figure, called a torus. Like the sphere, the torus has no boundary, and it is an orientable surface. When a "surface traveler" goes off an edge, he returns on the opposite side.



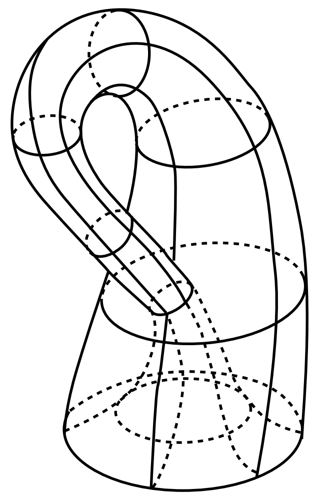
A nonorientable surface and an "impossible" object known as a Klein bottle can be created by changing how the edges of the square are glued. For this construction, an intermediate object known as a Möbius strip must first be created. Bring together the opposite sides of a square as before, but first twist them 180 degrees. The resulting object is almost a cylinder, but it is a cylinder with a twist. A point once on the upper or lower edge will now be glued to the point diagonally opposite it. For example, points on the left half of the top edge are attached to points on the right half of the bottom edge.

The name for this object is a Möbius strip, and it has many unusual properties. The Möbius strip has a boundary like a cylinder, but it has only one edge instead of two, and it has only one side. If you travel along the edge, you will cover the entire boundary, eventually returning to the place you started, but mirror-reversed. (Remember that traveling in a surface means traveling inside the surface, not on top of it.) Hence, the Möbius strip is considered nonorientable.

VIDEO GAMES AND TOPOLOGY

The surface of a torus is topological space found in most video games, where a vehicle (for example) goes off the right-hand side of the screen only to reappear on the left, or off the top only to reappear on the bottom.

The Klein bottle results from sewing together two Möbius strips along their single edge. Another way to make a Klein bottle is from a cylinder (as when constructing a torus), but by bringing together opposite edges of the boundary circles. But because of the twist, this is not as simple as before. The only way to picture this in three-dimensional space is to pass the surface through itself so that one circle can be glued to the other "from behind." However, this is not a realistic procedure because this self-intersection causes points to be attached to each other which were not supposed to be.



Thus the Klein bottle is a surface that cannot truly exist in three-dimensional space, even though a three-dimensional model is used to show its general shape. To get a general idea, stretch the neck of a bottle through its side and join its end to a hole in the base. But realize that a true Klein bottle requires four dimensions because the surface must pass through itself *without the hole*!

A Klein bottle therefore has one side and no boundary: it passes through itself; its inside is its outside; and it contains itself.

As with the Escher pictures, any small portion of the Klein bottle is perfectly easy to visualize in three-dimensional space; it is only the way in which these small pieces are attached to each other that yields a globally "impossible" object.

One more important "impossible" surface is called the projective plane. To construct it, start with a disc. Join points in the top half of the boundary (the circle) with points in the bottom half, but attach each point to the one diametrically opposite (that is, exactly halfway around the circle). This gives an "impossible" surface that cannot be constructed in three-dimensional space, even though it is locally two dimensional.

Compact Surfaces

It would seem as though there are an infinite number and variety of surfaces one could construct by gluing boundaries of planar regions. By drawing more elaborate shapes in the plane, one will come up with all sorts of combinations of edges to glue, with an occasional twist thrown in. But although there are an infinite number of surfaces that can be constructed, surprisingly enough their variety is limited and can be very precisely described.

If the entire boundary of the planar region is glued together in some way, the resulting object will be what is called a compact surface. A compact surface is a finite surface without a boundary—this rules out the plane itself and a surface such as a cylinder, which has a boundary. Every compact surface is topologically equivalent to one of the following: a sphere, a "sum" of finitely many tori (hollow doughnuts), or a "sum" of finitely many projective planes.

To "add" two surfaces means to cut a small hole in each one and glue the two surfaces together along the boundaries of these holes. For example, when three tori are added, the result is a three-holed torus. It is, of course, much harder to visualize the result of adding together a collection of projective planes, but interestingly enough, the sum of two projective planes is a Klein bottle. The remarkable fact that all conceivable surfaces can be described in this straightforward manner is called the Classification Theorem for Compact Surfaces. SEE ALSO Escher, M. C.; Möbius, August Ferdinand; Topology.

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Mathematics, New Trends in

What is new and different in the world of mathematics? Where is mathematics going in the future? What questions are mathematicians asking and exploring?

Tools Used in Calculation

Let's compare two of the instruments of calculation widely used during the last 50 years of the twentieth century. The slide rule is a portable calculator once carried by engineers so that they could quickly perform complicated calculations. Compare the slide rule, now a collector's item, to the graphing calculator used by high school students taking basic algebra. The graphing calculator can be used to do much more—more accurately and more easily.

The use of powerful calculators and computers will be an integral part of the mathematical problems and questions investigated in the future. Some classic problems, such as those involved with **prime numbers**, the **geometry** of soap bubbles, and the four-color theorem about how many colors are needed to distinguish neighboring colors on a map, are being extended to more complex questions, some involving three or more dimensions. Other problems of the future involve newer themes, such as **chaos theory** and how it can be applied to model various systems and computers and how they can be used to generate proofs.

prime numbers numbers whose only integral factors are themselves and one, numbers divisible without remainder only by themselves and one

geometry the branch of mathematics that deals with the properties and relationships of points, lines, angles, surfaces, planes, and solids

chaos theory the qualitative study of unstable aperiodic behavior in deterministic nonlinear dynamical systems

The Direction of Theories and Study in Mathematics

The advances in chaos theory made in the twentieth century will probably be extended into many areas of investigation as mathematical understanding of dynamical systems in biology, physiology, and clinical practice is increased. For example, according to Barry Cipra in *What's Happening in the Mathematical Sciences*, a yearly publication that reports on the latest mathematical research, mathematicians are working with scientists in a variety of fields to address many of our modern challenges, including how the human immune system works and how to deal with hazardous wastes. Chaos theory is also being used in questions about waves in all their forms, along with turbulence, complex fluid flows, and computational **fluid dynamics**. Ecology is another active field that is benefiting from the mathematics of chaos.

fluid dynamics the science of fluids in motion

Mathematicians continue to explore the nature of proof itself, and now this involves exploring whether a computer can develop a proof. The abilities of computers, the possible ability of computers to recreate themselves, and the **bioengineering** of computers continue to fascinate those at the cutting edge of developing technology and mathematics.

bioengineering the study of biological systems such as the human body using principles of engineering

Throughout the history of mathematics, games and **game theory** have fascinated mathematicians, and this trend continues today. A computer has defeated a human chess champion, but people continue to search for the "perfect" play in various games. In addition to studying games, mathematicians and many other people continue to work with codes and **cryptography** as they seek complete security for all types of messages—particularly those transmitted over computer networks and the Internet.

game theory a discipline that combines elements of mathematics, logic, social and behavioral sciences, and philosophy

cryptography the science of encrypting information for secure transmission

The Influence of Mathematics Beyond the Field

In the 1990s, Andrew Wiles solved what has been called "The World's Most Famous Math Problem" when he proved Fermat's Last Theorem. This accomplishment inspired people beyond the world of theoretical mathematics. Marilyn vos Savant wrote a book about Wiles's work in which she quotes a poem about Fermat's Last Theorem; there was a play presented in New York called *Fermat's Last Tango*; and there was a Fermat's Last Theorem Poetry Challenge.

QUIRKY MATH TOPICS

Mathematicians often enjoy exploring the mathematics of some rather peculiar topics. Some recent interests include finding the quickest way to untie a knot, figuring out how fish swim, and developing strategies to solve various puzzles.

Mathematics provided the subject matter for other achievements in the arts: The 2001 Pulitzer Prize for drama went to *Proof*, a mystery about a famous mathematician, and the book and movie titled *A Beautiful Mind*, which tells the story of John Nash, the mathematician who won a Nobel Prize in 1994 for his work on game theory.

So, solemn or frivolous, mathematics will continue to be used to model situations in every field of human endeavor where patterns and predictability pose challenges. And, at the same time, the search for useful models will continue to expand and benefit from the world of theoretical mathematics. SEE ALSO Chaos; Computers, Future of; Fermat's Last Theorem; Games; Gardner, Martin; Minimum Surface Area; Puzzles, Number; Slide Rule.

Lucia McKay

Internet Resources

American Mathematical Society. <http://www.ams.org/>.

Eric's Slide Rule Site. <http://www.sliderule.ca/>.

"John F. Nash." *Cepa.Newschool.Edu* <http://cepa.newschool.edu/het/profiles/nash.htm>.

Mathematics Teacher

Beyond a bachelor's degree in mathematics, most future math teachers must train for secondary education by taking extra classes and student teaching, which adds an additional year of college. In most states, teachers are required to pass a test in education and an examination in mathematics in order to receive a teaching certificate. After receiving a job, a teacher may be required to obtain a master's degree, be evaluated annually by the principal, and complete several hours of continuing education.

A math teacher's primary tasks are to present the topics of a math course in a manner that can be understood by all students, is useful in everyday life, and will be applicable on the job. For example, a math teacher may explain how to compare the mean, or average, grades to assess the progress of the entire class or the progress of a particular student. A teacher might use the mode to assess which topics need review by looking at the questions most frequently missed on a test.

In the 1988 movie *Stand and Deliver,* Edward James Olmos portrayed real-life math teacher Jaime Escalante, who motivated his students to study calculus.

One issue concerning math teachers is their high demand compared to their low pay. Schools districts need to balance limited financial resources and the educational needs of students. To address this issue, some states and districts offer signing bonuses, stipends for math teachers, and tax-free salaries. Although some math teachers leave the profession for higher-paying jobs, those who dedicate their careers to teaching mathematics know that they are playing an extremely important role in the education of future generations.

Michael Ota

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Mathematics, Very Old

From the dawn of civilization, humankind has needed to count and measure. Even the earliest civilizations developed effective and efficient number systems. Ancient mathematics is surprisingly sophisticated and, in many cases, quite similar to the mathematics used today.

Tally Bones

An early evidence of a "mathematical system" was found on a bone discovered in the Czech Republic. It dates from about 30,000 B.C.E. The bone contains fifty-five individual tally marks, divided into eleven groups of five marks each, just as tally marks might be grouped today. There is a dividing

HOW OLD IS THE OLDEST?

The oldest "mathematical artifact" currently known is a piece of baboon fibula with twenty-nine notches in it. It was discovered in the mountains between South Africa and Swaziland and dates from 35,000 B.C.E.

line, separating the first twenty-five marks from the remaining twenty, that makes totaling the tally marks even easier. No one knows just what the bone's owner was counting, but it may have been domestic animals such as sheep or some type of game animal.

Another ancient tally bone is the Ishango Bone, found in the Democratic Republic of the Congo. The Ishango Bone dates from 9000 B.C.E. to 6500 B.C.E. It is a bone tool handle, and it also contains tally marks that were probably used to keep a record of domestic items, perhaps sheep or cattle.

Early Number Systems

It is a short conceptual leap from a tally system to a number system. Early humans depended on body parts to keep an accurate count. Many civilizations developed a **base-10** system, which used ten fingers as a basis for counting. A base-10 system is still used today. Some ancient cultures included fingers and toes in a **base-20** system. The word *score* (twenty) is derived from the base-20 number system.

base-10 a number system in which each place represents a power of ten larger than the place to its right

base-20 a number system in which each place represents a power of twenty larger than the place to its right

The first concrete evidence of a numerical system comes from a ceremonial Egyptian weapon dating from King Menes (3000 B.C.E.). It contains hieroglyphics that described plunder taken by the king. One of the hieroglyphic figures is probably exaggerated, but it lists 1,422,000 oxen as part of the spoils of victory. Whether the mace head hieroglyphics give an accurate account or not is unimportant. What is significant about the hieroglyphic numbers is that more than 5,000 years ago the Egyptians could comprehend and represent extremely large numbers.

The Egyptian number system was additive. That is, each multiple of ten had its own symbol. The number of oxen, 1,422,000, was represented by a single symbol for 1,000,000, four symbols for 100,000, two symbols for 10,000, and two symbols for 1,000. In order to understand the hieroglyphic number, it was necessary to find the sum of all the symbols: $(1 \times 1{,}000{,}000) + (4 \times 100{,}000) + (2 \times 10{,}000) + (2 \times 1{,}000)$. Today's number system is positional. A number such as 345 is understood as $(3 \times 100) + (4 \times 10) + (5 \times 1)$. In the number 635, the number 3 now has the value of 3×10. There are only ten digits in today's system, and no number up to 9,999,999 requires more than seven digits. The Egyptian system required many more symbols to represent numbers. In the Egyptian system, a number like 45 would require four 10-symbols and five 1-symbols. Still, the Egyptian number system worked so well that there were essentially no changes for 3,000 years after its invention.

Many other number systems that followed the Egyptian number system were also additive, including the Roman numerals that are still used today. Several other early number systems, such as those developed by the Mayan, Incan, and Babylonian civilizations, rival, and in some cases surpass, the number system of today.

The Babylonians refined the number system that had been used by the Sumerians. The Sumerians left no written record to describe their number system, but it appears that the Babylonians adopted it without much alteration. By approximately 2500 B.C.E., the Babylonians were using a positional number system that is similar to the number system used today. The Baby-

lonians used a **base-60** system, with the value of a symbol dependent on its position in the number. For example, using modern symbols in the Babylonian system, the number 632 would represent $(6 \times 60^2) + (3 \times 60^1) + (2 \times 60^0)$, or $(6 \times 3600) + (3 \times 60) + (2 \times 1) = 21{,}782$ in our base-10 system. The Babylonian system continued to develop, and by about 500 B.C.E., clay tablets show numbers written with a symbol that is used as a zero, the first use of zero in a positional system.

base-60 a number system used by ancient Mesopotamian cultures for some calculations in which each place represents a power of sixty larger than the place to its right

The Babylonians also used base-60 for their fractions. By using base-60 for fractions and whole numbers, the Babylonians avoided much of the computation problems associated with fractions in today's number system. The following fractions ($\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{10}$) may be represented with the same denominator of 60 ($\frac{30}{60}, \frac{20}{60}, \frac{15}{60}, \frac{12}{60}, \frac{10}{60}, \frac{6}{60}$). Thus, the task of finding a common denominator was essentially eliminated in the Babylonian system. The Babylonians also extended the base-60 to units of weight, distance, and time. Today's sixty-minute hour and sixty-second minute are inherited from the Babylonians.

About 100 C.E. the Maya of Central America developed a number system that was positional like the Babylonian system. The Maya used base-20 instead of base-60. The Maya were also the first people to make full use of zero in their number system.

About 1,300 years later, the Incas of South America invented a record-keeping system that used a base-10 positional system and used zero. The Incas kept records on a series of knotted cords called *quipu.* The knots on the quipu cord indicated the digit, and the position of the cord indicated the value of the digit, much like today's number system. A missing cord was used to show zero.

Early Texts

The Babylonians left behind records about their mathematics. The Babylonians wrote on clay tablets in a simple script called cuneiform. One of the

Plimpton 322 is one of the earliest clay tablets dating to the Babylonian period. Written in base-60, it shows the most advanced mathematics before the development of Greek mathematics.

earliest clay tablets, called Plimpton 322, shows a table of whole number values that fit the Pythagorean theorem. Plimpton 322 dates from 1900 B.C.E. to 1600 B.C.E.

quadratic equations equations in which the variable is raised to the second power in at least one term

Other clay tablets contain what appear to be mathematics lessons. Among the problems found on the clay tablets are **quadratic equations** in the form (in modern notation) $x^2 + 6x = 16$, and problems that may be represented as cubic equations. The clay tablets show that the Babylonians could solve systems of equations such as $x + \frac{1}{4}w = 7$ and $1 + w = 10$.

Other tablets reveal that the Babylonians used 3.125 for the value of π, quite close to the present approximation of 3.1416. Some tablets contain tables of cube roots and square roots with great accuracy. The value of $\sqrt{2}$, for example, is correct to six places after the decimal.

Rhind Papyrus. While Babylonian mathematicians were inscribing cuneiform mathematics onto their clay tablets, Egyptian mathematicians were writing hieroglyphic mathematics on papyrus. One of the best sources of Egyptian mathematics is the Rhind Papyrus, named for archeologist Henry Rhind. It is 18 feet long and 13 inches wide and dates from 1650 B.C.E. It was written by copyist Ahmes, who claimed to be copying problems that had been known to Egyptians for at least 200 years. Thus, the eighty-five problems of the Rhind Papyrus date from about 2000 B.C.E. The Rhind Papyrus is the earliest arithmetic text ever written, and it is essentially a handbook of mathematics exercises.

The problems and solutions do not break any theoretical ground. The solutions are given because they work, and not because they are justified or logically shown to be correct. Some problems in the Rhind Papyrus, written in modern notation, are $100\,(7 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}) = x$, $x + 8x = 45$, $x + \frac{x}{7} = 9$, and $\frac{1}{3}(1 + \frac{1}{3} + \frac{1}{4}) = x$.

Although the problems are based on practical mathematics, the specific numbers are unlikely to be found in the everyday life of ancient Egypt, as this problem shows: Divide twenty-three loaves of bread among seventeen men. The Rhind Papyrus also contains formulas for the volumes of cylinders and prisms. Problems involving circles give the value of π as 3.1604, a value that is astonishingly close to the modern approximation of 3.1416.

Moscow Papyrus. Another source of Egyptian mathematics is the Moscow Papyrus, named for the Museum of Fine Arts in Moscow, where it is on display. It is much smaller than the Rhind Papyrus, at 18 feet long and 3 inches wide. The Moscow Papyrus dates from about 1850 B.C.E. and contains thirty exercise problems, written like those in the Rhind Papyrus. The problems were "already old" according to the copyist. One problem shows the formula for the area of a trapezoid, $A = \frac{1}{2}(b_1 + b_2)h$, which is identical to the formula used today. Another problem gives the formula for the area of a parallelogram in terms of its four side lengths: $A = \frac{1}{4}(a + c)(b + d)$. This formula, however, works only if the parallelogram is very close in shape to a rectangle. It is easy to show that the formula is incorrect for extreme parallelogram shapes, but the goal for this papyrus is the same as for the Rhind Papyrus: to provide exercises, not theoretical mathematics.

The Moscow Papyrus contains one problem that asks for the volume of the frustum of a pyramid. A frustum is the shape that is left if the top of a pyramid is cut by a plane parallel to the base. Math historians cannot ex-

plain how the Egyptians found this formula. The mathematics needed to derive the formula were not in existence until about 300 B.C.E., when Greek mathematicians rediscovered it. The only possible explanation is that, through a series of many problems, the Egyptians discovered an equation that gave the correct solution. Although the formula holds for all frustums, there was no attempt by the copyist to generalize it.

Other Papyrus Texts. Other mathematics papyrus texts include the Egyptian Mathematical Leather Roll, which contains twenty-six sums of unit fractions; the Kahun Papyrus, which contains six problems not fully deciphered; the Berlin Papyrus, which contains two problems that employ simultaneous equations; and the Reisner Papyrus, which contains volume calculations.

Practical Mathematics

The Egyptians developed practical **geometry**, necessitated by the Nile river floods every spring. The floodwaters erased all boundary markers, and when the waters receded, boundary lines had to be re-established. From at least 2500 B.C.E., the Egyptians used a special case of the **Pythagorean theorem** to set out the boundary lines. They used cords marked off by knots into twelve equal lengths. Then, the rope could be shaped into a right triangle with sides of the proportion 3-4-5. Two such triangles form a rectangle, and by replicating triangles and rectangles along the shores of the Nile, the Egyptians could reset the property lines.

geometry the branch of mathematics that deals with the properties and relationships of points, lines, angles, surfaces, planes, and solids

Pythagorean theorem a mathematical statement relating the sides of right triangles; the square of the hypotenuse is equal to the sums of the squares of the other two sides

In 2750 B.C.E. the Egyptians built a pyramid at Saqqara, Egypt. Inscriptions on the pyramid indicate that the builders used rectangular coordinates to erect the foundation of the pyramid. The Egyptian coordinate system is fundamentally the same system that is used today. The modern coordinate system was discovered in the mid-seventeenth century, some 4,000 years after the Egyptian system.

Another pyramid, the Great Pyramid, dates from about the same time period. The Great Pyramid demonstrates the practical mathematics of the Egyptians. The four base edges each measure within 4.5 inches of the average length of 751 feet. Each triangular face of the pyramid faces one of the cardinal compass points to within a tenth of a degree. Such precision is difficult to comprehend but nevertheless testifies to the practical development

The pyramids of the ancient Egyptians required high mathematical precision.

of mathematics that the Egyptians achieved. SEE ALSO Number System, Real.

Arthur V. Johnson II

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Matrices

A matrix, singular for matrices, is a rectangular array of numbers. Matrices naturally arise in describing a special class of functions called linear transformations. But the concept of matrices originated in the work of the two mathematicians Arthur Cayley and James Sylvester while solving a system of linear equations. In 1857, Cayley wrote *Memoir on the Theory of Matrices.*

A matrix can be seen as a collection of rows of numbers. Each number is called an element, or entry, of the matrix. An illustrative example of a matrix, C, is below.

$$\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 7 \\ 0 & 7 & \frac{3}{2} \end{bmatrix}$$

The order of the numbers within the row as well as the order of the rows are important. A matrix can be described by rows and columns. C has 3 rows and 3 columns, and hence it is a 3×3 matrix. A 2×3 matrix has 2 rows and 3 columns and a 4×2 matrix has 4 rows and 2 columns.

The size or dimension of a matrix is the number of rows and the number of columns, written in that order, and in the format $m \times n$, read "m by n." If $n = m$, which means that the number of rows equals the number of columns, then the matrix is called a square matrix.

Symbolically, the elements of the first row of a matrix are a_{11}, a_{12}, a_{13}, . . . The second row is a_{21}, a_{22}, a_{23}, . . . , and so on. The first digit in the subscript indicates the row number and the second digit indicates the column number. Therefore, the element a_{ij} is located in row i and column j.

Addition and Subtraction

Addition and subtraction are defined for matrices. To add or subtract two matrices, they must have the same dimension. Two matrices are added or

subtracted by adding or subtracting the corresponding element of each matrix. A matrix can also be multiplied by a real number. If C is a matrix and k a real number, then the matrix kC is formed by multiplying each element of C by k.

How can two matrices be multiplied? A useful definition of matrix multiplication involves a unique and unusual technique. To multiply two matrices A and B, the number of columns of the first matrix A has to equal the number of rows of the second matrix B. Let A be an $m \times n$ matrix, and B an $n \times p$ matrix. The product AB is a matrix C that has the dimension of $m \times p$. The first element c_{11} of the matrix C is obtained by multiplying the elements of the first row of A with the elements of the first column of B.

$$c_{11} = [a_{11}\ a_{12}\ .\ .\ a_{1n}] \begin{bmatrix} b_{11} \\ b_{21} \\ . \\ . \\ b_{n1} \end{bmatrix} = a_{11}b_{11} + a_{12}b_{21} + \cdot \cdot + a_{1n}b_{n1}$$

The second element in the first row, c_{12}, is obtained by multiplying the first row of A by the second column of B. Similarly, multiplying the row i of A with column j of B produces the element c_{ij} of matrix C.

Matrix multiplication is not commutative: that is, AB is not always equal to BA. The number 1 has a special property in arithmetic. Every number multiplied by 1 remains unchanged, hence 1 is called the multiplicative identity. Is there a special matrix which when multiplied by any matrix A leaves A unchanged? If A is a square matrix, then there exists an identity matrix I, such that AI = IA = A.

If A is a 3×3 square matrix, then I is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In an identity matrix, all elements are 0 except the diagonal elements that are all equal to 1. In arithemetic, a nonzero number has a multiplicative inverse, such that multiplying the number and its multiplicative inverse always produces 1, the identity. Does every square matrix B has an inverse B^{-1}? The answer is no. There is a special class of square matrices that have an inverse, such that $B\,B^{-1} = I = B^{-1}B$. These square matrices have a nonzero determinant.

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Mean *See Central Tendency, Measures of.*

Measurement, English System of

Early measurements of length were based upon parts of the human body, which for a long time were accurate enough for daily calculations. It is still common for us to use hand, finger, and arm movements to accompany statements regarding size or dimension.

Lengthy History

The first known standard length was used around 3000 B.C.E. in Egypt. The Egyptians created the cubit, which was the distance between the elbow and the tip of the extended fingers. The Egyptians overcame the variation from one person to another by making a standard cubit equal to what is now 20.62 inches.

The Greeks adopted the Egyptian cubit as the basis of their system, but also had their own measure of length, the fathom, which was the distance from fingertip to fingertip with arms outstretched. The Greek historian Herodotus (484 B.C.E.–424 B.C.E.) stated that the fathom was equivalent to four cubits (or six feet). From Plutarch (46 C.E.–120 C.E.) it is known that the Parthenon of Pericles in Athens had a platform length of 100 feet. When measured with modern units, it is found that the Greek foot was 12.14 of today's inches. The foot was passed to the Romans who subdivided it into 12 equal parts that they called the *uniciae*.

European Measures

Medieval Europe was greatly influenced by the Roman system of units. Nevertheless, regional variations over the centuries resulted in a number of standard lengths. In the ninth century, Charlemagne tried to impose a uniform unit of length throughout Europe but failed due to people's reluctance to accept new units.

Charlemagne (742–814), emperor of the West, wanted to standardize measurements to improve administrative procedures in his vast empire, which stretched from the Atlantic ocean to what are now the borders of Poland (east) and of the Czech Republic.

Charlemagne's efforts, however, did not go unrewarded and over the next few centuries the great trade fairs of Europe always had a Keeper of the Fair whose unit of length was compulsory for all commercial transactions within the fair grounds. The most prestigious unit of length was the Ell of Champagne, which was 2 feet 6 inches long, and was used in most of Europe as the standard measure for cloth, the most valuable of all trade goods at that time.

Royal Standards

An early attempt to standardize units in England resulted in a return to the royal standard, with the ell being equal in length to King Henry I's (1068–1135) own right arm. Present-day knowledge of the use of the royal arm is from the writings of William of Malmesbury (1095–1143), who has proven to be a reliable source for historical data. This ell was a larger measure than that used in cloth measuring, and eventually it became the yard.

By tradition, the yard of Henry I was the distance from the royal nose to the outstretched fingertips of the royal's right hand. The length was also chosen to ensure that $5\frac{1}{2}$ yards equaled a rod, the fundamental unit for measuring land. Because land was the greatest measure of wealth in the Middle Ages, this correspondence was of particular importance. The rod was the

Saxon measure for land and the Norman conquerors wanted to ensure that the land holdings of Saxon allies and new Norman landlords were precisely understood.

The Saxon word *aecer*, from which the word "acre" is derived, meant a field, or sown land. An aecer was 4 rods wide and 40 rods long, with the latter measurement being known as a furlong. The aecer represented about 5 hours of plowing by a team of oxen, the maximum the team could be expected to do in a day.

Hints of Change. By the time of the Magna Carta in 1215, the standards in England varied so widely that the thirty-fifth clause of the charter specified measurements for the unit of length as well as the gallon and the pound.

The next major review of the English units of measure occurred under the Tudor dynasty. In 1491 Parliament ordered the construction of new standards for length, weight, and capacity. These new Exchequer standards were stored in the Treasury, and in 1495, copies of these standards were supplied to forty-three shire towns in England. There was no change from the earlier yard and a later **recalibration** of measures in the reign of Elizabeth I also left the yard unchanged from the traditional length.

recalibration process of resetting a measuring instrument so as to provide more accurate measurements

To Change or Not to Change

Between the reign of Queen Elizabeth I starting in the late sixteenth century and 1824 there were many proposals to refine or change the system of measurement. Many were based on the scientific knowledge generated by the Renaissance and the Enlightenment. However, they all failed to gain acceptance because they lacked the support of the English Parliament. The arguments for rejecting any proposed change was the one still used in the twenty-first century: namely, that it would a very expensive change for merchants and manufacturers. These arguments were also used against moving toward a metric system that was tainted by its association with Napoleon's imposition of this system on Europe.

In the United States, Thomas Jefferson (1742–1826) reported to Congress in 1791 on weights and measures to be used in the new republic. He assumed that traditional measures would continue as they were well understood by the public, but he did try to recommend a decimal system of his own devising. In 1821 John Quincy Adams (1767–1848) reported to Congress that a uniform set of weights and measures was urgently required, and suggested that the metric system be adopted. This recommendation was not accepted.

In England, the Imperial Weights and Measures Act of 1824 repealed all previous legislation with respect to weights and measures. However, this new Imperial System of Measurement did not create a complete break with the past. The standard yard was defined as the distance between the centers of two gold studs on a brass rod that was in the custody of the Clerk of the House of Commons. The temperature of the rod when a length was to be verified was to be 62° F. All measures of length were derived from this standard yard.

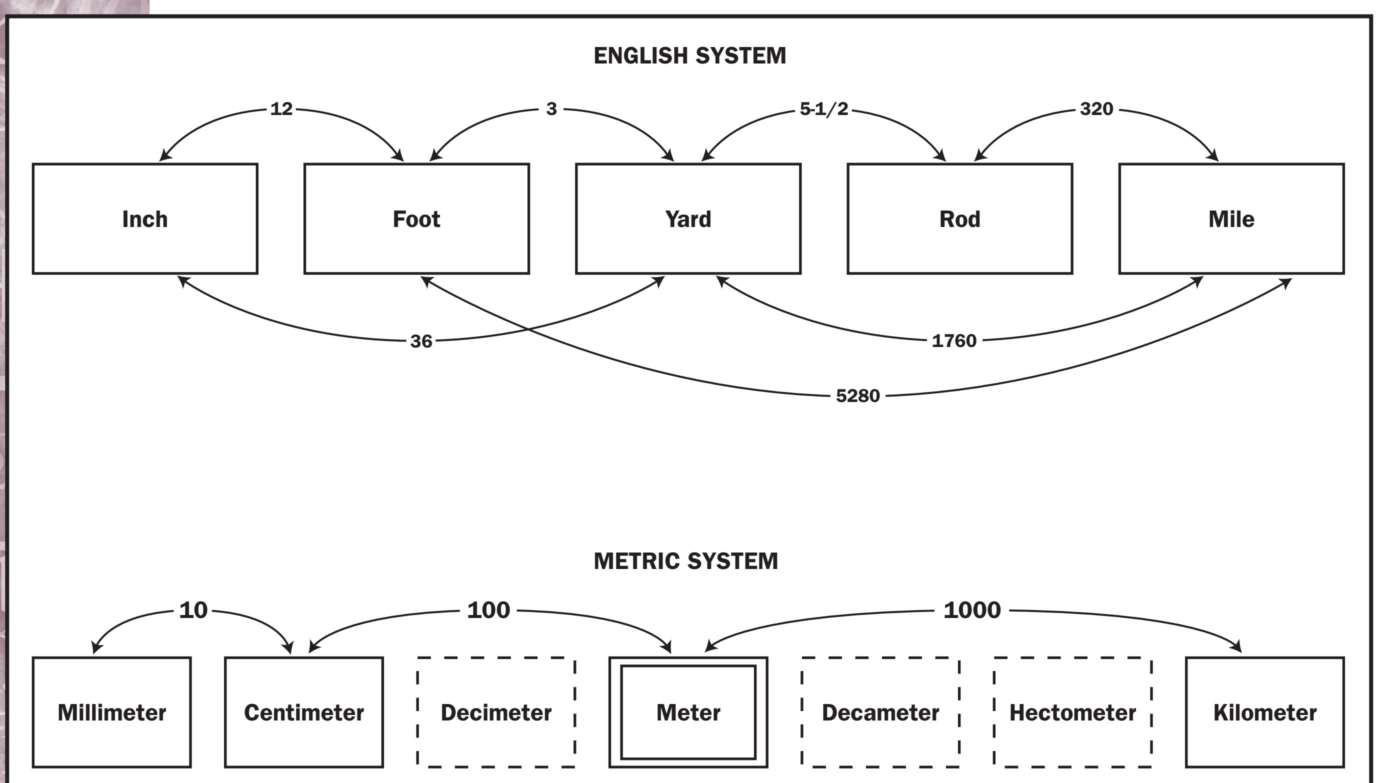
In 1834, the British Imperial Yard was destroyed by fire when the Houses of Parliament burned. A new Imperial Standard was designed and extensively tested to ensure that it matched perfectly the Elizabethan yard. The

imperial standard yard still exists and is a solid gunmetal bar 38 inches long. The material used is an alloy of copper, tin, and zinc in the ratio 16:2.5:1. This alloy was selected because it gave a stiff bar and also had a very small change in length when the temperature varied. Near each end of the bar is a hole $\frac{3}{8}$ inches in diameter with the two centers 36 inches apart. These holes were sunk halfway through the bar and have gold studs in them. On each of these gold studs are two lines parallel to the length of the bar and three other lines, 0.01 inch apart, at right angles to these parallels. The gap between the central line on each end is defined as the imperial standard yard. The entire bar rests on eight bronze rollers to avoid flexing. Copies of this new standard were presented to the United States, where the Office of Weights and Measures subsequently adopted it as a new standard, replacing all previous standards.

The Move to Metrics

In 1864 the use of metric units became permissible in England and in 1866 the metric system was made legal throughout the United States. Nine years later the United States went a step further and signed the Treaty of the Meter that established an international Bureau of Weights and Measures. In 1890 the signers of the Treaty each received exact copies of the International Prototype Meter and Kilogram. Three years later the Secretary of the Treasury issued the Mendenhall Order, which stated that the International Prototype Meter and Kilogram would be regarded as the fundamental standards from which all other measures in the United States would be derived. The yard was fixed at 0.91440183 meters and the pound at 0.4535924277 kilograms.

This figure illustrates the relationship between English and metric units.



During the next 50 years, comparisons were made between the English and American standards. Though they were supposed to be identical, the English standard was found to be minutely smaller than that of the United States. The slight difference did not create problems in commercial transactions between the countries. During World War II, however, the need for precise aircraft parts showed that the discrepancy made a difference in other areas.

In 1959, Australia, Canada, New Zealand, South Africa, the United Kingdom, and the United States adopted a common standard for the inch: 2.54 centimeters. However, the U.S. Coast and Geodetic Survey retained their established relationship of an inch equaling 2.540005 centimeters, which avoided extensive revisions to their charts and measurement records. The resulting foot is known as the U.S. Survey Foot. Since 1959, Australia, Canada, New Zealand, South Africa, and the United Kingdom have dropped the English yard as a legal unit of length and replaced it with the meter. SEE ALSO MEASUREMENT, METRIC SYSTEM; MILE, NAUTICAL AND STATUTE.

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Measurement, Metric System of

The metric system of measurement, more correctly called the International System of Units, is a system of weights and measures agreed through a network of international agreements. Using the first two initials of its French name *Système International d'Unités*, the International System is called SI. The foundation of the system was laid out in the Treaty of the Meter *(Convention du Mètre)*, signed in Paris on May 20, 1875. The United States was a founding member of this metric club, having signed the original document in 1875. Forty-eight nations have signed this treaty, including all of the major industrialized countries.

The *Bureau International des Poids et Mesures* (BIPM) is a small agency in Paris that supervises the SI units. The units are updated every few years by the *Conférence Générale des Poids et Mesures* (CGPM), which is attended by representatives of all the industrial countries and international scientific and engineering organizations.

Gabriel Mouton of Lyon originally proposed the metric system as early as 1670. His proposals arose out of attempts to measure nautical distance and to find a suitable unit of length that could be related to the degrees of the **arc** on lines of **longitude** and other **meridians**. Mouton's system was incomplete, but French scientists continued work on his ideas throughout the next century.

In 1790, Talleyrand introduced the subject of standard measurements to the French National Assembly. The result of the ensuing debate was a

arc a continuous portion of a circle; the portion of a circle between two line segments originating at the center of the circle

longitude one of the imaginary great circles beginning at the poles and extending around Earth; the geographic position east or west of the Prime Meridian

meridian a great circle passing through Earth's poles and a particular location

directive to the French Academy of Sciences to make recommendations to the government. Talleyrand invited both England and the United States to collaborate on the new system of measures, but both declined. England declined because it thought that importing a revolutionary new system of measurement would bring with it the social aspects of the French Revolution. The United States, upon hearing of the English response, also declined to join the French initiative. In addition to substantial opposition in Congress, the United States did not want to damage the fragile relationship between the two countries so soon after American independence.

Birth of the Meter

After much deliberation, the academy recommended that the length of the line of longitude passing through Paris be determined from the North Pole to the equator. This distance was to be divided by one million and formed the length now known as the meter. This single unit of measure was used to determine all other units, both subdivisions and multiples. In an attempt to unify length and weight, the standard weight was defined as the amount of water in a cube whose side was one-hundredth of the new unit of length.

The experimental determination of the new unit of length was deemed too impractical. As a result, the meridian that ran through Dunkirk, France, and Barcelona, Spain was used as the basis of determining the standard meter. Initially, there was debate over whether to use a decimal or **duodecimal** system of subdivisions, with the decimal system being eventually adopted. There was also a recommendation for the decimalization of time with 10-hour days, 100-minute hours, and 100-second minutes, but this last proposal never left the committee rooms.

duodecimal a numbering system based on 12

The final law of 1795 organized the basic standards and their prefixes. For multiples of the standard unit, Greek prefixes were used: *kilo* for thousand, *hecto* for hundred, and *deca* for ten. For subdivisions of the standard, Latin prefixes were assigned: *milli* for one-thousandth, *centi* for one-hundredth, and *deci* for one-tenth.

In 1798, three platinum standards, together with several iron copies, were manufactured. Just one year later the French Republic collapsed, and the Consulate and Empire that ruled after the Republic had little interest in popularizing the new system of measurement. When the monarchy was restored in 1814, the new system had made little impact on everyday life, as there had been a total failure to distribute copies of the standards throughout France. The new units of measurement were met with great skepticism from the public and the timetable for change was continually extended. There was also a second system, closely aligned to the traditional units of France, enacted to give the populace a feeling of continuity. It was not until 1840 that the metric system became the official single method of measurement in France.

SI PREFIXES

Multiple or Submultiple	Prefix	Symbol
10^{18}	exa	E
10^{15}	peta	P
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{2}	hecto	h
10	deca	da
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	mu
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

Although the metric system took a long time to gain popular support, it did achieve rapid acceptance within the scientific community. No sooner had the meter been adopted as the official unit of measure than a problem arose. In 1844 the German scientist Bessel conducted experiments on the shape of Earth and determined that it was not spherical, which was what the construction of the meter was based on. Bessel's finding was confirmed

by George Everest of the British Army, von Schubert of the Russian army, and Clarke of the British Ordinance Survey. These findings did not affect the metal standards constructed in 1798, but they did remove the original connections between Earth and the unit of length.

Call for a New Standard

In 1867, the International Geodetic Association met in Berlin and recommended the construction of a new standard to reflect the scientific improvements since the construction of the original meter. To ensure that there was continuity, the convention recommended that the new physical standard should be as close as possible to the original standard meter held in the French Archives. This first recommendation was followed by other scientific organizations petitioning the French Academy to construct a new standard for the meter.

In 1870, the delegates of twenty-four nations met in Paris to begin work on a set of identical physical standards that would be distributed to each country. The conference was cut short by the Franco-Prussian War, but reconvened in 1872, with thirty nations taking part.

The results of these meetings included the construction of standard meter bars of 90 percent platinum and 10 percent iridium created from a single ingot produced at a single casting. The temperature at comparison to the standard meter was to be 0° C. The original attempt to connect the unit of weight to the volume of a cubic centimeter of water was abandoned. Instead, the kilogram standard that had been in the Archives was made the basis of the kilogram, which was copied for the new standard. This new kilogram was to be made of the same alloy as the standard meter and was to be compared to the original kilogram in a vacuum. This is the kilogram still in use in the twenty-first century, with the standard held in Paris being considered the primary standard. The participating nations signed the Treaty of the Meter in Paris on May 20, 1875, and each nation was presented with a standard meter and kilogram. The treaty also established and maintained a permanent international bureau to help with the maintenance and propagation of the metric system.

Recent History

In 1960, the International System of units was adopted by most countries as the basis for all measurement. The unit of length, the meter, was redefined as 1,650,763.73 wavelengths in a vacuum of the orange-red line of the spectrum of krypton-86. This somewhat awkward definition had two advantages: First, it did not necessitate any change in the length of legally certified meter standards already in existence, and second, it removed the need to keep a physical standard in a vault under a particular climatic condition.

As science progressed through the twentieth century, however, this light wavelength definition became inadequate. At the 17th CGPM in 1983, the meter was again redefined to its current definition:

> *The meter is the length of the path traveled by light in vacuum during a time interval of 1/299,792,458 of a second.*

This 1983 definition depends on the definition of a second, which was also refined in the twentieth century. In 1955, at the National Physical Labora-

tory in Teddington, England, a clock was made that used a stream of atoms from the element cesium. The clocks could be so accurately tuned that they were accurate to one part in a million million for commercially produced instruments and one part in fifty million million for instruments in laboratories dedicated to making physical measurements for standards.

cesium a chemical element of the alkali metal group that is the most electropositive element known

atomic weight a quantity indicating atomic mass that tells how much matter exists or how dense it is, rather than its weight

From 1955 to 1958 a joint experiment between the U.S. Naval Observatory and the National Physical Laboratory established that a particular transition of a **cesium** atom whose **atomic weight** was designated as 133 was associated with a photon whose frequency was 9,192,631,770 cycles per second. In 1960, the General Conference on Weights and Measures redefined the second in terms of the transition of the cesium-133 atom.

To prevent any problems that might become associated with the new second, it was set as equal to 9,192,631,770 periods of radiation emitted or absorbed in the transition of a cesium atom. The measurement was taken at sea-level under clearly defined conditions. The second thus defined corresponds with the second that had been in use before that date and had been calculated by making observations of the Moon's movement, which had been known as the Ephemeris second. SEE ALSO MEASUREMENT, ENGLISH SYSTEM OF.

Phillip Nissen

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Measurements, Irregular

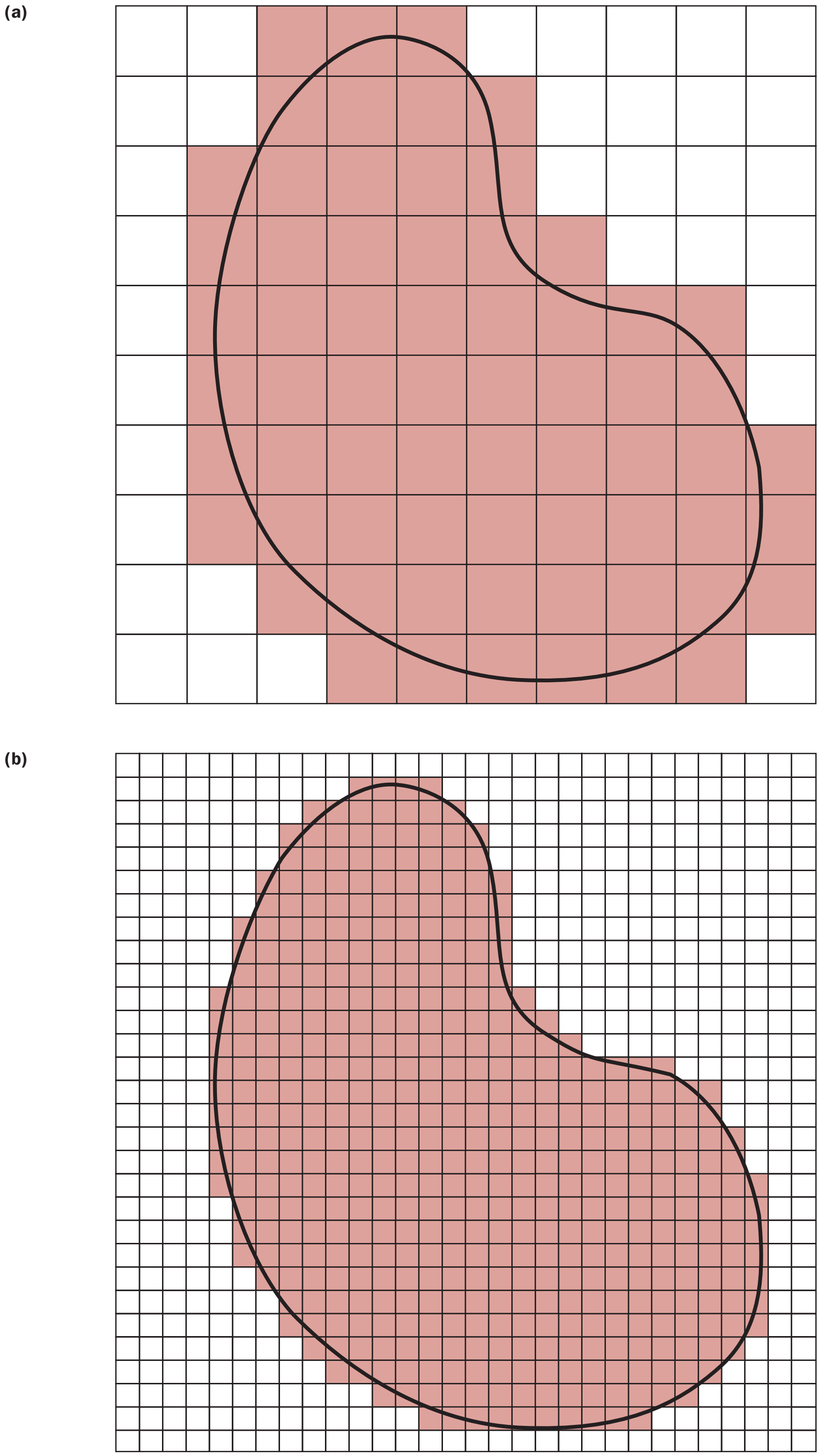
Many shapes in our world are irregular and complicated. The boundary of a lake or stream can be curved and winding. A stock market graph can be highly erratic, with jumps and dips of various sizes. The path of a fly ball is a smooth arc affected by gravity and wind resistance. Clouds are filled with cotton-like bumps and ripples. Measuring our world requires us to confront a variety of irregular shapes, ranging from graceful curves to sharp, jagged edges.

Measuring Irregular Shapes

calculus a method of dealing mathematically with variables that may be changing continuously with respect to each other

Irregular shapes can be measured with a technique called discrete approximation, a powerful method that provides the foundation for **calculus** and a means by which computers perform calculations. Making a discrete approximation involves representing a continuous quantity through a collection of distinct pieces. We live with such approximations every day. For example, a movie reel is a collection of picture frames, shown to us rapidly on a screen to give the impression of a continuous flow of events. Computer screens and laser printers represent images by a collection of small, tightly packed cells (called pixels), joined together to give the impression of a continuous image. If one looks closely at a computer or television screen, the small cells can become noticeable.

Measurement with discrete approximation involves dividing an irregularly shaped object into a collection of smaller pieces whose measurement is more manageable. For example, consider an overhead view of a lake, whose boundary rests upon a square grid (see figure below). Every square that con-

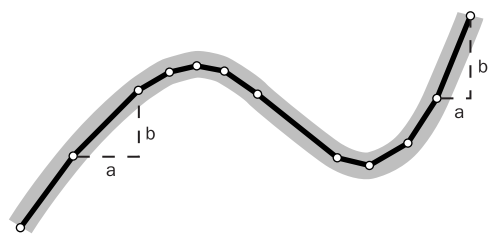


This figure represents a lake using a collection of shaded squares:
(a) a larger grid and
(b) a smaller grid.

tains a portion of the lake is completely shaded. The resulting collection of shaded squares is a discrete approximation of the lake, much like a collection of pixels providing a digital image of the lake. The lake's area can be computed by simply adding up the areas of the shaded squares. The larger grid in figure (a) results in a sizable overestimate of the lake area, but the smaller grid in figure (b) shows a notable decrease in the approximation error. Further refinement of the grid leads to increasingly better approximation of the lake area, much like how a digital image becomes sharper with an increasing number of pixels.

In a similar manner, the figure below shows a discrete approximation of a curve using a collection of line segments. The curve might represent a portion of a roller coaster track, a path through snow-covered hills, or the trajectory of a helicopter. The length of each line segment is determined by applying the Pythagorean theorem to a standard right triangle construction. Using this relation, the length of each line segment is equal to $\sqrt{a^2 + b^2}$. The curve's length is approximated by the sum of the lengths of the line segments. Using an increasing number of smaller line segments gives an increasingly better representation of the curve, just as a regular **polygon** appears more like a circle as the number of polygon sides increases.

polygon geometric figure bounded by line segments



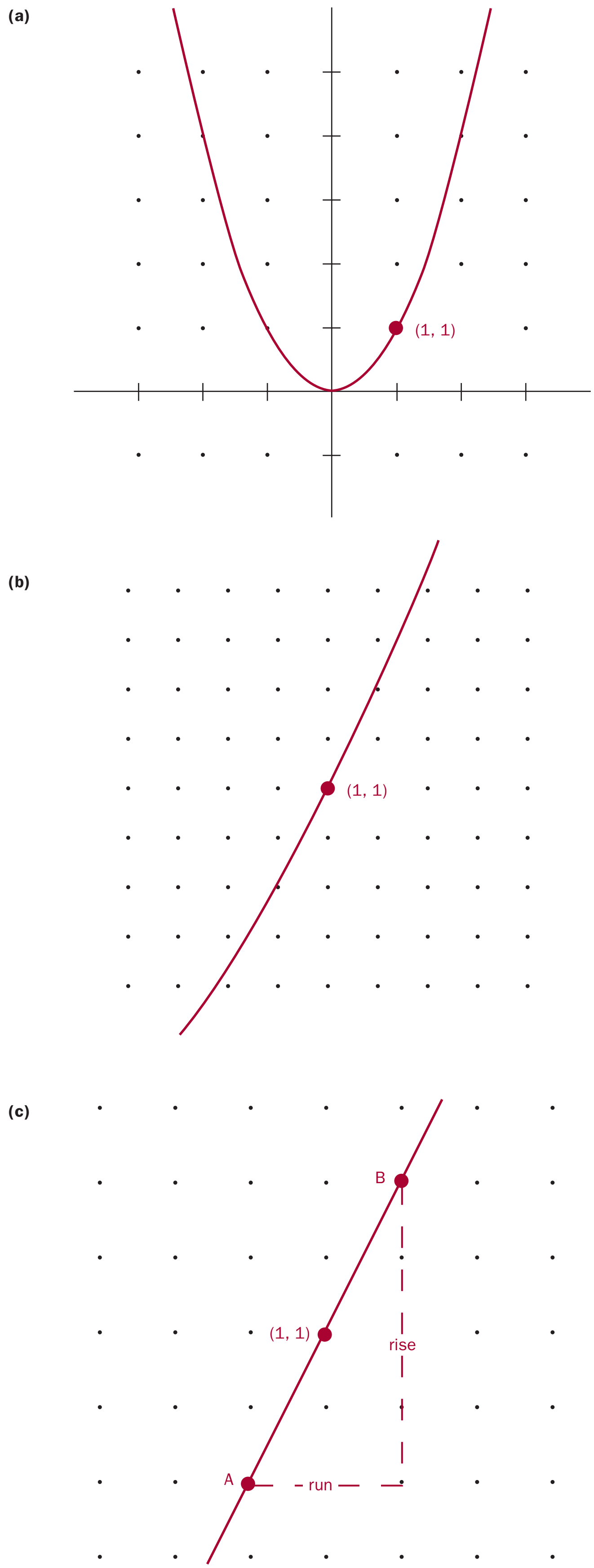
Approximating irregular shapes by collections of squares or line segments may seem unsophisticated and cumbersome. Such a measurement approach, however, is intimately connected with the tendency of curves to look simpler, indeed straighter, as we magnify our view of the curve. This fact has a profound implication for measurement, allowing for the definition of a **slope** at any point along a curve. Consider, for example, the graph of the **parabola** $y = x^2$ shown in figure (a) on the next page. Let us focus on the ordered pair (1, 1), and keep enlarging the region around this point using the zooming capabilities found in graphing calculators and other graphing software [see (b) and (c)]. The closer we zoom into a point on the graph, the straighter the graph looks, much like the curved surface of Earth can appear flat from close range. By the eighth magnification, the graph looks effectively linear and a two-point slope calculation makes sense. Using the two points in the eighth-zoom window, the slope of the graph is calculated to be

slope the angle of a line relative to the x-axis

parabola a conic section; the locus of all points such that the distance from a fixed point called the focus is equal to the perpendicular distance from a line

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{0.020}{0.010} = 2.000.$$

The eighth-zoom calculation gives a slope of 2 to within three decimal places, and with enough magnification, a slope calculation gives this result to within any number of decimal places. With the zooming visualization as an aid, the notion of approximating irregular shapes by small line segments or squares should seem more reasonable. Upon magnification, complicated shapes tend to look increasingly like the simple shapes upon which measurement is based.



Zooming in on the point (1, 1) on the graph $y = x^2$:

(a) original window $-4 \leq x \leq 4$ and $-2 \leq y \leq 6$;

(b) 3-zoom window $0.5 \leq x \leq 1.5$ and $0.5 \leq y \leq 1.5$;

(c) 8-zoom window $0.98436 \leq x \leq 1.0156$ and $0.98436 \leq y \leq 1.0156$.

The two ordered pairs used for the slope calculation are A = (0.995, 0.990) and B = (1.005, 1.010).

Nature shows many examples of small building blocks in the fabric of its construction, like cells for organisms, atoms and molecules for substances, and photons for light. The mathematical technique of breaking down complicated objects into simpler pieces has both historical context and modern relevance. Using the idea of a limit, calculus offers a method for shrinking simple measurement pieces and making approximation error disappear. The result is a robust mathematical language applied by mathematicians and scientists since the late-seventeenth century in a wide variety of contexts, from direct measurement of physical quantities to the creation of equations that describe physical processes.

The modern digital revolution is intimately connected with the discrete representation of sound, images, and information. Computerized processes are founded upon discrete packages of information (called bits), and rapidly increasing technological power has enabled realistic, high-quality discrete representations of our world through computer graphics, compact discs, digital cameras, laser printers, and so on. Simpler building blocks for complicated objects are not only part of nature's fabric, they provide an increasingly pervasive means by which we measure, describe, and model the complexity of our world. SEE ALSO Calculus; Limit.

Darin Beigie

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Median ***See Central Tendency, Measures of.***

Mile, Nautical and Statute

The statute mile has it origins in Roman times where a measure of a thousand paces, *mille passum*, was used for large distances. For the Romans, a pace was the distance between the same foot touching the ground— that is the distance covered in two steps. The pace was taken as being equal to five Roman feet, a length that historians have calculated to be 11.65 inches.

The study of Anglo-Saxon texts reveals that the early English used a measure for long distances called a mil (plural: mila), which was also equal to 5,000 feet. However, the foot used by the Saxons was measured by using an adult's actual foot, and so was about 80 percent of the length of the Roman foot.

Although the confusion between Roman and Saxon feet was resolved by the Statute for Measuring Land of 1305, making the foot close to our modern measure, this did not get translated into resolving the conflict between the two miles for measuring long distances. It was nearly 200 years later, in 1593, that Elizabeth I signed into law a statute titled *An Acte againste newe Buyldinges.* The act prohibited any new construction within three miles of the gates of the City of London. In this statute the mile was declared to be 8 furlongs, a furlong being 40 rods, which was itself $16\frac{1}{2}$ feet long. Thus the statute mile was a total of 5,280 feet, the length it is today.

The Nautical Mile

A nautical mile is a distance of 1,852 meters, as recommended by the International Hydrographic Conference of 1929. Before the sixteenth century, navigation at sea used landmarks, seamarks, and seabed samples to estimate the depth of an area of water. It was rare for ships to sail in waters deeper than 100 fathoms (600 feet), and if they did, it was for short distances. Navigational information was exchanged between ships' pilots, each of whom kept his own set of notes called a Rutter.

There were a variety of measures of distance in different areas of the world. In the Mediterranean, Roman miles, or leagues equal to $1\frac{1}{2}$ Roman miles, were used. In waters off the western European coast, distances were in kennings, about 20 miles, which was the farthest one could be from a coastline before losing sight of it.

A full understanding of the spherical nature of Earth became widely accepted at the end of the fifteenth century and led to an improvement in map making. Pilots began showing **meridians** of **longitudes** and parallels of **latitudes** on the maps that accompanied their Rutters. As sailors began to venture farther from land, they began to use a new measure, which they called a league, but which was twice its counterpart on land, being three Roman miles in length.

In 1484, King John II of Portugal formed a commission to tackle the problem of finding latitude in the Southern Hemisphere where the North Star was not available for navigation. The solution was to use the solar tables of the astronomer Zacuto of Salamanca, which allowed navigators to use the Sun as a means of determining latitude. It should also be noted that the Portuguese had their own version of the league, which was four Roman miles in length.

Compounding the confusion over which league was more accurate, there was disagreement over the size of Earth, and consequently the distance between each degree of latitude. The Portuguese used a measurement of 25,200 Roman miles for the circumference of Earth given by the ancient Greek astronomer Eratosthenes. For political reasons, the Spanish adopted the circumference measurement, given by Ptolemy, of 18,000 Roman miles. This allowed the Spanish to claim the Moluccas in the East Indies in the early part of the sixteenth century. In 1617 the Dutch scientist Snell completed a new assessment of the circumference of Earth and found it to be 24,630 Roman miles (24,024 statute miles).

The will of Sir Thomas Gresham established a college in London in 1596, with one of the principal aims being the teaching of astronomy in relation to navigation. Gresham College soon attracted the best mathematical minds of the day, either as teachers or researchers. Edmund Gunter (1581–1626) was a mathematician and also a church minister whose church was across the Thames river from Gresham College. Gunter reasoned that the navigator was concerned with two principle problems: the ship's position, and the distance it had sailed, or had to sail. Gunter believed that the most acceptable unit of distance for a navigator would be one in which the angle measurements of latitude could be related to the distance traveled. The result of this reasoning was to take one minute of a meridian as being equal to the unit of distance. In accepting Snell's measurement of the cir-

A DIFFERENT KIND OF KNOT

The knot is a measurement of speed equal to a rate of one nautical mile per hour. The term knot comes from the original method of measuring speed at sea, which was to throw a shaped log overboard attached to a rope with equally spaced knots. The log was weighted so that the shaped face gave the maximum resistance to towing, and was practically stationary in the sea. By counting the number of knots that passed overboard during a period fixed by a sand timer, navigators were able to calculate the speed of the ship.

meridian a great circle passing through Earth's poles and a particular location; the Prime Meridian passes through Greenwich, England

longitudes the geographic position east or west of the Prime Meridian

latitude the number of degrees on Earth's surface north or south of the equator; the equator is latitude zero

cumference of Earth, Gunter defined his "nautical mile" to be equal to 6,080 feet, the length of one minute of **arc** at 48°. Because Earth is not a perfect sphere, the minute of circumference measurement actually varies between 6,046 feet at the equator to 6,108 feet at the pole. Other countries used a measurement for the nautical mile at 45° of latitude, which put the nautical mile at a length of 6,076 feet. Finally, in 1929 the nautical mile was agreed internationally as being equal to 1,852 meters. SEE ALSO Distance, Measuring; Measurement, English System of; Navigation.

Phillip Nissen

arc a continuous portion of a circle; the portion of a circle between two line segments originating at the center of the circle

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Millennium Bug

The millennium bug refers to the existence in many computer software packages of a six-digit date rather than an eight-digit date. Computers speak the language of mathematics. Every question posed to a computer program is answered either "yes" or "no" and represented in **binary** code as either a one or a zero. Since computers recognize math symbols, a computer can add 97 + 1 and get 98. When the computer is presented the problem 99 + 1 and the answer does not fit in the required two-digit field, the computer's response is an error, or worse, a shut-down.

binary existing in only two states, such as "off" or "on," "one" or "zero"

What caused this error in planning to occur? In the 1960s and 1970s, when computers were coming into widespread use, one of the largest concerns to programmers was the amount of memory available to the user. With this concern in mind, programmers searched for ways to cut memory requirements. Reducing dates to six digits (01/31/99, for example) rather than using the eight-digit international date (01/31/1999) was one way to decrease memory requirements. A second reason is that computers were so new and programmers were so inexperienced that they believed that any software produced in the 1960s and 1970s would be archaic by the 1980s and certainly long before the year 2000. They produced software designed to answer immediate needs, failing to predict that software programs would be used far longer than intended.

Many businesses and industries were impacted by the millennium bug. All entities that work from a mainframe were vulnerable, including many government agencies, such as the Internal Revenue Service and the Social Security Administration among others. Any business or entity using personal computers (PCs) was vulnerable. Computer experts estimated that 93 percent of the PCs made before 1972 would fail. When PCs are part of a network, the risk of problems developing is much greater.

Many doomsday predictions were made prior to January 1, 2000. Books, videos, and web sites, all designed either to sell consumers the equipment needed to get them through the "apocalyptic" event or to warn computer users what to expect, began to appear. Almost weekly, newspaper articles were published reporting who had done—or had not done—what to pre-

pare and what types of tests had succeeded or failed. During the final week of 1999, local governments suggested that citizens stockpile drinking water. It was predicted that businesses dependent upon computer systems would fail. Some people feared runs on banks by panic-stricken people that would put the national economy at risk of collapsing. Public and private transportation would shut down due to fuel shortages resulting from delivery problems and electrical failures. Hospitals, governing agencies, the food industry, news, communications, and education were predicted to suffer due to the millennium bug.

Many people worried that they would not have access to their bank accounts if the Millenium Bug disrupted the normal activities of ATMs, which contain computer chips.

The millennium bug dilemma appeared deceptively simple: After all, how difficult could it be to change programming to read a four-digit year rather than a two-digit year? The situation became more complex due to the magnitude of programs that needed to be changed. Every mainframe computer, every program created before 1995, the vast majority of PCs made before the mid-1990s, every microchip embedded in every car, every pacemaker—all of these only begin a list of the things that could potentially be affected by the millennium bug. The challenge of fixing all of the potential problems before January 1, 2000, seemed impossible. Information technologists would have to check every system for compliance. The problem, which existed not just in the United States but across the entire globe, seemed astronomical.

At the end of 1999, the computer industry was expected to spend between $300 to $600 billion to deal with the problem. The director of the Millennium Watch Institute said: "Responsible estimates of what we collectively paid . . . range from 250 gigabucks to 600 gigabucks—enough to give $100 to every human alive. Another estimate mentions one terabuck as a reasonable figure." (A gigabuck is equivalent to one billion dollars and a terabuck is equivalent to one trillion dollars.)

So what did occur on January 1, 2000? A German businessman found an additional $6.2 million more dollars in his bank account than he truly had. A U.S. defense satellite was out of commission for three hours. Amtrak lost track of some trains until the date was reset in the company's computer system. A newborn baby in a Denmark hospital was registered as 100 years old. The doors on a federal building in Nebraska flew open. A government official in Slovenia was forced to resign for overemphasizing the potential problems of the millennium bug, and a few small inconveniences occurred elsewhere in the world. But amazingly, none of the predicted disasters came to pass. Some contend that the Y2K (Year 2000) problems were not severe due to the extensive upgrades that businesses and individual consumers made to their hardware and software prior to the date change.

Some good did come from the preparation. Many companies and government offices are now much more efficient and have developed better communication skills. Software quality has improved. Old hardware has been updated or replaced. Audits have been performed at many companies and agencies, leading to streamlining processes. Information technology experts have acquired much more precise knowledge of systems. According to experts, the modernization that came with the millennium bug preparation will bring economic gains. In the words of one columnist, the millennium bug preparation was "the greatest technological housecleaning of all time."

The extensive hardware and software upgrades by businesses and private citizens also meant that many technology firms saw greatly increased

profits during that time. Once systems were Y2K compatible and the intense demand for new products and services decreased, some technology companies began to experience much lower sales and profits than expected in 2000 and 2001. SEE ALSO Computers and the Binary System.

Susan Strain Mockert

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Minimum Surface Area

Nearly everybody has, at some point, been fascinated by soap bubbles. A fun experiment is to dip a loop of wire into a soap solution and pull it out. As if by magic, a thin, transparent film of soap will form across the wire. Trying to predict what shape the soap film will be often yields a surprise. What makes the soap bubble take the shape of a perfect sphere, and what determines the shape of a soap film?

Soap Bubbles and Soap Films

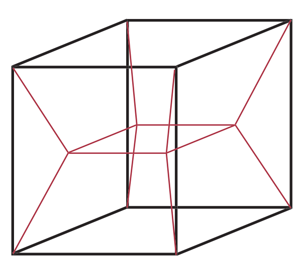
Like many other physical systems, soap bubbles and soap films try to minimize energy. The energy in a soap bubble comes from the force of surface tension that holds it together and keeps it from popping. This energy is proportional to the area of the soap film.

Soap bubbles and films try to minimize their surface area.* A soap bubble takes the shape of the surface with the smallest area that encloses a certain volume of air. That surface is a sphere. (For example, the smallest surface that encloses 1 cubic inch of air is a sphere with a radius of 0.62 inches.) If a wire is dipped into a soap solution, the soap film takes the shape with the smallest area that will not require it to let go of the wire.

***Centuries ago, architects discovered that experimenting with soap bubbles could help them define the most economical form for the actual structure.**

Shapes of Films. Although soap bubbles come in only one shape, soap films come in a staggering variety. A circular wire loop will produce a soap film that is a simple disk. A loop of wire folded into a **tetrahedral** frame will produce a soap film consisting of six triangular pieces that meet at a point in the middle. One might expect that a cubical frame would produce twelve flat sheets (one for each edge) that meet at a point. Twelve flat sheets are indeed the result, but surprisingly there is also a thirteenth sheet—a small square—in the middle.

tetrahedral a form with four triangular isosceles faces on each side of a cube



In the 1830s, the Belgian physicist Joseph Plateau conducted such experiments and arrived at two general rules that govern the way sheets of soap films join together.

- Three sheets may join along an edge, and they must always form 120° angles with each other.
- Four edges and six sheets may join at a point (as in the tetrahedral configuration), and the edges must always form 109° angles with each other.

These are the only two possibilities. This explains why the cubical frame did not produce twelve triangles that met in the middle: Plateau's laws forced the film into a less symmetric arrangement where only six sheets meet at a time.

A MINIMAL SUMMARY

An area-minimizing surface is one that has the smallest possible area of any surface with the same boundary. Minimal surfaces might not be strictly area-minimizing: that is, there may be another surface with the same boundary and less area. However, minimal surfaces are always "locally" area-minimizing in the sense that any small change to the surface will increase its area.

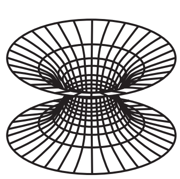
Minimal Surfaces

A minimal surface is a surface satisfying the zero mean curvature property. In a soap film, for example, for any curve in the surface that bends one way, there is another curve perpendicular to it that bends the other way. The reason is surface tension: Each of the curves pulls the surface in the direction that it bends, so in order for the surface to be stationary, the amount of bending in the two directions must cancel out. This is called the zero mean curvature property of soap films.

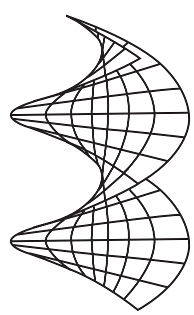
The zero mean curvature property prevents a soap film from ever closing up. Note that soap bubbles do not have the zero mean curvature property, because the surface tension in a soap bubble is counterbalanced by the air pressure of the air trapped inside.

Minimal surfaces have interesting properties, and some are so striking or attractive that they have been given names. The catenoid is the minimal surface bounded by two parallel circles; the helicoid resembles a strand of DNA; and Enneper's surface✶ is a beautiful, potato-chip shaped surface.

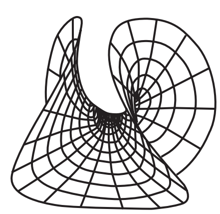
✶**A sculpture in the shape of Enneper's surface won the Peoples' Choice award at the 2000 snow sculpture contest in Breckenridge, Colorado.**



Catenoid



Helicoid

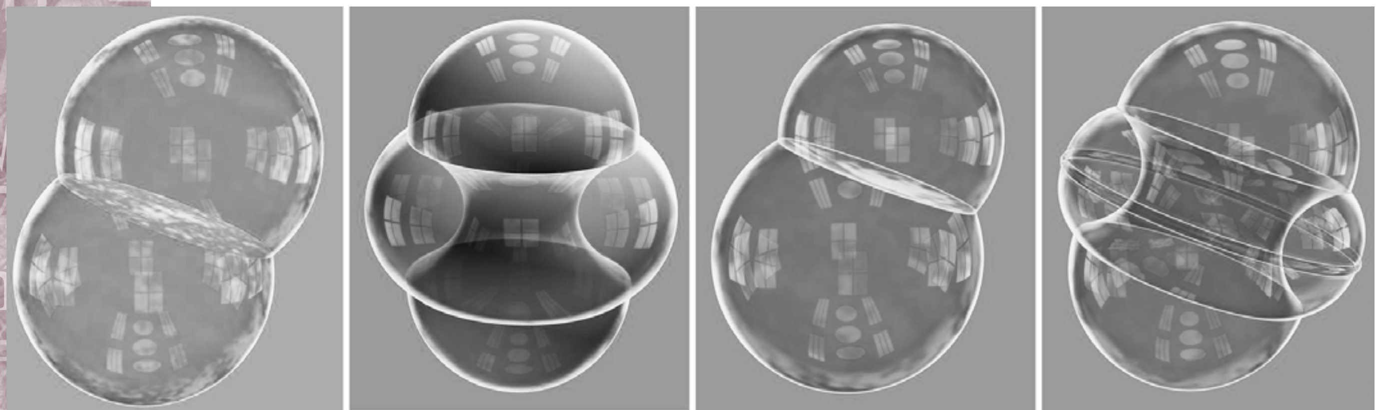


Enneper's Surface

With computer graphics, new kinds of minimal surfaces have been found that could not even be imagined before. For example, the Costa-Hoffman-Meeks surfaces, which have a stack of three or more circles as their boundary, were discovered in the 1980s.

Solving Area Minimization

Though Plateau's laws were easy to demonstrate experimentally, it took mathematicians over a century to prove that these were indeed the only pos-



Double bubbles illustrate principles of area minimization. From left to right are the following: a standard double bubble of equal volume; a nonstandard cluster with a torus "waist"; a standard double bubble of unequal volume; and a nonstandard bubble.

sibilities. The American mathematician Jean Taylor finally met the challenge in 1970. This has been a recurring pattern throughout the history of minimal surface theory: The area-minimization principle is easy to state, and the actual soap-film configurations are elegant and simple, but it is very difficult to prove that natural laws have solved the area-minimization problem correctly.

As another example, anyone who has blown soap bubbles has noticed that they sometimes join together in pairs, forming two spherical caps with a thin sheet between them. But the first proof that this configuration has the least area (the "double bubble conjecture") did not come out until 2000. The difficulty is that there are many other bizarre configurations, never seen in nature, that nevertheless obey Plateau's laws and therefore have to be ruled out on other grounds.

Applications of Minimization. Although many mathematicians study minimal surfaces for purely aesthetic reasons, the principle of energy minimization has many practical applications. For example, crystals minimize their surface energy as they grow, the same as soap bubbles. They do not grow into spheres, however, because their surface energy is direction-dependent. (It takes less energy to cut a crystal with the grain than against the grain.) Energy minimization also governs the shape of fluid in a capillary or the shape of a drop of water resting on a table. Although the shapes in all these problems are slightly different, the mathematical methods used to solve them are very similar.

Dana Mackenzie

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Mitchell, Maria

American Astronomer
1818–1889

Maria Mitchell, America's first prominent woman astronomer, was one of ten children born to William Mitchell and Lydia Coleman. Mitchell showed a talent for mathematics and an enthusiasm for the field of astronomy at an early age. In 1838, at the age of 20, she became a librarian at the Nantucket Atheneum, which gave her the opportunity to further study astronomy.

On the evening of October 1, 1847, in the observatory built by her father, Mitchell sighted a comet using her father's 2-inch telescope, which earned her international recognition and a gold medal from the king of Denmark.

Maria Mitchell championed the education and professional advancement of women.

Throughout her career, Mitchell was the first woman elected to many prestigious organizations, such as the American Academy of Arts and Sciences (1848) and the American Philosophical Society (1869). She served as president of the Association for the Advancement of Women (1873). In 1865, she accepted an appointment to Vassar College to become director of the observatory and became the first woman to join the faculty as a professor of astronomy—posts she held until her retirement in 1888. The observatory, built in 1864 for Mitchell, housed her original telescope until 1964, at which time it was donated to the Smithsonian Institution.

MARIA MITCHELL HOUSE: A NATIONAL HISTORIC LANDMARK

The Maria Mitchell House, a National Historic Landmark, is located on Nantucket Island, Massachusetts. The Nantucket Maria Mitchell Association, founded in 1902, maintains the property, which is open to the public.

In addition to this recognition, she received many honors including honorary degrees from Hanover College (1853), Columbia University (1887), and an honorary doctorate from Rutgers Female College. The Boston Public Library and a public school in Denver, Colorado are named in her honor, as well as a crater on the moon. She died in Lynn, Massachusetts, on June 28, 1889. SEE ALSO Astronomer; Astronomy, Measurement in.

Gay A. Ragan

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Mode *See Central Tendency, Measures of.*

Möbius, August Ferdinand

German Mathematician and Astronomer
1790–1868

An astronomy professor at the University of Leipzig, German mathematician and theoretical astronomer August Ferdinand Möbius made special contributions to theoretical astronomy and analytical geometry. Möbius studied law at the University of Leipzig but soon discovered he disliked the subject, and in his first year decided to follow his interests in mathematics, astronomy, and physics.

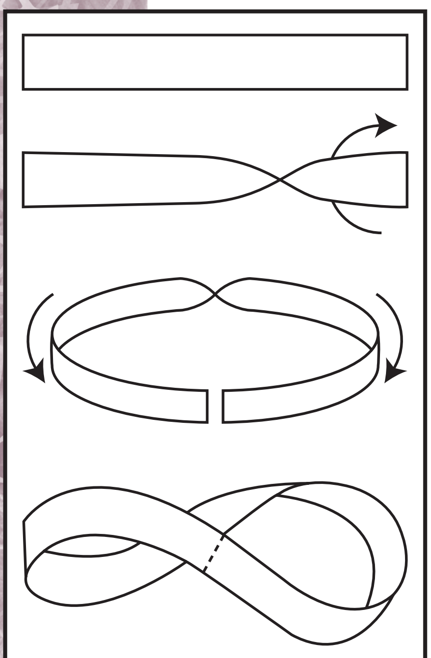
Upon graduation in 1815, Möbius was appointed chair of astronomy at the University of Leipzig. However, he was not granted a full professorship, as he was a poor lecturer and was unable to attract many fee-paying students. Yet Möbius's publications spanned a broad range of topics in both astronomy and mathematics. His writings were influenced by his astronomy teacher Karl Mollweide, theoretical astronomer Carl Friedrich Gauss, and mathematician Johann Pfaff, who was Gauss's teacher.

Möbius was also a pioneer within the then-developing mathematical field of **topology**. He is best known for his description of the curious-looking, one-sided Möbius strip—a loop that has no inside or outside, no top or bottom.

topology the study of those properties of geometric figures that do not change under such nonlinear transformations as stretching or bending

The Möbius Strip

The Möbius strip is a surface that is formed by taking a long, rectangular strip of paper, rotating the ends 180 degrees (a half-twist) with respect to one another, and then gluing the edges together. The result is a one-sided, one-edged Möbius strip. Theoretically, it is a two-dimensional surface with only one side, but it has been constructed in three dimensions.



The Möbius strip is a rectangular, flat strip with a half twist, whose ends connect to create a continuous, single-edged loop.

Strange as it sounds, if a continuous line is drawn along the middle of the loop, the line will eventually end up where it began, thus showing that a Möbius strip has only one side. Even stranger is the result when the Möbius strip is cut along the line down the middle of the loop. Instead of falling apart into two loops, the strip becomes a single, two-sided loop twice as long as the original strip. In contrast, an ordinary paper ring cut in half would give two separate rings, each of them the same length as the original.

Applications of the Möbius Strip

Applications of the Möbius strip concept show up in a variety of settings: monumental sculptures, literature, music, art, magic, science, engineering, synthetic molecules, postage stamps, knitting patterns, skiing acrobatics, and even the recycling symbol.

For example, freestyle skiers have named one of their acrobatic stunts the "Möbius Flip." Author Martin Gardner wrote a humorous short story called "The No-sided Professor" based on the Möbius strip. Artist M.C. Escher included in his many drawings a march of ants on a Möbius strip.

The Möbius strip also has practical applications. A Möbius strip conveyor belt will last twice as long as a normal two-sided belt because two-sided belts quickly wear out on one side. Similarly, a Möbius filmstrip that records sound on continuous-loops, or a Möbius tape in a tape recorder, will double the playing time. SEE ALSO Mathematics, Impossible; Topology.

William Arthur Atkins (with Philip Edward Koth)

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Morgan, Julia

American Architect
1872–1957

Julia Morgan, America's first prominent women architect, was the second of five children born to Charles and Parmelee Morgan on January 20, 1872, in San Francisco, California. Morgan pursued an interest in architecture, and was one of the first women to graduate from the college of engineering at the University of California at Berkeley in 1894. On the advice of her professor, architect Bernard Maybeck, Morgan went to Paris and became the first woman to be accepted into the architecture department at the Ecole des Beaux-Arts, a respected school of fine arts in France. In 1902, she also became the first woman to be granted the Ecole des Beaux-Arts certificate.

Following her studies in France, Morgan returned to the United States and was the first woman to be a licensed architect in California. She worked for John Galen Howard on two Berkeley structures—the Hearst Mining Building (1901–1907) and the Greek Theater (1903). Morgan was also recognized for her design and use of reinforced concrete to build the El Campanil Bell Tower (1903–1904) and the Margaret Carnegie Library (1905–1906) at Mills College in Oakland, California. She gained further recognition with the reconstruction of San Francisco's Grand Fairmont Hotel after the fire of 1906. In 1907, she established her own practice in the Merchants Exchange Building in San Francisco.

Julie Morgan designed San Simeon, the lavish residence of William Randolph Hearst. Referred to as the Hearst Castle, the residence quickly became a masterpiece of American architecture, and remains a famous tourist attraction.

In 1919, Morgan was hired by newspaper tycoon William Randolph Hearst to design his ranch, San Simeon, in central California (1920–1939). On her retirement in 1951, Morgan destroyed most of her blueprints and drawings, resulting in the architectural anonymity of many of the buildings she designed and constructed. Throughout her forty-seven-year career, she is noted as having said, "My buildings will be my legacy . . . they will speak for me long after I'm gone." She lived privately in San Francisco until her death on February 2, 1957. SEE ALSO ARCHITECT; ARCHITECTURE.

Gay A. Ragan

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Mount Everest, Measurement of

A process began around 1800 that would ultimately establish Mount Everest as the world's tallest mountain. Started by Englishman William Lambton, this process was referred to as "The Great Trigonometrical Survey," and led to a rigorous mapping of India.

In 1796, Lambton was posted to India as a British lieutenant. Lambton's arrival coincided with increasing subcontinent colonization, and maps and surveys of these British-conquered territories were of great interest. Lambton proposed a more exacting survey than any attempted before in Asia. The resulting measurements would yield, for example, more accurate values for India's width. Lambton believed that the most important outcome would be a better understanding of Earth's **geodetic** shape.

Early Mapping of the Earth

Fifty years before Lambton's proposed survey, French scientists determined that Earth is better described as an **oblate spheroid** instead of a sphere. This meant that the distance from Earth's center to the equator is greater than the distance from its center to either pole. Lambton's survey, renamed "The Great Trigonometrical Survey of India," would help calculate the amount of this **equatorial bulge**, and thereby result in a better model of Earth's shape. As an unintended by-product, a precise height of the Himalayan Mountains would also be sought.

Points on Earth are given a measurement of **latitude** and **longitude**. The longitudinal line passing through Greenwich, England, is called the **Prime Meridian**, and all lines of longitude are measured from it. Lambton's survey centered on the longitudinal line 78 degrees east from the prime meridian and running from India's southern tip to the Himalayan foothills. This arc-of-longitude became known as "The Great Arc." From the central hub of the Great Arc, an accurate survey of all of India could be performed.

geodetic of or relating to geodesy, which is the branch of applied mathematics dealing with the size and shape of Earth, including the precise location of points on its surface

oblate spheroid a spheroid that bulges at the equator; the surface created by rotating an ellipse 360 degrees around its minor axis

equatorial bulge the increase in diameter or circumference of an object when measured around its equator; usually due to rotation, all planets and the sun have equatorial bulges

latitude the number of degrees on Earth's surface north or south of the equator; the equator is latitude zero

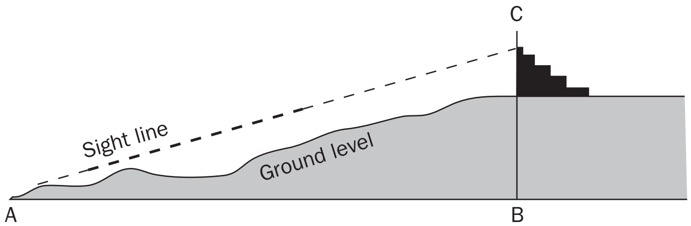
longitude one of the imaginary great circles beginning at the poles and extending around Earth; the geographic position east or west of the Prime Meridian

Prime Meridian the meridian that passes through Greenwich, England

Trigonometry in Mapping

The survey Lambton headed from around 1800 until his death in 1823 ran more than 1,000 miles northwards from India's southern coast. The survey consisted of a web of triangles whose **vertices** were located precisely.

vertex the point on a triangle or polygon where two sides come together



The basic idea behind traditional surveying is to use **trigonometry** (the mathematics of triangles) and measuring devices to locate exactly "new" points on Earth from points whose locations are already accurately known. A right triangle is used to determine the location of new point *C* relative to a known point *A*, as shown in the above illustration. There are six measurements for any triangle: three angles and three lengths of the sides. Given the lengths of two of three sides and one angular measurement, or given two angles and one side, any triangle is uniquely determined. Since the triangle is a right triangle, angle *ABC* is 90 degrees. Using a device called a **transit**, angle *CAB* can be measured. The length of side *AC* is now measured by using, for example, a chain. The length of sides *AB* and *BC* can now be computed. *AB* is the lateral distance from *A* to *C*, while length *BC* is the elevation of point *C* above point *A*. Point *C* can now be used as the starting point for a new triangle, and the procedure is repeated.

trigonometry the branch of mathematics that studies triangles and trigonometric functions

transit to pass through; in astronomy, the passage of a celestial object through the meridian (a line that runs from the north pole to the south pole and that goes directly overhead)

This method is a much-simplified version of what took place in the Great Survey. For example, the orientation and length of a baseline had to be determined laboriously. Lambton's first baseline near the eastern Indian coast measured approximately 7.5 miles, and it was measured using a specially constructed 100-foot iron chain. In terrible subtropical heat and humidity, the first baseline took 57 days to complete. Lambton then turned west and plunged into the Indian jungle. Because of the jungle canopy, large numbers of trees had to be cut down so that towers could be built; people standing on the towers could then make the necessary angular measurements.

Into this scene, British Lieutenant George Everest* arrived in 1819. Everest joined the Great Survey as one of many engineers reporting to Lambton. For the next several years, Everest worked in terrible jungle conditions of heat, humidity, and monsoon. Eventually, he collapsed from malaria and fever, and he left India in 1822, but he returned to his duties to become head of the survey after Lambton's death. Everest developed many new and innovative techniques, and despite poor health and the terrible climate, he pushed the Great Arc to the Himalayan foothills in northern India by the early 1840s. In 1843, Everest retired from his duties and returned to England.

***Unlike the pronunciation used today, George Everest pronounced his name "Eve-rest," like "evening."**

Early attempts to measure the height of Mount Everest used principles of trigonometry. Today's methods use satellites and Global Positioning System (GPS) technology.

Measuring Mountain Heights

Everest never attempted to measure any of the heights of the Himalayan range. However, two of his subordinates, Andrew Waugh and John Armstrong, made measurements from the Himalayan foothills. Many of the apparently loftiest mountains lay north of India within Nepal or Tibet. Nepal had closed its borders to foreigners, so the team could only estimate mountain distances. Nevertheless, with estimated distances and several angular measurements, the surveyors computed heights for various mountain peaks. From the present-day town of Darjeeling, Waugh measured a height for "Kangchenjunga," which is now known to be the world's third tallest mountain.✶ Waugh's measured height of 28,176 feet is within seven feet of today's accepted value. In 1847 both Waugh and Armstrong, from different locations, took measurements of a mountain suspected to be even taller than Kangchenjunga. Since no local name could be determined, it was simply listed in survey records as "Himalaya Peak XV."

✶ The most common translation of *Kangchenjunga* is "Five Treasuries of the Great Snow," from the five high peaks that rise from its surrounding glaciers.

Since Peak XV measurements were taken from a great distance away, **terrestrial refraction** (the bending of light in Earth's atmosphere) could have had a profound effect on any angles measured. Mathematical constants called "coefficients of **refraction**" had to be included in the elevation computations to correct for this phenomenon. By 1856, armed with better coefficients of refraction and with more accurate angular measurements, Waugh communicated his finding that Peak XV was computed at 29,002 feet above sea level. Moreover, Waugh recommended that this mountain be officially named Mount Everest to honor Everest's important role in the Great Survey.

terrestrial refraction the apparent raising or lowering of a distant object on Earth's surface due to variations in atmospheric temperature

refraction the change in direction of a wave as it passes from one medium to another

In 1953, about a century after Mount Everest's height was first clearly ascertained, Edmund Hillary and Tenzing Norgay became the first to reach its summit. Since that time, mountaineers have placed various devices on

Mount Everest to more accurately determine its height. For example, in 1992, an American expedition placed a reflector atop the mountain to bounce laser light off its surface. This device led to a measurement of 29,031 feet. However, like earlier measurements, this one included the snowcap's depth. Since the snowcap can vary, it would be advantageous to determine Everest's height minus this layer, estimated at between 30 and 60 feet. It has been proposed that a future expedition use ground penetrating radar to find the snow pack depth and thereby determine the height of Everest's rocky apex.

The latest surveying method used to measure Mount Everest's elevation makes use of the Global Positioning System (GPS). GPS uses satellite signals to determine the coordinates of points on Earth's surface. The National Geographic Society announced in November 1999 a revised height of 29,035 feet (using GPS) for Mount Everest, but that measurement, as others before it, includes the ice and snow layers. SEE ALSO Angles of Elevation and Depression; Angles, Measurement of; Global Positioning System.

Philip Edward Koth (with William Arthur Atkins)

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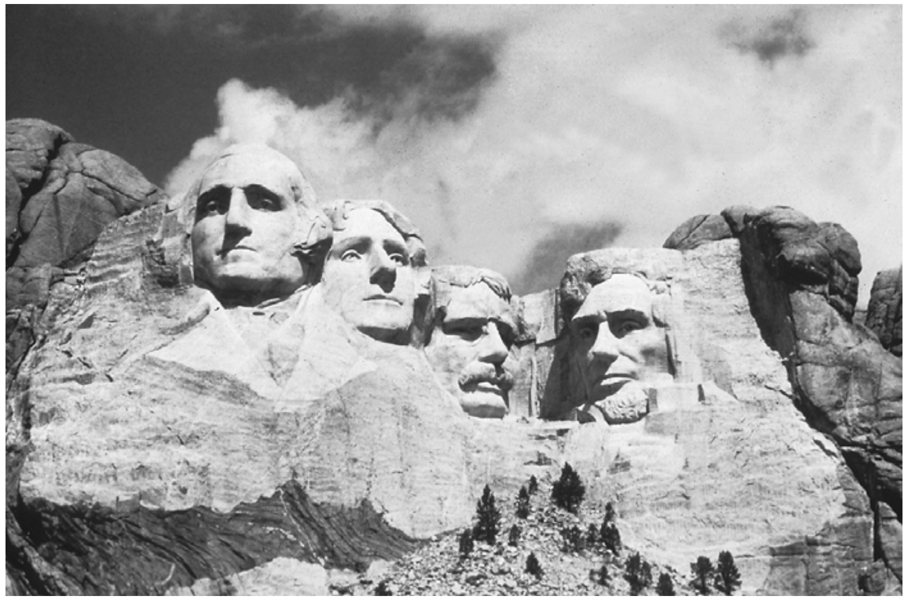
Mount Rushmore, Measurement of

The Mount Rushmore National Memorial is one of the world's largest sculptural and engineering projects. Sculptor-designer John Gutzon de la Mothe Borglum (1867–1941) was contracted in 1927 to carve the solid-granite memorial. Borglum conceived the model figures, brought them to life within the mountain's stone, and directed 400 artisans until his death in 1941. Later that year, his son Lincoln finished the project, which had spanned 14 years (6.5 years of actual carving and 8.5 years of delays due to lack of money and bad weather) at a cost of $1 million.

Mount Rushmore is considered a priceless U.S. treasure, memorializing the first 150 years of the country's struggle for independence and the birth of the republic as represented by George Washington; the idea of representative government in an expanding nation as envisioned by Thomas Jefferson; the preservation of the union of states and the equality for all citizens as championed by Abraham Lincoln; and the twentieth-century developmental role of the United States in world affairs and economy as promoted by Theodore Roosevelt.

Mount Rushmore would not have been possible without Borglum's earlier carving experience of Confederate leaders on Stone Mountain. In the Rushmore project, Borglum employed few conventional sculpturing meth-



The 60-foot tall portraits of U.S. Presidents George Washington, Thomas Jefferson, Abraham Lincoln, and Theodore Roosevelt are blended into the face of a mountain in the Black Hills of South Dakota and are visible for 60 miles.

ods in what was a unique engineering accomplishment. Borglum knew that he needed a cliff 400- to 500-feet in height; it had to lay at an angle so the main wall would face toward the sun; and there had to be sufficient amounts of even, unblemished stone to provide an acre of upright surface.

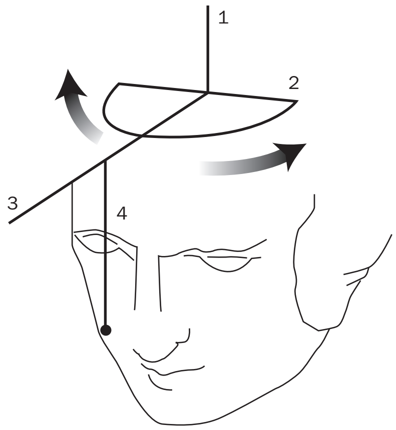
The presidential models were based on descriptions, paintings, photographs, life masks, and Borglum's interpretations. Initially, Borglum climbed down the mountain to locate Washington's nose line, eyebrows, and chin, marking each with red paint. From these points, he studied and mathematically calculated the scale necessary for the first head. Realizing the importance of models at the worksite, Borglum displayed a five-foot mask of each figure as a guide for the workers. However, Borglum did not simply transpose the models directly onto the granite; rather he fine-tuned the heads into artwork. In fact, Borglum realized that to transfer accurately his models into finished heads, he needed mathematical and engineering concepts. What he had to be, besides a sculptor, was an explosives expert, a geologist, a miner, an engineer, and a mathematician.

Scale in Mount Rushmore

Borglum constructed a "pointing machine" that enabled the transfer of mathematical dimensions of his models onto the mountain with a simple ratio of 1:12. That is, 1 inch on the model equaled 12 inches (1 foot) on the mountain. Each model was measured the same way (as shown in the diagram below). A metal shaft (1) was placed upright at the center of the top of the head, the "master point." A **protractor** (2), in degrees, was attached at the shaft's base. A horizontal bar (3) was placed on the protractor's central axis and pivoted to measure right and left angles from the centerline of the model's face. A weighted **plumb-bob** (4) was dropped from the horizontal bar and slid back and forth to measure the horizontal distance from the master point to the position on the bar where the plumb-bob touched a head-point being measured. The plumb-bob was also raised or lowered to measure the vertical distance from the bar on the top of the head to that particular head-point.

protractor a device used for measuring angles, usually consisting of a half circle marked in degrees

plumb-bob a heavy, cone-shaped weight supported point down on its axis by a strong cord, used to determine verticality in construction or surveying



Each reference point on the model received three measurements: (a) a rotational (angular) measurement along the protractor, (b) a horizontal distance (linear) measurement along the bar, and (c) a vertical distance (linear) measurement along the plumb-bob. The two linear measurements for each reference point were multiplied by 12, with the angular measurement remaining constant, and then transferred to the mountain with a large-scale "Rushmore pointing machine."

This machine was secured at the mountaintop with a vertical mast, horizontal steel boom, and steel-slab protractor. Borglum used it in this way: If the model measure from the top of Washington's wig to his nose measured 20 inches, that length was multiplied by 12 to find that his nose on Rushmore measured 240 inches, or 20 feet, from the top of his head. However, the angular measurement remains the same between the model and the mountain since the definition of an angle is the ratio of a circle's **arc** length (s) and its **radius** (r), and that ratio remains constant for both model and mountain. That is, if the model ratio of arc length to radius is $\frac{s}{r}$, then the mountain ratio of arc length to radius is $\frac{(12 \times s)}{(12 \times r)}$, and the 12s cancel each other out. This system of transferring measurements from small to large pointing machines proved to be so effective that it was the only necessary measuring system.

arc a continuous portion of a circle; the portion of a circle between two line segments originating at the center of the circle

radius the line segment originating at the center of a circle or sphere and terminating on the circle or sphere; also the measure of that line segment

pneumatic drill a drill operated by compressed air

Carving Mount Rushmore

An oval-shaped volume was first dynamited to remove sheets of excess stone for each roughened head. This "egg" was divided into three sections—one at the eyebrow line, another at the nose end, and a third at the chin end. Rough shaping of heads began as the surface was removed with smaller charges of dynamite until good carving stone was reached. After a reference point, such as a nose tip, was located, lines of holes were drilled from 2 to 6 feet deep. Excess rock was removed with mini-charges of dynamite (sometimes only a half ounce) inserted into the holes, sometimes within inches of the finished surface.

Drillers suspended over the mountain's face by cables in swing seats then shaped the features. **Pneumatic drills** punctured the surface with a honeycomb series of holes at intervals of about 3 inches. The holes' depths ultimately shaped the finished "skin." The remaining rock was later chiseled with a drill, or a hammer and wedging tool, to the finished depth. Fi-

MOUNT RUSHMORE'S NEIGHBOR

Another monumental sculpture is taking shape in the hard rock of the Black Hills. Honoring Native American Chief Crazy Horse, the work depicts the warrior on horseback. The sculpture was begun in 1947 by Korczak Ziolkowski, who had assisted Borglum on Mount Rushmore. A celebration was held in 1998 when the chief's face (measuring nine stories high) was completed.

When the entire work is finished, the tribute is slated to be the largest sculpture in the world, standing more than 560-feet high and 640-feet long. Ziolkowski, who died in 1982, took steps to ensure that his vision could be completed by other carvers, leaving detailed plans and specifications for others to follow.

nally, the surface was smoothed with pneumatic hammers in a process called bumping to create a white surface as smooth as a concrete sidewalk. Measuring-drilling-blasting-drilling-wedging-bumping became the work cycle as 450,000 tons of rocks were removed.

The work on Mount Rushmore was Borglum's way to preserve a symbol of a great national ideal. Borglum never felt anyone (including himself) was endowed at birth with superior talents. His ability as a successful artist was due to trained observation, hard work, and the ability to use engineering and mathematical methods to produce his artistic creations, including the impressive Mount Rushmore. SEE ALSO Ratio, Rate and Proportion; Scale Drawings and Models.

William Arthur Atkins (and Philip Edward Koth)

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Music Recording Technician

Musicians and singers rely on technicians to produce high-quality studio recordings.

A music recording technician operates the soundboard and other electrical equipment required to make a recording. The technician works closely with recording artists before, during, and after tapings to produce sound recordings that meet the goals of the artists.

Recording technicians oversee all recording procedures and operate the studio equipment. The different audio controls they use create various sounds. During the recording they select and position microphones and instruments. They organize and execute **overdubs** (additions to the basic recording) and determine levels and balances for the final mix (when all instruments and other sounds are combined). Finally, they assemble the finished mixes in the proper order for release. In certain studios the technician must also know how to repair the equipment when it malfunctions.

Recording technicians must keep up with the latest electronic and recording technology. High-tech developments in the industry include synthesizers, sequencers, computers, automated mixing consoles, and digital recording.

A college or technical school background in sound engineering or recording technology is helpful, but not necessary, for most technician jobs.

Courses in computer literacy are recommended to help students learn how to use computerized music production equipment. Many recording engineers start in entry-level positions at a studio and then get on-the-job training in apprenticeship positions. SEE ALSO Compact Disc, DVD, and MP3 Technology; Sound.

Denise Prendergast

overdubs adding voice tracks to an existing film or tape

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Nature

N

Mathematics is widespread in nature, and mathematical concepts are essential to understanding the biosphere, the rocks and oceans, and the atmosphere. This article explores a few examples.

The Fibonacci Series

In 1202 a monk in Italy, by the name of Leonardo Pisano Fibonacci, wanted to know how fast rabbits could breed in ideal circumstances. Suppose a newly born pair of rabbits, one male, one female, are put in a field. Rabbits are able to mate at the age of 1 month. So at the end of its second month, a female can produce another pair of rabbits. Suppose that these rabbits never die and that the female always produces one new pair (one male, one female) every month from the second month on. The puzzle that Fibonacci posed was: How many pairs would there be after 1 year?

1. At the end of the first month, they mate, but there is still only one pair.
2. At the end of the second month the female produces a new pair, so now there are two pairs of rabbits in the field.
3. At the end of the third month, the original female produces a second pair, making three pairs in the field.
4. At the end of the fourth month, the original female has produced yet another new pair, the female born two months ago produces her first pair also, making five pairs.

The resulting series of numbers, 1, 1, 2, 3, 5, 8, 13, 21, 34, . . ., is known as the Fibonacci series. Fibonacci's experiment is not very realistic, of course, because it implies that brothers and sisters mate, which leads to genetic problems. But the Fibonacci series is puzzlingly common in nature.

Bees. The Fibonacci series is evident in generations of honeybees. For instance, in a colony of honeybees there is one special female called the queen. There are many worker bees who are female too, but unlike the queen bee, they do not produce eggs. Then there are drone bees who are male and do no work. Males are produced by the queen's unfertilized eggs, so male bees have only a mother but no father. In contrast, females are produced when the queen has mated with a male, and so females have two

parents. Females usually end up as worker bees but some are fed with a special substance, called "royal jelly," which makes them grow into queens ready to start a new colony when the bees form a swarm and leave their hive in search of a place to build a new nest.

Let's look at the family tree of a male drone bee ("he").

1. He had one parent, a female.
2. He has two grandparents, since his mother had two parents, a male and a female.
3. He has three great-grandparents: his grandmother had two parents but his grandfather had only one.
4. How many great-great-grandparents did he have?

Here is the sequence:

Number of	parents:	grand parents:	great-grand parents:	great, great grand parents:	gt, gt, gt grand parents:
of a male bee:	1	2	3	5	8
of a female bee:	2	3	5	8	13

Flowers and Other Plants. Another example of the Fibonacci series is the number of petals of flowers: lilies and iris have three petals; buttercups have five petals; some delphiniums have eight; corn marigolds have thirteen petals; some asters have twenty-one whereas daisies can be found with thirty-four or fifty-five petals. The series can also be found in the spiral arrangement of seeds on flowerheads, for instance on sunflowers, and in the structure of pinecones. In both cases the reason seems to be that this forms an optimal packing of the seeds (or cone studs) so that, no matter how large the seedhead (or cones), they are uniformly packed, and about the same size.

The Fibonacci series also appears in the position of a sequence of leaves on a stem. It should be noted that among plants there are other number sequences and aberrations. In other words, the Fibonacci series is really not a universal law, but only a fascinatingly prevalent tendency in nature.

The Golden Number (Phi)

If we take the ratio of two successive numbers in a Fibonacci series (1, 1, 2, 3, 5, 8, 13, . . .), dividing each number by the number before it, we will find the following series of numbers:

$$\frac{1}{1} = 1, \frac{2}{1} = 2, \frac{3}{2} = 1.5, \frac{5}{3} = 1.666\ldots,$$

$$\frac{8}{5} = 1.6, \frac{13}{8} = 1.625, \frac{21}{13} = 1.61538\ldots$$

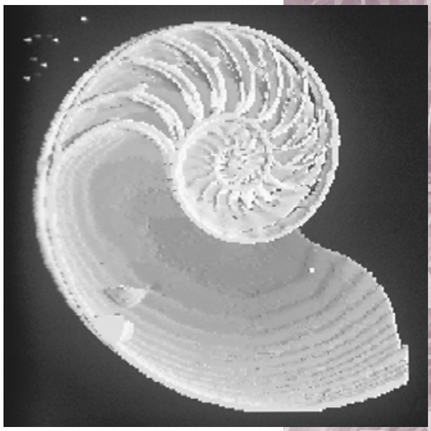
The ratio seems to be approaching a particular value known as the golden number, or Phi (ϕ). It has the value of ≈ 1.61804. The golden number is an amazingly universal constant. It turns out that $\phi = 1 + 1/\phi$, or $\phi^2 = \phi + 1$.

Plants grow from a single tiny group of cells right at the tip of any growing plant, called the meristem. There is a separate meristem at the end of each branch or twig and it is here that new cells are formed. Once formed,

they grow in size. Cells earlier down the stem expand and so the growing point rises. These cells grow in a spiral fashion, as if the stem turns by an angle and then a new cell appears, turning again and then another new cell is formed and so on. These cells may then become a new branch, or perhaps on a flower become petals and stamens.

The amazing thing is that a single fixed angle can produce the optimal design no matter how big the plant grows. If this angle is an exact fraction of a full turn, for example, $\frac{1}{3}$ (120°), then leaves of a vertical branch will be on top of each other. The fraction needs to be an irrational number. It turns out that if there are ϕ (or approximately 1.6) leaves per turn, then each leaf gets the maximum exposure to light, casting the least shadow on the others. This also gives the best possible area exposed to falling rain so the rain is directed back along the leaf and down the stem to the roots. For flowers or petals, it gives the best possible exposure to insects to attract them for pollination. And this angle optimizes the seeds on a sunflower. The Fibonacci numbers merely form the best whole number approximations to the golden number, ϕ. SEE ALSO Chaos; Fibonacci, Leonardo Pisano; Fractals; Golden Section.

Bart Geerts



The nautilus shell is an example from nature of the ratio that yields the golden number. The chambered shell demonstrates this number as it grows, sequentially building upon itself in a spiral patterns with dimensions determined by the golden ratio.

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Navigation

In the broadest sense, navigation is the act of moving about from place to place on land, sea, in air, or in outer space. Navigation, with its primitive beginnings, has evolved to become a sophisticated science.

Early Navigation

Prior to the fifteenth century, European mariners were reluctant to sail out of sight of land, partly because they feared getting lost and partly because they did not know what lay beyond the horizon. Thus, sailing voyages by Europeans were largely confined to the Mediterranean Sea or close to shore in the Atlantic Ocean. The high and broad continental shelf of Northern Europe, where the continent ends and the ocean begins, allowed for shallow sailing waters within sight of land from the Iberian Peninsula (Portugal and Spain) to Scandinavia (Norway, Sweden, and Denmark).

The Vikings of Scandinavia were renowned coastal navigators. Not only did the Vikings sail the coast of Europe, but they also followed the continental shelf into the Northern Atlantic to Iceland, Greenland, and ultimately to North America. Although such extended voyages were remarkable accomplishments, they involved no sophisticated navigational techniques.

Today's tall ships, shown here during the Tall Ships 2000 transatlantic race, preserve the history of early sailing vessels.

In about the year 1000, the Norseman Leif Ericson made a transatlantic voyage to North America with the midnight sun lighting his way. Using the pole star as his only navigational guide, he followed the North Atlantic's generous continental shelf to the northeastern coast of mainland North America.

While ambitious open sea voyages such as Ericson's were possible in the extreme northern latitudes, the South Atlantic was not as accommodating. Africa's continental shelf was narrow, and left very little room for navigational error before a ship could be swept into the deep currents and unfamiliar winds off the African coast. These currents and winds were unpredictable and tended to flow to the north and east, exactly the opposite direction from that in which sailors wanted to go.

The Europeans, including the Vikings, remained essentially coastal navigators until the first half of the fifteenth century. The situation was the same in all parts of the world at that time. All navigation was local rather than global. Sailing on the open sea was possible only where there were predictable winds and currents or a wide continental shelf to follow.

Medieval Navigation

In the early part of the fifteenth century, Portuguese sailors began to sail farther out into the Atlantic using favorable winds, currents, and the paths of birds as guides. By the 1440s, they had reached as far as the Azores, an archipelago of small islands some 800 miles west of Portugal. To venture farther than this would require the beginnings of a more scientific and math-

ematical type of navigation. This more scientific approach took two forms. The first was a type of navigation known as "dead reckoning" and the second was the application of astronomy and mathematics to what is known as "celestial navigation," or navigation by the stars.

In the process of dead reckoning, a triangular wooden slab, called a chip log, attached to a rope with evenly spaced knots along its entire length, was tossed into the ocean from the stern of the ship. Sailors would then count the number of knots pulled out by the log in a given amount of time, usually measured by sand glasses calibrated for one minute or less. From this observation, an approximation of the speed of the ship could be calculated. Such measurements were taken each time the ship changed course due to a change in wind direction.

This was an early attempt to measure what we now call the **longitude** of the ship at a given moment. The method was not very accurate, but it was the best that could be done at the time. The captain's log of Christopher Columbus's 1492 journey to the Americas suggests that Columbus relied almost exclusively on dead reckoning to navigate to the New World. Truly accurate measures of longitude would have to wait until the invention of the **chronometer** in the eighteenth century.

Celestial navigation could help in estimating a ship's **latitude**. In the Northern Hemisphere, mariners could use the pole star as a reference point. At the north pole the star would be directly overhead at all times, but as one moves farther south it appears lower and lower in the sky until, at the equator, it dips below the horizon.

An instrument called a quadrant could be used to measure the angle of the pole star above the horizon. The quadrant was a quarter circle with degree markings from 0 to 90 along its arc. A plumb line hung from the point at the center of the circle and the observer would then line up the edge of the quadrant with the pole star. The plumb line would then cross the arc of the circle at the position that would indicate the number of degrees above the horizon at which the pole star was located. In this way latitude could be approximately determined.

Of course this method worked only at night, but an alternative method for determining latitude in the daytime made use of the astrolabe, a heavy brass disk with degrees marked around its edge. An observer would move a rotating arm attached at the center of a disk until sunlight shone through a hole at one end of the arm and fell on a hole at the other end. The arm would indicate the altitude of the Sun by the degrees marked around the edge of the disk.

In 1473, the astronomer Abraham Zacuto created a book of tables called *Rules for the Astrolabe* that allowed mariners to determine the latitude for any day of the year. Use of the tables depended upon knowing in which constellation of stars the Sun rose on the day of the measurement. An observer would view the eastern horizon before sunrise and note the constellation in which the Sun rose. Later in the day, when the Sun reached its highest point in the sky, the observer would take a reading with the astrolabe. Zacuto's *Rules for the Astrolabe* could then be used to look up the latitude with a degree of accuracy never before possible.

The use of the astrolabe revolutionized navigation because of the accuracy it afforded to determinations of latitude.

longitude one of the imaginary great circles beginning at the poles and extending around Earth; the geographic position east or west of the prime meridian

chronometer an extremely precise timepiece

latitude the number of degrees on Earth's surface north or south of the equator; the equator is latitude zero

Zacuto constructed this extensive set of tables using mathematics, specifically trigonometry, developed between the ninth and thirteenth centuries by Judeo-Arab mathematicians and astronomers in Portugal and Spain. To produce these tables, Zacuto needed, in addition to trigonometry, an accurate solar calendar giving the location of the Earth with respect to the Sun at any time during the year. Such a calendar had been constructed in the eleventh century by Muslim astronomers in Spain. Making use of this calendar, the Sun's position relative to the constellations, and the height of the midday Sun above the horizon, Zacuto produced the first scientifically accurate method for determining latitude. This method was used by European navigators for more than a century.

By the 1520s, the ability to determine latitude at sea with reasonable accuracy was well established, but the problem of finding longitude with an acceptable degree of precision remained intractable for another 300 years. Whereas latitude measures positions north and south of the equator, longitude uses imaginary "great circles" passing through the north and south poles to measure positions east and west of a predetermined great circle called the **Prime Meridian**.

Prime Meridian the meridian that passes through Greenwich, England

The first prime meridian was established by the Portuguese map-maker Pedro Reinel in 1506. It passed through the Portuguese Madeira Islands. Reinel's prime meridian would remain the world's standard for more than 300 years, but with the decline of Portuguese sea power and the rise of England in the seventeenth century, a British prime meridian was established passing through Greenwich, England. In 1884, a conference of European nations ratified the new prime meridian as the world's standard. It remains so to this day.

The problem of determining longitude involves knowing the time at the prime meridian and the time aboard the ship on which one is traveling. Earth rotates on its axis once every 24 hours. One revolution is 360 degrees of longitude, so $360 \div 24$ gives 15 degrees per hour. Thus if the ship has a clock which accurately gives the time at the prime meridian and the time on board the ship, then the longitude of the ship can be calculated.

This may seem a trivial matter to people of the twenty-first century who possess incredibly stable and accurate time-pieces, but such was not the case for navigators of the fifteenth, sixteenth, and early seventeenth centuries. Clocks of that time period were of the pendulum type and were useless on the deck of a rocking ship. An obscure English clockmaker, John Harrison, would finally solve the longitude problem in 1764 with the invention of a clock that could keep time to within less than a second of accuracy per day and could withstand the rocking and temperature extremes experienced aboard a ship on the open sea. Harrison's invention was the forerunner of the modern chronometer that is present on all ocean-going vessels today.

At about the same time that Harrison was creating his chronometer, a more stable and accurate version of the astrolabe, called the sextant, was invented. Together, these two inventions ushered in a new, more scientifically based era of navigation.

Modern Navigation

In the cold-war era of tension between the United States and the former Soviet Union, the U.S. Department of Defense authorized about $12 bil-

lion for research and development to devise and perfect a navigational system that could provide an almost instantaneous and accurate reading for the location of any point on the surface of the Earth. The military's purpose was to allow pinpoint accuracy in the launch of its missiles from submarines in the ocean. Yet in the mid-1990s, this Global Positioning System technology was made available to the civilian population.

GPS Technology. The Global Positioning System (GPS) utilizes satellites in orbit around Earth to send signals to Earth-based devices for the purpose of calculating the exact latitude and longitude of the Earth-based unit. The ability of computer-chip makers to pack more memory onto smaller and smaller chips has resulted in GPS devices that can be held in the palm of a hand and are reasonably priced.

The mathematics behind GPS is essentially the same as that used by Abraham Zacuto to develop his tables for use with the astrolabe except that the calculations are done by computer through the trigonometric idea of **triangulation**. The distances from a handheld GPS receiver to three of the orbiting satellites is determined by the time-encoded signals traveling at the speed of light from each satellite to the receiver. Then using the familiar "rate × time = distance" equation, the GPS device calculates the distance to each satellite from the device's position on the ground. With these three measurements, the GPS can calculate this position to within a few meters of accuracy. Essentially, the distances to the three satellites can be thought of as the radii of three imaginary spheres. These three spheres will intersect in two points, only one of which will be a reasonable position on Earth's surface. The GPS device will give a reading of the latitude and longitude of this position.

triangulation the process of determining the distance to an object by measuring the length of the base and two angles of a triangle

With the introduction of the Global Positioning System, the age-old problem of knowing where you are on Earth's surface at any given time has essentially been solved, assuming that you are carrying a GPS receiver at all times. That is now the case for most ocean vessels and airplanes, both commercial and military. Many of the latest model cars come equipped with navigation systems powered by GPS technology. This may well become standard equipment on all vehicles in the near future.

Maps and Planning. Even without sophisticated technology, it is still possible to plan trips on land using a map of the area you are interested in navigating. Using a well-marked map, you can decide whether you want to take a scenic route or a more direct and quicker route. Using the mileage markings on the map or the legend that gives the scale of the map, you can determine how far you must travel using each route. With a little mathematics, you can determine the approximate length of time required to reach the destination.

If you know that you can average about 60 miles per hour on the direct route, which is 240 miles long, then you calculate 240 miles divided by 60 miles per hour to get 4 hours as the approximate time to make the trip. If the scenic route is 280 miles and you can only average 40 miles per hour then it will take 7 hours to travel the scenic route, by calculating 280/40.

You can also estimate the gasoline cost for each route. If your car gets about 24 miles to the gallon when traveling at 60 miles per hour on the open highway, and if gasoline is $1.50 per gallon, then you can calculate the

gasoline cost for the direct route as approximately 240 miles divided by 24 miles per gallon times \$1.50 per gallon = \$15.00. Of course if you are coming back by the same route, you could double this to \$30.00 for the round trip. Similar calculations would allow you to compare the cost of this route to that of the scenic route, taking into account that your car may get poorer gas mileage on the scenic route due to frequent starts and stops, climbing hills, and the like. Perhaps future generations of GPS devices will do these calculations as well as letting you know where you are at each second of your trip. SEE ALSO Angles of Elevation and Depression; Angles, Measurement of; Distance, Measuring; Flight, Measurements of; Geometry, Spherical; Global Positioning System; Mile, Nautical and Statute.

Stephen Robinson

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Negative Discoveries

While solving problems and constructing proofs, mathematicians use many different approaches. A common technique for proving a statement is by contradiction. In this approach, it is supposed that the converse of the statement, or its opposite, is true. If this supposition leads to an absurd result, or contradiction, then it can be said the original statement is true. Hence, exploring incorrect answers and assumptions can often lead to new correct results.

Euclidean and Non-Euclidean Geometry

In 300 B.C.E., Euclid of Alexandria put forward a logical construction of a geometry, which has come to be known as Euclidean geometry. Until the middle of the nineteenth century mathematicians believed that Euclid's geometry was the only type of geometry possible. Euclidean geometry is based on a number of fundamental statements called postulates, or axioms.

In his book *Elements*, Euclid based his geometry on five axioms. The fifth axiom, also known as the parallel axiom, states the following: Given a line m and a point P not on m, there is only one line through P which is parallel to m.

In mathematics, a set of axioms has to fulfill two conditions: consistency and independence. A set of axioms is consistent if its use does not produce an absurd result that contradicts a statement derived from the axioms. A set of axioms is independent if none of the axioms can be logically deduced from the others.

Since Euclid, a number of mathematicians have thought that the parallel axiom was not independent and could be logically derived from the rest of the axioms. In 1763, the German mathematician Georg Klügel noted nearly thirty attempts to prove the dependence of the parallel postulate. But all attempts failed.

In 1733, a noteworthy attempt was made by Giovanni Girolamo Saccheri. After failing to show the dependence of the parallel axiom, Saccheri declared that Euclid's five axioms are indeed independent. But Saccheri's approach contained all the clues to invent or discover a new type of geometry. However, he failed to see the consequence of his own work because he thought that Euclid's geometry was the only geometry possible.

After Sacherri, a few more mathematicians continued to work on the parallel axiom problem. What Saccheri failed to discover, the young Hungarian mathematician Jénos Bolyai discovered by making a bold declaration. He proposed the first non-Euclidean geometry by replacing the parallel axiom with its "opposite" or negation. In Euclid's fifth axiom, instead of limiting to one parallel line, Bolyai's geometry stated that there is more than one parallel line.

By keeping Euclid's first four axioms the same and combining them with the modified fifth axiom, Bolyai discovered a new consistent geometry. This geometry is known as hyperbolic geometry. Bolyai's bold idea expanded the narrow world of Euclidean geometry. This important discovery occurred partly because of the negative attention that the parallel axiom received for several centuries!

New Negative Solutions

The process of eliminating the wrong answers occasionally results in the discovery of new mathematics. A famous problem, referred to as the four-color hypothesis, proposed in 1852, was partly solved by eliminating incorrect solutions. The four-color hypothesis states that a map drawn on a plane can be colored with four (or fewer) colors so that no two regions that share a common boundary have the same color.

To prove the four-color hypothesis, mathematicians tried to construct maps in which two or three colors were not sufficient. Also, a proof was constructed that showed that any map in the plane can be colored with five colors, so that neighboring regions do not have the same color.

Essentially, this approach showed that three colors are too few and five are more than enough. Therefore, the question that remained was whether four is just enough. In 1976, with the help of a computer, Kenneth Appel and Wolfgang Haken of the University of Illinois presented a proof of the four-color theorem. SEE ALSO CONSISTENCY; EUCLID AND HIS CONTRIBUTIONS; PROOF.

Rafiq Ladhani

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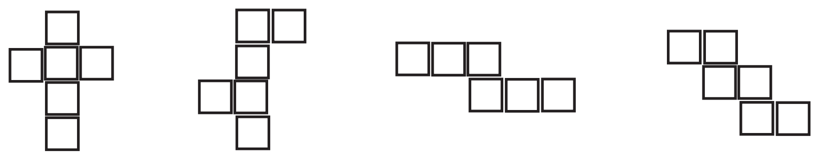
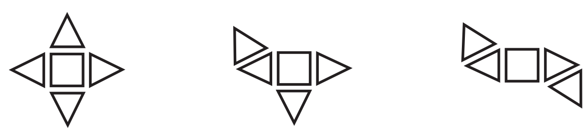
Nets

Nets are two-dimensional representations of three-dimensional shapes. German painter Albrecht Dürer (1471–1536) probably had fishing nets in mind when he first used the term net to name these two-dimensional figures that may be "folded" to form three-dimensional solids. Some of the nets of a cube are shown in the figure below. Although they are each formed with six squares, they are different nets of a cube.



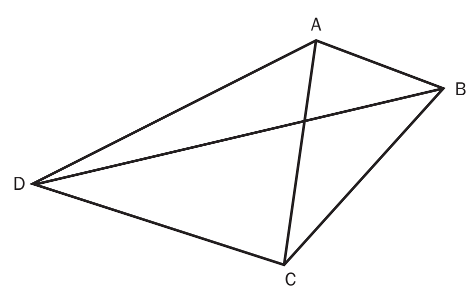
Each of these nets may be folded along its edges to form a cube.

Any solid with polygons for faces, called a polyhedron, may be unfolded to form a net. The polyhedron known as a square pyramid has several different nets. Some of them are shown below. Each of these nets can be folded up to form a square pyramid.



vertex the point on a triangle or polygon where two sides come together

Nets are used to study the properties of various polyhedrons. A more valuable use of nets is in the field of networks. A network is a series of segments, or edges, that joins a given number of **vertex** points, also called nodes. The network below contains four vertex points and six edges.



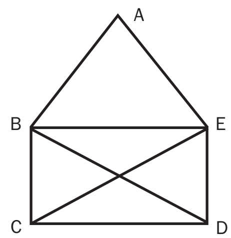
The Bridges of Königsburg

The first mathematician to make a study of networks was Swiss mathematician Leonhard Euler (1707–1783). His interest in networks was sparked by the town of Königsburg, in Prussia. This town was built on the banks of the Pregel River, near two islands. Seven bridges joined the islands to each other and to the river banks. A favorite Sunday afternoon recreation for the

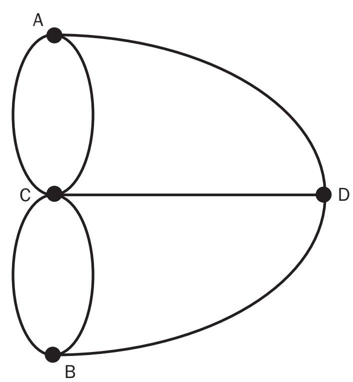
citizens of Königsburg was to try to cross all seven bridges without walking across the same bridge twice. No one could find a route that would take them across each bridge, once and only once.✶

✶**A path in which a person travels upon each segment once and only once is called a Euler Walk in honor of the eighteenth-century mathematician, Leonhard Euler.**

The diagram below shows an eight-segment Euler Walk. Starting at Point C, the network may be toured by the path CBECDBAED. This tour travels every path once and only once. It also returns to the departing point. Such a path is called a "closed" Euler Walk.



When Euler began to study the problem of crossing the seven bridges of Königsburg, he made an abstract model of the seven bridges.



In Euler's simplified diagram, the riverbanks are represented by Point A and Point B, and the two islands in the river are represented by Point C and Point D. Each of the lines connecting the vertex points represents the bridges, seven in all. After some study of the problem, Euler concluded it was impossible to cross each bridge once and only once. The problem was with how the islands and the river banks were connected by the bridges. Euler used the term "valence" to describe how many paths or edges met at a single vertex point. In the diagram, Point D has a valence of 3, and Point C has a valence of 5.

Euler stated that the network formed by the bridges was impossible to travel so that each bridge was crossed only once, because of the valence of the points. Euler discovered that only networks with all even valance points or exactly two odd valence points could form an Euler Walk. The reason is that if a vertex has an odd number of paths terminating at that point, then one of the paths must be used to either start or finish a journey.

In Euler's simplified diagram, three paths terminate at vertex A. If the Sunday walk began at point C and continued to Point A, then to Point D, that would leave one path still terminating at vertex A. The walk that started at Point B would have to eventually stop at Point A in order to travel along

the remaining path that terminates at Point A. If the walk began at Point A and traveled to Point B, then there would be two paths left terminating at Point A. Sometime later in the walk, one path could be taken to Point A, and the remaining path could be used to leave Point A. If there were five paths terminating at Point A, then the additional two paths could be used to arrive at, and then depart from, Point A. Thus any vertex with an odd number of paths could be used only as the starting point or as the ending point of an Euler Walk.

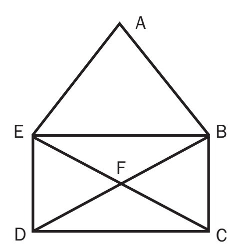
The Königsburg bridges formed four vertex points, each with an odd number of terminating paths. According to Euler it was impossible to travel across each and every bridge once and only once. Although Euler's discovery may be used to determine if it is possible to travel each path in a network once and only once, there is no **algorithm** for determining the shortest path that will travel across all the edges in a network with a large number of vertex points.

algorithm a rule or procedure used to solve a mathematical problem

Euler's discovery has been helpful for a variety of real-life situations. The route for a snow plow, a postal deliverer, or a utility meter reader requires that all roads or paths in a network be traveled once. The best tour for each of these situations is a Euler Walk that maps a route over each road once and only once. But a Euler Walk is possible only if there are either two vertex points or no vertex points with an odd valence.

In 1875 an eighth bridge was built across the Pregel River, joining Point A and Point D in Euler's diagram. This combination of eight bridges left only two vertex points with an odd valence, Point B and Point C. The new bridge made it possible to cross all eight bridges once and only once. The resulting walk is called an *open* Euler Walk because it does not end at the starting point.

The Hamilton Walk. A network problem related to the Euler Walk involves visiting all the vertex points in a network once and only once. Irish mathematician William Rowan Hamilton (1805–1865) studied such networks. A path that visits all vertex points once and only once is called a Hamilton Walk. The network below shows a Hamilton Walk. One Hamilton Walk for this network is FCBAEDF. This path visits each vertex once and only once, and ends up at the starting point. It is called a closed Hamilton Walk.



Like the Euler Walk, the Hamilton Walk also has applications in real-life activities. A salesperson visiting various business sites or a delivery service vehicle follows a route that requires visiting individual locations or vertex points. However, as with a Euler Walk, it is not possible mathematically to determine the shortest Hamilton Walk for a large number of vertex points. SEE ALSO Euler, Leonhard; Polyhedrons.

Arthur V. Johnson II

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Newton, Sir Isaac

British Mathematician, Physicist, and Astronomer
1642–1727

Many consider Isaac Newton to be among the best mathematicians who ever lived, and some scholars consider him the best mathematician to date. But in addition to his mathematical prowess, his insight into the physical world was no less a part of his greatness. Newton invented and applied a broad range of mathematics both to describe natural phenomena as well as to gain a deeper insight into it. He is also considered to be one of the greatest physicists who ever lived. Probably the only other man in history who could rival Newton's contributions to both mathematics and physics was the Greek mathematician and inventor Archimedes (287 B.C.E.–212 B.C.E.).

Sir Isaac Newton developed the system of advanced mathematical calculation that is now called calculus. And through his experiments with light beams (shown here), he also discovered that white light is composed of a spectrum of colors.

Newton was born Christmas Day, 1642, on a farm in Woolsthorpe, England. Newton does not seem to have enjoyed a happy childhood. His father had died three months before he was born. While still a small boy, his mother remarried and Isaac was reared for some time by a grandmother. Nevertheless, Newton showed some academic promise, and he entered Trinity College at age 18. It is thought that his education there was initially leading to a law degree. Newton did not distinguish himself academically, but his thirst for knowledge can be seen in this excerpt from his college notes: "Plato is my friend, Aristotle is my friend, but my best friend is truth." Eventually an interest in mathematics was sparked, and Newton read both contemporary books on mathematics as well as classical works like Euclid's *Elements.*

Newton quickly went from reading about mathematics to developing it. By age 23, Newton had devised the **binomial theorem** and formed a framework for what is known today as **differential calculus**. At about this time Newton left for home because the bubonic plague had closed Trinity College. Over the next year or so Newton further developed his calculus, laid the foundation for his theory of gravitation, and performed optical experiments. The latter led Newton to pronounce that white light is actually composed of a combination of many colors.

binomial theorem a theorem giving the procedure by which a binomial expression may be raised to any power without using successive multiplications

differential calculus the branch of mathematics primarily dealing with the solution of differential equations to find lengths, areas, and volumes of functions

The invention of infinitesimal calculus, what he termed as the "method of fluxions," is Newton's greatest claim to fame. (Somewhat later than Newton, but independently, Gottfried Wilhem Liebniz [1646–1716] also developed an infinitesimal calculus.) Infinitesimal calculus encompasses differential calculus and integral calculus. Newton was the first to understand that although the purpose of each seems quite different, differentiation and integration are inverse procedures of one another.

tangent a line that intersects a curve at one and only one point in a local region

radius the line segment originating at the center of a circle or sphere and terminating on the circle or sphere; also the measure of that line segment

In his book *The Method of Fluxions and Infinite Series*, Newton considered the curve generated by a point in motion. Given an arbitrary curve formed in this way, Newton used his methods of "fluxions," now called differentiation, to find (calculate) the **tangent** and **radius** of curvature at any point on that curve. Newton went on to produce many simple analytical methods to solve apparently unrelated problems, such as finding the areas of various figures, the lengths of curves, and the maxima and minima of functions.

Though so many of his discoveries were of great importance to science and the world at large, Newton was reluctant to publish his findings. For example, his book *The Method of Fluxions and Infinite Series*, though written in 1671, was not published until 1736, well after his death. Fortunately, Newton's friend Edmund Halley encouraged Newton to write and publish (at Halley's expense) the *Philosophiae naturalis principia mathematica (Mathematical Principles of Natural Philosophy)*, which includes the formulation of Newton's three laws of motion and his Universal Theory of Gravitation. The *Principia*, as it is widely known, became Newton's masterpiece, and it made him internationally prominent. SEE ALSO ARCHIMEDES; CALCULUS; EUCLID AND HIS CONTRIBUTIONS; LIMIT.

Philip Edward Koth (with William Arthur Atkins)

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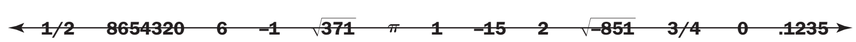
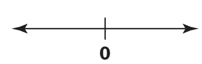
Number Line

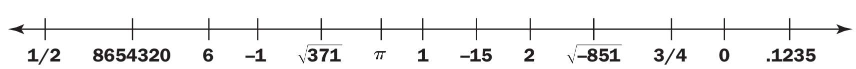
What is a number line? You might reasonably think that it is a line with numbers on it, like this:



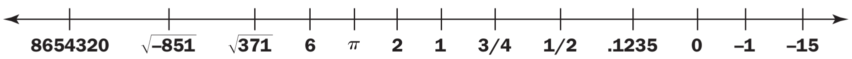
But when mathematicians draw a number line, the numbers are not actually *on* the line. A number's location is marked like this:



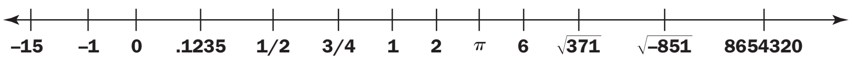
On a number line, a number represents a point on the line, and every point on the line represents a number. So you might think a number line would look like this:



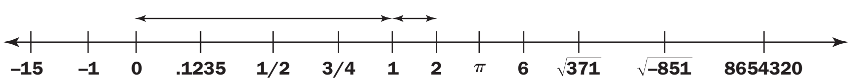
However, this is not quite correct. The numbers need to be in order from smallest to largest, or, as a mathematician might say, from least to greatest. The next line shows this progression from right to left.



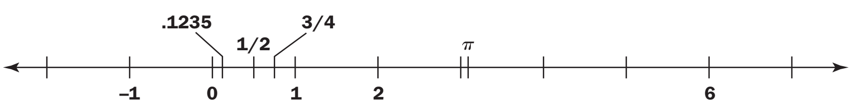
But the preceding example is still not the number line used in mathematics. The conventional way to draw a number line is horizontal, with numbers increasing from left to right.



But something is wrong with this line. Look at 0, 1, and 2. Notice how the distance between 0 and 1 is bigger than the distance between 1 and 2.



Distances between numbers should be consistent. The difference between 0 and 1 is 1, the same as the difference between 1 and 2. Therefore, the distance between 0 and 1 should be same as the distance between 1 and 2. Let's try again.



But this line seems to have lost some numbers. Actually, once the size of the unit was chosen (as the distance between 0 and 1) it was not possible to fit all of the numbers previously shown on this number line.

You can pick any unit you like for a number line. That is why we speak of *a* number line, rather than *the* number line. You can draw many number lines, but they all have the characteristics discussed so far. You can make the unit on a number line one millimeter or one mile. However, you should probably choose a unit that works for what you are doing.

For example, you would have had to have chosen a small unit to fit in all the numbers from -15 to 8,654,320. The unit chosen for the last number line drawn in the preceding discussion allows most of your numbers to fit and is large enough to show the difference between 0 and .1235, the two closest numbers.

It is clear that the distance from 0 to 1 is one unit on a number line, but what about the distance from 1 to 0? Going backwards, the distance is the same, but $0 - 1 = -1$.

For this situation on a number line, you can use the absolute value function, so it does not matter which direction you go when subtracting numbers to measure distance.

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Applying the absolute value function gives $|1 - 0| = |0 - 1| = 1$.

So a number line is a line with numbers represented by points on the line located at a distance from 0 equal to the number in some arbitrarily chosen unit (typically, the distance from 0 to 1). The unit distance between two points on the number line is the absolute value difference between the numbers representing those two points.

Number lines have arrows on both ends. The arrows show that a number line goes on forever in both the positive and negative directions. You can always take a very large counting number, such as 123,456,789,102,131,141 and make a still larger counting number by adding 1. In the same way, all the numbers on a number line keep going without stopping or having a largest number. Any number that can have a point on a number line is called a real number, and all the numbers in the set of real numbers have a corresponding point on a number line. Such numbers as π and $\sqrt{2}$ are real numbers, and you can find points that correspond to them on a number line. Imaginary numbers, such as $\sqrt{-1}$ are not real numbers, and you cannot find a point for them on a real number line such as the ones described here. SEE ALSO Absolute Value; Integers; Number System, Real; Numbers, Complex.

Stanislaus Noel Ting

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Number Sets

Numbers are at the heart of mathematics and have been a source of fascination and curiosity for mathematicians, astronomers, scientists, and even theologians, magicians, and astrologers. Some mathematicians pursue this fascination in a field of mathematics called number theory, in which number sets are used.

The emergence of the whole numbers 0, 1, 2, 3, 4, . . . naturally stems from the fundamental idea of counting. If a group of bananas and pineapples can be paired off, one banana for every pineapple, then the two groups, or sets, have the same "number" of objects. The members of the two sets do not resemble each other in appearance—one is the set of bananas, whereas the other is a set of pineapples—but both sets share a common property.

The common property that results from comparing the "size" of sets leads to the counting numbers. Any two sets whose members can be paired

off has the same "number" of elements and this equality in "size" of the two sets is denoted by an appropriate numeral: 1, 2, 3, ...

Whole numbers are adequate for counting, but they are not adequate for many other purposes. For instance, dividing an odd number of objects, like 3, into an even number of parts is not possible with just whole numbers. The need to evenly divide odd number of objects leads to fractions like $\frac{3}{2}$, $\frac{5}{2}$, and $\frac{7}{2}$. The fraction $\frac{7}{5}$ means dividing 7 pieces in 5 equal parts. But what does $\frac{3}{4}$ indicate? Does this mean dividing 3 pieces in 4 equal parts? Another interpretation is to think of $\frac{3}{4}$ as taking 3 equal pieces of something that has already been divided into 4 equal parts.

In mathematics, numbers are removed from particular descriptions of objects or sets. For instance, 7, $\frac{3}{4}$, and $\frac{5}{3}$ are numbers that can stand alone without describing objects or sets quantitatively. Some primitive languages have different names for a number of particular objects. For example, in a language spoken by Fiji Islanders, ten boats is "bolo," but ten coconuts is "koro." The idea of removing number from particular objects is called "abstraction"—the essential idea of number is removed, or abstracted, from counting a group of particular objects.

The Greeks made another important discovery that forced mathematicians to extend the number system beyond fractions. The length of the diagonal line of a square with sides of 1 unit cannot be expressed as a whole number or fraction. They called the length of the diagonal an "incommensurable quantity."

Using the **Pythagorean Theorem**, the length of the diagonal line equals $\sqrt{2}$. Mathematicians further extended the number set to include numbers of the form $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, and so on. In decimal notation, these numbers have a nonterminating and nonrepeating form; $\sqrt{2}$ is 1.41421. . . and the numbers after the decimal point do not end or contain a repeating pattern. These numbers are called irrational because they cannot be written as the ratio of two integers. A simple nonterminating but repeating decimal is 0.333. . . . In fractions, 0.333. . . is expressed as $\frac{1}{3}$, and therefore it is a rational number.

Pythagorean Theorem a mathematical statement relating the sides of right triangles; the square of the hypotenuse is equal to the sums of the squares of the other two sides

The set of all rational and irrational numbers is called the real number set, **R.** Are real numbers adequate for all mathematical needs? Consider the equation: $x^2 = -1$. The equation requires a number x whose square is negative one. But the square of all real numbers is positive. Therefore, the equation, $x^2 = -1$ has no solution in the real number set. But that did not stop mathematicians from further extending the number set.

Suppose i is a number whose square is negative 1. The number i, with the defined property of $i^2 = -1$, is called an imaginary number. This name is unfortunately misleading and an historic accident. The number i is no less a number than -3, $\sqrt{2}$, or $\frac{4}{7}$. In fact, the name "real" numbers is also misleading because numbers that are not in the set of real numbers are just as "real" as any other number. The following is a summary of the sets of numbers.

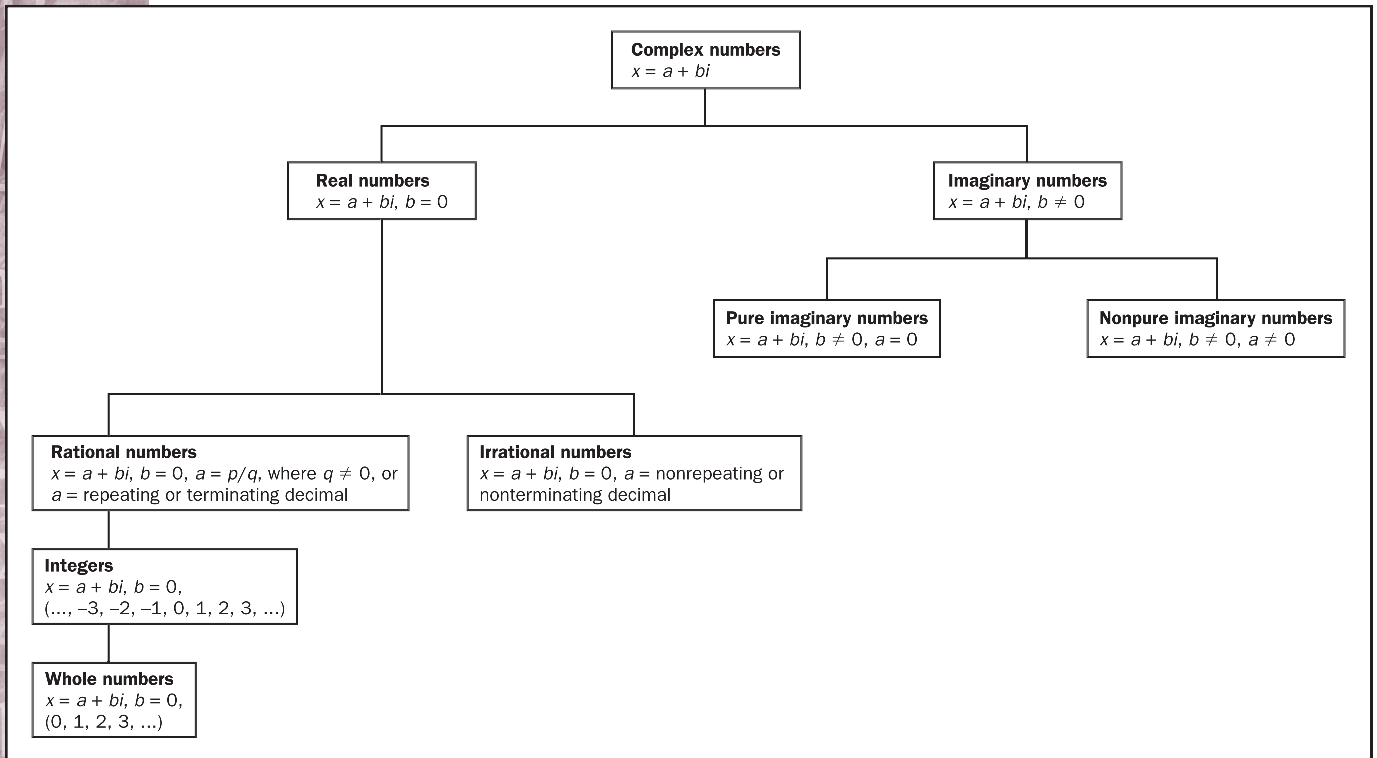

Chart of number systems.

Whole Numbers

The set of whole numbers, **W,** consists of the counting numbers and zero: {0, 1, 2, 3, . . .}. Every whole number can be generated from 1 and the operation of addition: $2 = 1 + 1$, $3 = 2 + 1$, and so on. The sum of two whole numbers is also a whole number. Mathematicians describe this property by saying that the whole number set is closed under addition. In symbols, the closure property is described as the following: If $a, b \in \mathbf{W}$, then $a + b \in \mathbf{W}$. The whole number set is also closed under multiplication, but it is not closed under subtraction because the result of $3 - 7$ is not a whole number.

Integers

The integer number set contains the positive counting numbers, their corresponding negative numbers, and 0. The integer number set **I** is written as $\{. . . ,-2, -1, 0, 1, 2,. . .\}$ or $\{0, \pm 1, \pm 2,. . .\}$. In other words, the set of integers consists of the set of whole numbers and their negatives. Like the whole number set **W, I** is closed under addition and multiplication. But **I** is also closed under subtraction. For instance, $3 - 9$ is -6, which is an integer. However, **I** is not closed under division. Dividing 8 by 4 equals 2, which is an integer, but dividing 7 by 2 does not equal an integer.

Rational Numbers

All numbers of the form *a/b* where *a* and *b* are integers and *b* is not equal to 0 are called rational numbers. Rational numbers can be written as the ratio of integers. The rational number set, **Q,** is closed under all the four operations: addition, subtraction, multiplication, and division (provided division by 0 is excluded). All rational numbers have a terminating or nonterminating-but-repeating decimal form.

Irrational Numbers

Numbers like $\sqrt{2}$ and π that cannot be expressed as a ratio of integers are called irrational numbers. In decimal form, irrational numbers have a nonterminating and nonrepeating form. The sum of two irrational numbers is also an irrational number, so the irrational set is closed under addition. But the irrational number set is not closed under multiplication; for example, consider the product of two irrational numbers, $\sqrt{2}$ and $3\sqrt{2}$. Their product is $3\sqrt{4}$ or 6, which is not an irrational number.

Real Numbers

The real number set, **R,** is constructed by combining the sets of rational and irrational numbers. In the language of sets, **R** is the union of the rational number set and irrational number set.

Complex Numbers

A number of the form $a + bi$, where a and b are real numbers and $i^2 = -1$, is called a complex number. For $a = 0$ and $b = 1$, the result of $a + bi$ is the imaginary number, i. All real numbers are also complex numbers. For instance, $\sqrt{2}$ can be expressed as $\sqrt{2} + 0i$. Therefore, by taking $b = 0$, all real numbers can be expressed as complex numbers. A nontrivial example of a complex number is $5 + 6i$.

Imaginary Numbers

An imaginary number is that part of a complex number which is an even root of a negative number. Examples are $\sqrt{-2}$ or $\sqrt[6]{-15}$. Imaginary numbers typically are represented by the constant, i. Like real numbers, imaginary numbers have relative magnitude and can be plotted along a number line. Imaginary numbers may also be negative, with $3i > i > -i > -3i$. SEE ALSO Integers; Numbers, Complex; Numbers, Irrational; Numbers, Rational; Numbers, Real; Numbers, Whole.

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Number System, Real

The question "How many?" prompted early civilizations to make marks to record the answers. The words and signs used to record how many were almost surely related to our body parts: two eyes, five fingers on one hand, twenty fingers and toes. For instance, the word "digit," which we use for the symbols that make up all our numerals (0, 1, 2, 3, 4, 5, 6, 7, 8, 9), is the Latin word for finger.

This ancient Egyptian calendar with Zodiac symbols was used by Greek scholars in Alexandria for astronomical calculations.

These first numbers are now called the set of counting numbers: {1, 2, 3,. . .}, and sometimes this set is called the natural numbers. Notice that the counting numbers do not include 0. Whereas some early cultures, including the Egyptians, the Chinese, and the Mayans, understood the concept of zero, the digit 0 did not appear until some time after the other nine digits.

In the earliest development of counting, many languages used "one, two, many," so that the word for three may have meant simply "much." People everywhere apparently developed counting systems based on repeating some group, as in counting "one group of two, one group of two and one, two groups of two." We know that a scribe in Egypt used number signs to record taxes as early as 2500 B.C.E.

Hundreds of unbaked clay tablets have been found that show that the Babylonians, in the region we know today as Iraq, were using marks for one and for ten in the 1,700 years before the birth of Christ. These tablets show that the idea of place value was understood and used to write numbers. Number signs were needed not only to count sheep or grain but also to keep track of time.

Many civilizations developed complex mathematical systems for astronomical calculations and for recording the calendar of full moons and cycles of the Sun. These earliest records included fractions, or rational numbers, as well as whole numbers, and used place value in ways similar to the way decimal fractions are written today. In a manuscript, possibly from the sixth century, fractions were written with the numerator above and the denominator below, but without the division line between them. The bar used for writing fractions was apparently introduced by the Arabs around 1000 C.E.

Early forms of the Arabic-Hindu numerals, including 0, appeared sometime between 400 C.E. and 850 C.E., though recent evidence suggests that 0 may have been invented as early as 330 B.C.E. The zero sign began as a dot. It is possible that the late development of 0 was because people did not see zero as a meaningful solution when they were doing practical problems.

About 850 C.E., a mathematician writing in India stated that 0 was the identity element for addition, although he thought that division by 0 also resulted in a number identical to the original. Some 300 years later, another Hindu mathematician explained that division by 0 resulted in infinity.

Number rods were used by the Chinese as a computational aid by 500 B.C.E. The Koreans continued to use number rods after the Chinese and the Japanese had replaced the counting rods with beads in the form of an abacus. Red rods represented positive numbers, and black rods represented negative numbers.

The book *Arithmetica*, by Diophantus (c. 250 C.E.), calls an equation such as $4x + 20 = 4$ "absurd" because it would lead to $x = -4$. Negative numbers are mentioned around 628 C.E. in the work of an Indian mathematician, and later they appear in all the Hindu math writings. Leonardo Pisano Fibonacci, writing in 1202, paid no attention to negative numbers. It was not until the Renaissance that mathematics writers began paying attention to negative numbers.

The idea of letting a variable, such as *a* or *x*, represent a number that could be either positive or negative was developed around 1659. The neg-

ative sign as we know it began to be used around 1550, along with the words "minus" and "negative" to indicate these numbers.

The idea of square roots, which leads to irrational numbers such as $\sqrt{2}$, apparently grew from the work of the Pythagoreans with right triangles. Around 425 B.C.E., Greeks knew that the square roots of 3, 5, 6, and 7 could not easily be measured out with whole numbers. Euclid, around 300 B.C.E. classified such square roots as irrational; that is, they cannot be expressed as the ratio of two whole numbers.

The history of the development of human knowledge of the real numbers is not clearly linear. Different people in widely separated places were thinking and writing about mathematics and using a variety of words and notations to describe their conclusions. The development of numbers that are not real—that is, of numbers that do not lie on what we today call the real number line—began around 2,000 years ago.

The square root of a negative number, which leads to the development of the complex number system, appears in a work by Heron of Alexandria around 50 C.E. He and other Greeks recognized the problem, and Indian mathematicians around 850 stated that a negative quantity has no square root. Much later, in Italy, after the invention of printing, these roots were called "minus roots."

In 1673, Wallis said that the square root of a negative number is no more impossible than negative numbers themselves, and it was he who suggested drawing a second number line perpendicular to the real number line and using this as the imaginary axis. SEE ALSO Calendar, Numbers in the; Integers; Mathematics, Very Old; Number Sets; Numbers and Writing; Numbers, Complex; Numbers, Irrational; Numbers, Rational; Numbers, Real; Numbers, Whole; Radical Sign; Zero.

Lucia McKay

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Numbers: Abundant, Deficient, Perfect, and Amicable

What are "abundant" numbers? Why would such numbers be called plentiful? "Abundant" is a strange way to describe a number, and equally strange are descriptions such as "deficient," "perfect," and "amicable." But these descriptions of numbers came about because the ancient Greek mathematicians were intrigued by certain characteristics of positive **integers**.

integer a positive whole number, its negative counterpart, or zero

The Greeks discovered, for example, that some numbers are equal to the sum of their divisors; for instance, 6 is equal to the sum of its proper divisors 3, 2, and 1. (Although 6 is a divisor of 6, it is not considered a

"proper" divisor.) Greek mathematicians discovered a sense of balance or perfection in such numbers, and labeled them "perfect."

As an extension of the idea of perfect numbers, the concept of "abundant" and "deficient" numbers emerged. If the sum of the proper divisors of a number is greater than the number itself, then the number is called abundant or excessive. The proper divisors of 12 are 1, 2, 3, 4, and 6. Because the sum of its proper divisors ($1 + 2 + 3 + 4 + 6 = 16$) is greater than 12, 12 is an abundant number. Numbers like 8, whose proper divisors have a sum that is less than the number itself, are called deficient or defective.

The Greeks, who regarded the proper divisors of a number to be the number's "parts," were the first to refer to perfect numbers—numbers that are the exact sum of their parts. Later, the philosopher Nicomachus,* in his *Introduction to Arithmetic*, would coin the terms "abundant" and "deficient," attaching moral qualities to these numbers. From a mathematical point of view, abundant, deficient, and perfect numbers are "abundant," "deficient," and "perfect" only in how the sum of their proper divisors compares to the numbers themselves.

***Nicomachus' treatise was the first work to treat arithmetic independent of geometry.**

Abundant Numbers

Twelve is the first abundant number. The next abundant number is 18 because the proper divisors sum to 21 ($1 + 2 + 3 + 6 + 9$). The first five abundant numbers are 12, 18, 20, 24, and 30. As it turns out, the twenty-one abundant numbers under 100 are all even. Not all abundant numbers, however, are even; the first odd abundant number is 945.

Every multiple of an abundant number is itself abundant, so there is an infinite number of abundant numbers. In 1998, the mathematician Marc Deleglise showed that roughly one-quarter of all the positive integers are abundant.

Deficient Numbers

Deficient numbers occur more frequently than abundant numbers. In other words, the sum of the proper divisors of most numbers is less than the numbers themselves. Examples of deficient numbers include 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, and 23.

Perfect Numbers

Perfect numbers, which occur infrequently, are the most interesting of the three types. The ancient Greeks apparently knew the first four: 6, 28, 496, and 8,128. The fifth perfect number, however, is 33,550,336, and the sixth is 8,589,869,056. As of June 1999, the thirty-eighth and largest known perfect number is $2^{6972592}(2^{6972593} - 1)$, a number roughly 4 million digits long!

Perfect numbers have some interesting properties. For example, the sum of the reciprocals of the divisors of a perfect number is always equal to 2. Consider the sum of the reciprocal of the divisors of 6 (1, 2, 3, and 6) and 28 (1, 2, 4, 7, 14, and 28).

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 2$$

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{7} + \frac{1}{14} + \frac{1}{28} = 2$$

The Historic Search. The Pythagoreans encountered the oldest problem in number theory—the problem of finding all of the perfect numbers. Euclid was the first to produce a mathematical result concerning perfect numbers. In 300 B.C.E., he proved in his book *Elements* that any number of the form $2^{n-1}(2^n - 1)$ is a perfect number whenever $2^n - 1$ is a prime number (a number that has no proper divisors except 1). In 1757, the Swiss mathematician Euler made further progress by proving the converse of this statement: every even perfect number is of the form $2^{n-1}(2^n - 1)$, where $2^n - 1$ is a prime number.

Primes of the form $2^n - 1$ are mathematically interesting in their own right; they are known as "Mersenne primes," named after the French monk Marin Mersenne. Because of the results of Euclid and Euler, every Mersenne prime corresponds to a perfect number and vice versa. Thus, the problem of finding all the even perfect numbers is reduced to the problem of finding all the Mersenne primes.

Mathematicians have conjectured, but not proven, that there is an infinite number of Mersenne primes and hence an infinite amount of even perfect numbers. It is not known if there are any odd perfect numbers. The largest perfect number currently known, $2^{6972592}(2^{6972593} - 1)$, was discovered by using computer programs, and this number corresponds to the thirty-eighth known Mersenne prime.

THOSE EVASIVE PERFECT NUMBERS

Despite being tackled by such great mathematicians as Pierre de Fermat and Leonhard Euler, the problem of finding all the perfect numbers remains unsolved today. Although mathematicians have been unable to solve this problem, their attempts to do so have resulted in many advances in number theory.

Amicable Numbers

Related to abundant, deficient, and perfect numbers are *amicable numbers*. Amicable numbers are described in pairs. Two numbers are amicable if the sum of the proper divisors of each number equals the other number in the pair. The first pair of amicable numbers is 220 and 284. The sum of the proper divisors of 220 is $1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$, whereas the sum of the proper divisors of 284 is $1 + 2 + 4 + 71 + 142 = 220$.

Amicable numbers occur infrequently; there are only a total of 350 amicable pairs whose smaller number has less than ten digits. The amicable numbers were also first noted by the Pythagoreans, who would be the first, among many, to attach mystical qualities to these "friendly" numbers.

There currently are more than 7,500 known pairs of amicable numbers. There is no mathematical formula describing the form of all of the amicable numbers. As with perfect numbers, today's searches for amicable pairs continue on computers. It is conjectured that the number of amicable pairs is infinite. SEE ALSO Euclid and His Contributions; Numbers, Forbidden and Superstitious; Primes, Puzzles of; Pythagoras.

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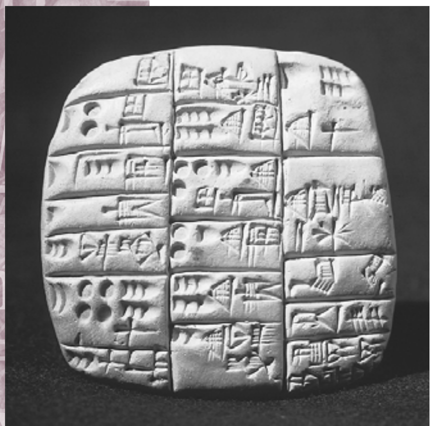

This Sumerian clay tablet from ancient southern Mesopotamia shows a tally of sheep and goats using incised cuneiform characters.

Numbers and Writing

Denise Schmandt-Besserat made the great discovery that the origins of writing are actually found in counting. This discovery began as most groundbreaking work does, with an unrelated pursuit. The University of Texas at Austin professor began her academic career in the 1960s. At that time, the main concern was not where writing came from but little bits of clay and their role in ancient Middle Eastern life. At nearly every Middle Eastern archaeological site, researchers found these perplexing little pieces of fired clay.

Though it was sure that these bits of clay played an important role in ancient civilization, for years no one knew what they were. Then along came Denise Schmandt-Besserat. The French-born graduate student, through a number of fellowships and grants, began what ended up being more than two decades of combing archives and sites all over the world trying to discover, and then to prove, what the clay cones, cylinders, spheres, and disks might have been. She referred to them as "tokens."

Since her discovery, Schmandt-Besserat has written article after article to build her ironclad case. She has explained a mystery that frustrated archaeologists, anthropologists, and philosophers for years.

She found that the tokens made up an elaborate system of accounting that was used throughout the Middle East from around 8000 to 3000 B.C.E. Each token represented a certain item, such as a jar of oil or a sheep. The tokens were used to take inventory and keep accounts. The ancient people sealed the tokens in clay envelopes, and then the envelopes were marked with the debtor's personal "cylindrical seal." The seal acted as a kind of signature.

After using this system for some time, a new one emerged. People began to impress the token into the side of the envelopes before sealing them. This way they would not have to break the seal, and thus the bargain, to check the contents of the envelope. It later occurred to people that they did not need to put the tokens in the envelopes at all. They could just impress the image from the tokens onto the clay so they could keep track of the account.

Another later change also occurred. The ancient Sumerians realized it was possible to simply inscribe the image on the token. They used a stylus to do the inscribing. This served as the earliest type of written sign. Schmandt-Besserat had found her answer to the mystery of how writing began: from counting.

Her theory helped solve another mystery. Schmandt-Besserat had explained why so many pictographs did not look like what they were supposed to represent. She filled in the missing link between the object and its sign. For example, a sheep was a circle with an X on it. It became apparent that the mark was not supposed to represent a sheep; it was supposed to represent the counter for the sheep.

The origin of writing has always been a sensitive subject. It is a far more heated area of debate than the origin of counting. Schmandt-Besserat's findings have proven that writing began with counting. She has also dealt with the transition from writing as accounting to writing as literature.

Max Brandenberger

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Numbers, Complex

The set of complex numbers includes all the numbers we commonly work with in school mathematics (whole numbers, fractions, decimals, square roots, etc.), plus many more numbers that are generally not encountered until the study of higher mathematics. Complex numbers were invented centuries ago in order to provide solutions to certain equations that previously had seemed impossible to solve.

Imagine trying to find a solution to $x + 6 = 4$ but being able to look for a solution only in the set of whole numbers. This is impossible. However, if we expand our domain to all integers, -2 provides a solution. Similarly, it is impossible to find a solution to $2x = 7$ using only integers, but we can expand our domain to the set of rational numbers, and $x = \frac{7}{2}$ or 3.5 provides a solution. Now suppose you wanted to find a solution to $x^2 = 2$ using only rational numbers. This, too, is impossible. However, the set of rational numbers can be expanded to create still another new set of numbers—the real numbers. Clearly, $\sqrt{2}$ is one solution to the equation $x^2 = 2$ because by definition the square root of any number multiplied by itself equals the number: $(\sqrt{2})^2 = 2$. Another solution is $-\sqrt{2}$ because it is also true that the negative square root of a number multiplied by itself equals the number: $(-\sqrt{2})^2 = 2$.

In about 50 C.E. another seemingly impossible problem emerged when Heron of Alexandria, a Greek mathematician, was trying to evaluate the square root of a negative number. Consequently, the square root of a negative number cannot be evaluated using only real numbers. To find a solution, another new number system needed to be invented. In the mid-sixteenth century an Italian mathematician named Girolamo Cardano began to do just that. He is recognized as the discoverer of the imaginary numbers that play an essential role in understanding the complex number system.

Imaginary Numbers

Contrary to their name, imaginary numbers are not imaginary at all. Imaginary numbers were invented in an effort to evaluate negative square roots, a mathematical operation that before their existence was impossible. Thus, the argument to justify the existence of imaginary numbers is similar to the argument for the existence of integers, rational numbers, and real numbers.

Leonhard Euler, an eighteenth-century Swiss mathematician known for his prolific writing in mathematics and his standardization of modern mathematics notation, chose the symbol i to stand for the square root of -1. Since that time, the imaginary number set has included i and any real number times i. So, for example, $\sqrt{-16} = i \times 4$ or $4i$; $\sqrt{-25} = i \times 5$ or $5i$; and $\sqrt{-3} = \sqrt{3} \times i$ (which we write as $i\sqrt{3}$).

It is interesting and important to observe the behavior of i when it is multiplied by itself. To begin, i multiplied by itself is i^2 (also written as $\sqrt{-1} \times \sqrt{-1}$, which is -1). So $i^2 = -1$. To continue this line of reasoning,

$i^3 = -i$ (because i^3 can be written as $i^2 \times i$ and thus $-1 \times i$ or $-i$). Similarly, $i^4 = 1$ (because it can be thought of as $i^2 \times i^2$ or -1×-1). Continuing, we find that $i^5 = i$, and this is where the pattern begins again (i, -1, $-i$, 1).

Incredibly, by using imaginary numbers it is possible to solve many equations that were deemed impossible for centuries. Consider $x^2 + 4 = 0$. Using algebraic manipulation (subtracting 4 from both sides of the equals sign), the equation becomes $x^2 = -4$, and x can be either $\sqrt{-4}$ or $-\sqrt{-4}$. So $x = 2i$ or $-2i$. To check your answer, substitute the $2i$ or $-2i$ for x. So, for example, the equation becomes $(2i)^2 + 4 = 0$, which is $(2^2 \times i^2) + 4 = 0$. Because $i^2 = -1$ and $2^2 = 4$ then $(4 \times -1) + 4 = 0$, or $-4 + 4 = 0$.

Complex Number System

Carl Friedrich Gauss, a nineteenth-century German mathematician, is credited with inventing and naming the complex number system. Complex numbers are generally expressed in the form $a + bi$, where a and b are real numbers and i is the imaginary number described above (that is, $i = \sqrt{-1}$). The a part is considered the real part of the complex number and the bi part is the imaginary part of the complex number. Upon further inspection, we can see that the set of complex numbers includes all the pure real numbers, together with all the pure imaginary numbers, together with many more numbers that are sums of these. In other words, whenever a complex number has $b = 0$, it is actually a pure real number too because it is equal to $a + 0i$, which is just a (a real number). Whenever a complex number has $a = 0$, it is actually a pure imaginary number because it is equal to $0 + bi$, which is just bi, an imaginary number.

The complex number system consists of all complex numbers, $a + bi$ (where a and b are real numbers), together with the rules that define the four basic operations on this set of numbers (addition, subtraction, multiplication, and division). Indeed, in order to define any number system, there are certain rules that must be obeyed. First, addition and multiplication must be well defined (that is, it must be clear how to add and how to multiply any two numbers in the set). Addition of complex numbers occurs by adding their real parts and their imaginary parts separately. For example, $(4 + 2i) + (6 + 3i) = (4 + 6) + (2i + 3i) = 10 + 5i$. In general, $(a + bi) + (c + di)$ is equal to $(a + c) + (b + d)i$. Subtraction is performed similarly.

To multiply any two complex numbers, use the distributive property (in a method sometimes referred to in elementary algebra classes as the "foil" method) and then combine real terms and imaginary terms separately. Thus, $(a + bi) \times (c + di) = ac + (ad)i + (bc)i + (bi)(di) = (ac + bd) + (ad + bc)i$. For example, $(4 + 5i)(6 + 3i) = 24 + 12i + 30i + 15i^2 = 24 + 42i - 15 = 9 + 42i$. Remember from previous discussions that $i^2 = -1$. Multiplying a complex number by a constant is a simpler case of this process in which the distributive property is similarly invoked. For example, $4(4 + 2i) = 16 + 8i$. In general, $c(a + bi) = ac + (bc)i$.

Every number system also must have both an additive identity and a multiplicative identity—that is, numbers that when added to (or multiplied by) any number in the set produce the same number started with. For the complex numbers—just as for the whole numbers, the integers, the rational

numbers, and the real numbers—the additive identity is 0 and the multiplicative identity is 1.

For any number system, there must also be both additive and multiplicative inverses—that is, numbers that when added to (or multiplied by) a number in the set produce the additive (or multiplicative) identity, respectively. For the real numbers, the opposite of any number is its additive inverse. That is, $-a$ is the additive inverse of a (since, for example, $-4 + 4 = 0$, and 0 is the additive identity). For the complex numbers, we must take the opposite of both the real and imaginary parts of a number to find its inverse. Thus, $(-a - bi)$ is the additive inverse of $(a + bi)$ (because adding these together gives $0 + 0i$, or simply 0, the additive identity).

Finally, as for the real numbers, multiplication and addition of complex numbers must be commutative, associative, and distributive. This is indeed the case because the operations in complex numbers are based on the operations in real numbers.

Dividing any complex number $a + bi$ by a real number (say, r) is done by dividing each part of the number by r. Thus, $(a + bi) \div r = a/r + (b/r)i$. To divide a complex number by a complex number is somewhat more complicated. However, the process is similar, in some ways, to a process you probably learned for dividing by a decimal.

Consider the example, $135.5 \div 0.25$. To perform this division, you were probably taught to multiply both dividend and divisor by 100 before proceeding with the division. Thus, the problem is transformed to $13550 \div 25$. The purpose of performing this transformation is to create a new (easier) problem that will have the same answer as the original problem.

The process of dividing by a complex number proceeds in a similar fashion. To divide by a complex number, we choose first to multiply both dividend and divisor by something to make the divisor non-complex. In the case of division by a complex number, we choose to multiply by a number known as the "complex conjugate" of the divisor. The complex conjugate of any complex number $(a + bi)$ is simply the complex number $(a - bi)$. Multiplying these two numbers together produces a product that is not complex (that is, the product has imaginary part equal to 0).

For example, the complex conjugate of $2 + 3i$ is $2 - 3i$. When you multiply these two numbers you obtain 13 because $(2 + 3i)(2 - 3i) = 4 + 6i - 6i - 9i^2 = 4 - 9i^2 = 4 - 9(-1) = 4 + 9 = 13$. The product (13) has imaginary part equal to 0 because the middle (imaginary) terms cancel out.

Thus, if you want to perform $(3 + 4i) \div (2 + 3i)$, you must first multiply both of these numbers by $(2 - 3i)$ to produce an equivalent division problem: $(3 + 4i)(2 - 3i) \div (2 + 3i)(2 - 3i)$. This works out to $(18 - i) \div 13$. Now the division is easily performed by dividing both real and imaginary parts by 13, producing $\frac{18}{13} - (\frac{1}{13})i$ as the answer.

Complex conjugates also arise when finding the roots of **polynomials**. When a polynomial is factored, the total number of roots is always equal to the degree of the polynomial, as proven by the Fundamental Theorem of Algebra. For example, $x^2 + 5x + 6 = 0$ is an equation of degree 2 (since the highest power of x is 2). By factoring this equation into $(x + 3)(x + 2) = 0$, we can see that it has two solutions, -3 and -2. Thus, the polynomial $x^2 + 5x + 6$ has exactly two roots.

polynomials expressions with more than one term

It is quite possible, however, that some of the roots of a polynomial will not be real because (as seen above) some equations can be solved only by appealing to the set of complex numbers. For instance, a polynomial of degree 4 might have two real roots and two complex roots.

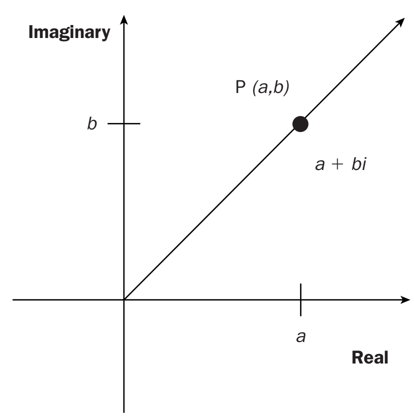
Consider, for example, the equation $x^4 - 16 = 0$. This can be factored into $(x^2 - 4)(x^2 + 4) = 0$. Thus, we see that either $x^2 - 4 = 0$ or $x^2 + 4 = 0$. The former is true only when $x = 2$ or -2. The latter is true only when $x = 2i$ or $-2i$. Thus, $x^4 - 16$ is a polynomial of degree 4 with two real roots and two imaginary (or complex) roots. In fact, complex solutions always come in pairs of complex conjugates. That is, whenever $a + bi$ is a root of a polynomial, then $a - bi$ will also be a root.

Geometric Representations of Complex Numbers

In the early nineteenth century, Jean Robert Argand, an amateur French mathematician; Caspar Wessell, a Norwegian cartographer; and Carl Friedrich Gauss, a German mathematician, all worked on developing geometric representations of complex numbers in a plane. Because complex numbers have both a real part and an imaginary part, a two-dimensional plane (rather than a one-dimensional line) is needed to plot them.

The real part and the imaginary part of a complex number can be written as a coordinate pair. For example, $3 + 2i$ can be written as (3,2), where the first coordinate represents the real part of the complex number and the second coordinate represents the imaginary part. Then, we can use this ordered pair to produce a geometric representation of the complex number. In fact, mathematicians use two different, but related geometric representations of complex numbers.

First, the number $a + bi$ or (a,b) can be thought of as the point $P(a,b)$ in the complex plane (or Argand diagram) by starting at the origin and plotting a point over a units and up b units. Or, second, (a,b) can be thought of as a vector from (0,0), the origin, to P in the complex plane.



Notice that construction of the complex plane is similar to that of the Cartesian plane. However, for the purpose of plotting complex numbers, the horizontal axis is considered to be the real axis and the vertical axis to be the imaginary axis. Thus, ordered pairs for complex numbers are plotted in the same manner as plotting ordered pairs of real numbers on a Cartesian plane.

Many relationships can be defined using a geometric model of complex numbers. For example, consider the notion of absolute value. Recall that for real numbers, the absolute value of any number a is just the distance of a from 0 (or, geometrically, the length of the segment from 0 to the point a on the number line). For example, the absolute value of -3 is 3, since -3 is 3 units from 0 on the number line. For the same reason, the absolute value of ± 13 is 13.

In the complex number system, the absolute value of a number $a + bi$ or (a,b) is defined in an analogous way as the distance from the origin (0,0) to the point $P(a,b)$ or length of the vector that (a,b) represents. Thus, the absolute value of $a + bi$ is determined by computing the length of the vector from the origin to P. This distance can be found by dropping a perpendicular line segment from $P(a,b)$ to the x-axis.

Doing so forms a right triangle whose hypotenuse is the length of the vector and whose horizontal leg has length a and whose vertical leg has length b. The length of the vector, c, can be computed using the Pythagorean Theorem ($c^2 = a^2 + b^2$). In other words, the absolute value of $a + bi$ is equal to the square root of $(a^2 + b^2)$. For example, the absolute value of the complex number $3 + 4i$ is the square root of $3^2 + 4^2$, which is $\sqrt{25}$, or 5.

Uses of Complex Numbers

Complex numbers are used both in the study of pure mathematics and in a variety of technical, real-world applications. When we study the real number system, many of its properties are easier to illustrate by looking at the real numbers within the more inclusive set of complex numbers. In other words, complex numbers are important in the study of number theory. Another example of the utility of complex numbers in mathematical study is that some functions, which generate fractals, contain complex numbers.

Complex numbers also have practical applications in technical fields. For instance, complex numbers are used in the study of electromagnetic fields. An electromagnetic field has both an electric and magnetic component, so two measures are required: one for the intensity of the electric field and one for the intensity of the magnetic field. Complex numbers help to describe the field's strength.

Electrical engineers use complex numbers to measure electrical current and to explain how electric circuits behave. Mechanical engineers use complex numbers to analyze the stresses of beams in buildings and bridges. Complex numbers appear when the engineers look for the **eigenvalues** and **eigenvectors** of the matrix that the engineers configure to explain numerically the stresses of the beams. SEE ALSO Euler, Leonhard; Integers; Number System, Real; Numbers, Rational; Numbers, Real; Vectors.

Rose Kathleen Lynch

eigenvalue if there exists a vector space such that a linear transformation onto itself produces a new vector equal to a scalar times the original vector, then that scalar is called an eigenfunction

eigenvector if there exists a vector space such that a linear transformation onto itself produces a new vector equal to a scalar times the original vector, then that vector is called an eigenvector

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The Math Forum. *Ask Dr. Math.* Swarthmore College. <http://forum.swarthmore.edu/dr.math/>.

Numbers, Forbidden and Superstitious

In trying to understand the world, mankind has created many superstitions and myths that have slowly led to what we consider modern science. For example, alchemy, the attempt to change other materials into gold, helped establish important principles of modern chemistry. The same is true of mathematics.

Many mathematical theories have been proposed over the centuries. Some have become the foundation of major branches of mathematics. Others have been disproved and discarded. But like many other superstitions, some erroneous theories have unfortunately persisted even after the science of mathematics has examined and invalidated them.

The Number 13

Thirteen frequently is called an unlucky number. Many modern buildings, particularly hotels, will not have a floor numbered as the thirteenth floor because so many people consider it unlucky and wish to avoid it. Of course, any building of sufficient height does have a thirteenth floor; it is simply skipped in the numbering of floors, which will go from 12 directly to 14. But this is enough either to satisfy or to delude most superstitious people. Many airlines will skip numbering across the thirteenth row.

Similarly, some race car drivers reject the number 13 to identify their cars, and some street numbers are altered to avoid the use of the number 13. The irrational fear of the number is so common that it even has a scientific name: triskaidekaphobia. The thirteenth day of the month is considered unlucky, particularly if it falls on a Friday, a day also considered unlucky for religious and cultural reasons. Throughout history and many different cultures, superstitious meaning has often been assigned to numbers and to groups and arrangements of numbers.

The reasons that the number 13 is considered unlucky derive from various sources. Some of those reasons come from religious references, such as the Last Supper, which was comprised of thirteen people, although other references seem to predate this. Other reasons have associations with mythology and the occult. Witches are said to gather in covens of thirteen, a belief found in the Teutonic mythology of Scandinavian folklore and widespread in the Middle Ages. Indeed, the occult has embraced the number 13 and now often uses thirteen candles or other forms of symbolism based on this number.

Many other numbers are believed to hold special meaning. The number 666 was claimed by the ancient Greeks to represent the mortal mind. In the Bible the number 666 is referred to as the number of the beast and has come to be known as a symbol of Satan and the Antichrist:

LUCKY 13?

Though the number 13 has many unlucky connotations, it is considered lucky to be born on the thirteenth day of the month. A child born on the thirteenth is expected to prosper in anything he or she begins on that day later in life.

> Rev 13:18 Here is wisdom. Let him that hath understanding count the number of the beast: for it is the number of a man; and his number is Six hundred threescore and six.

The belief that there is a hidden meaning in numbers is so prevalent that special meanings steadily continue to be assigned to numbers by different groups of people with different belief systems; hence, the meanings assigned vary widely. Numerology designates meanings to all numbers from 1 to 9, as well as to some other numbers such as 11, 22, and 33. The occult assigns different meanings to these same numbers. The number 13, an unlucky number to some and a satanic number to others, represents Mary, the mother of Christ in Christian belief.

Numerology

Numerology is a belief that all things can be expressed in numbers and that numbers have special meanings because of special "vibrations" that they give off or are associated with. The ancient Greek mathematician and religious leader Pythagoras of Samos is credited with having originated this pseudoscience and is believed to have assigned numbers to letters, adding the numeric values of each letter to find the symbolism of any word.

In the process of this numerology technique, the digits of a resulting number are added together until only a single digit remains (24 yields $2 + 4 = 6$). Numerology claims that when a name is assigned to anything, it instantly releases occult forces expressed in numbers. Pythagoras did study music and learn important concepts about vibration and frequency from this study, but he made an unfortunate leap in forming his conclusions in regard to numerology.

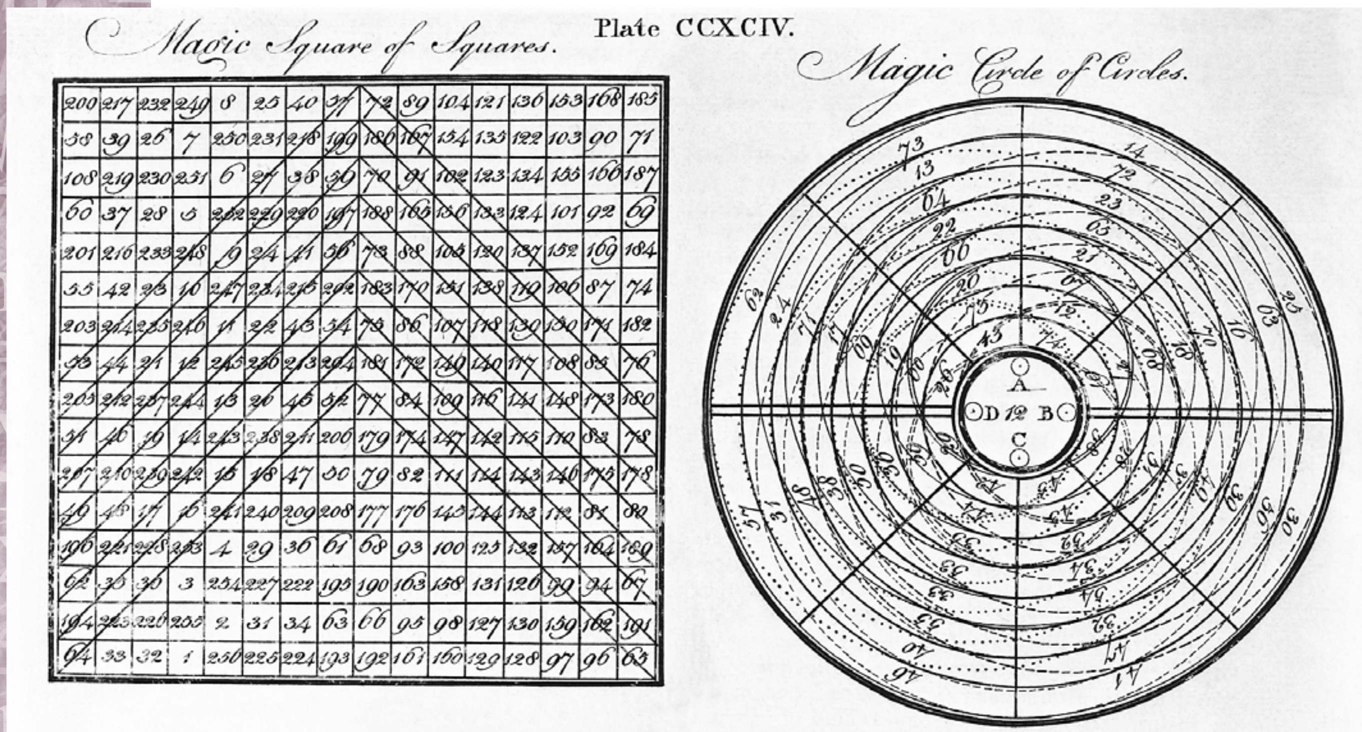
Numeric Value of Letters. When adding up the values of the letters in a name or other word, the resulting "revelations" are much like the information found in a horoscope. The number 1 indicates individuality, 2 indicates balance, and 3 is said to indicate acceptance of life as it comes. The number 4 indicates dependability, 5 a desire for freedom, and 6 denotes friendship. The number 7 is said to be an indication of a deep thinker, 8 indicates discipline, and 9 is said to be governed by love. Though science considers it groundless and no scientific basis for numerology has been shown, numerology continues to attract a wide range of disciples even in current times.

Magic Squares

An interesting and novel mathematical device is often called a magic square. It is a two-dimensional array of numbers in which each row and each column, as well as the major diagonals, all add up to the same number.

6	1	8
7	5	3
2	9	4

The above example is a simple third-order magic square in which the sum of 15 is found in any row, column, or major diagonal. The order of the magic square is simply its size along one side. Squares of any order can be constructed. Magic squares such as this one date back as far as ancient China



This eighteenth-century engraving shows two mathematical charts: the "Magic Square of Squares" (left) and the "Magic Circle of Circles" (right).

in 2200 B.C.E., where legend has it that Emperor Yu first saw a magic square on the back of a divine tortoise.

Magic squares have been a mathematical curiosity since then, and creation of larger order magic squares is an interesting diversion. In some cultures magic squares have been linked to spiritual or supernatural powers. They have been applied to astrology, which claims to be able to predict the future based on the positions of stars and celestial objects, asserting that a mysterious life energy originates in the cosmos and is controlled or filtered by nine stars. Magic squares also show up in some forms of the Oriental practice of Feng Shui, in which objects are arranged in a certain way to produce a mysterious "energy flow."

The Golden Ratio

In addition to the integers and rational numbers, mathematics includes many interesting irrational numbers. Pi (π), the ratio of the circumference of a circle to its diameter, is perhaps the best known. The letter *e*, used to represent 2.71828182845. . . , is another; it is the base of the natural logarithm and its inverse, the exponential function. Logarithms based on *e* have much more practical use than common logarithms built on a number base of 10 because the value *e* shows up in many real world rates of change. Another irrational number that has interested many cultures over the centuries is the irrational number Phi, also known as the golden ratio. The golden ratio can be illustrated by a line that is cut into two segments, A and B, in such a way that the ratio of the entire line to segment A is the same as the ratio of segment A to segment B.

Phi is the only positive number from which 1 can be subtracted to get its own reciprocal. This phenomenon is also seen in the Fibonacci sequence,

or any similar sequence: If one adds any two numbers to produce a third, adds the second and third to produce a fourth, and the fourth and fifth to produce a sixth, and so on, the ratio of each pair of adjacent numbers approaches the golden ratio.

There was much interest in Phi during the Renaissance, and since that time a considerable amount of literature has been written about it. Many misconceptions about the golden ratio have arisen. Probably the most repeated one is that it is the most pleasing ratio for a rectangle. While some modern architecture and art have been based on this statement, there seems to be no basis for it. Blind studies have, in fact, shown that when large numbers of people are asked to pick the most pleasing rectangle from a group of rectangles, those rectangles based on the golden ratio are selected no more often than other similar rectangles.

Some have tried to establish Phi as a basis of the works of Leonardo da Vinci, but there is no real evidence that Leonardo da Vinci used Phi as a basis for any of his works. Similarly, claims that the pyramids, the sphinx, and other ancient structures are based on the golden ratio cannot be confirmed. Some measurements taken to substantiate these claims may have come close to this ratio, but in those cases it seems more likely that those people who were trying to validate the claim were too willing to give unwarranted significance to the measurements.

Dangers of Superstitious Numbers

While there is no evidence that the numbers 13, 666, or others are unlucky, the belief that numbers have special metaphysical influence has affected math and science in very unfortunate ways. Galileo was put on trial twice by the Catholic church during the Inquisitions and forced to recant what he had learned of the solar system. The Catholic church was primarily upset that he contradicted their teaching that the Sun revolved around the Earth.

Another of Galileo's "transgressions" was his discovery of several moons of Jupiter, thus increasing the number of known observable objects in the solar system from seven to eleven. The church insisted that there had to be seven such objects because seven was a mystical number. While Galileo was able to retract his statements, others were not so lucky and were executed. SEE ALSO Fibonacci, Leonardo Pisano; Golden Section; Pythagoras.

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Numbers, Irrational

integer a positive whole number, its negative counterpart, or zero

The set of irrational numbers is the set of real numbers that cannot be expressed as the ratio, or quotient, of two **integers**. Thus, an irrational number cannot be written in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. A real number is irrational if its decimal representation is nonterminating and nonrepeating. Irrational numbers contrast with rational numbers, which *can* be expressed as the ratio of two integers. Every rational number, when changed to a decimal number, will be either a repeating or a terminating decimal number.

Irrational numbers are found everywhere. The square roots of natural numbers that are not perfect squares, such as $\sqrt{2}$, $\sqrt{5}$, $\sqrt{19}$, and $\sqrt{21}$, are irrational numbers. In any circle, the ratio of the distance around the circle to the distance across the circle is π, another irrational number. Finally, another instance of an irrational number is e, the base for natural logarithms, which is used in solving problems such as the population growth of bacteria and the rate of decay of radioactive substances such as uranium. The value of $e = 2.7182818284\ldots$ If you substitute large numbers for the value of n in the formula $(1 + \frac{1}{n})^n$, you can approximate the value of the irrational number, e. SEE ALSO Integers; Numbers, Rational; Numbers, Real.

Marzieh Thomasian

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Numbers, Massive

Many questions about computers are answered with numbers that have the prefixes “kilo,” “mega,” “giga,” and even “tera” in their name. An understanding of these prefixes, and the massive numbers they represent, will help consumers navigate their way through the world of computers and other technologies.

Understanding Massive Numbers

Massive numbers are part of the metric system. The French Academy of Science introduced the idea for this system in 1790. It was originally created to measure length, mass, volume, temperature, and time. In 1960, countries that used the metric system agreed on rules for its use. It is now called “SI,” which stands for *Système International d’Unités*—French for “International System of Units.”

byte a group of eight binary digits, representing a single character of text

In the SI system, a basic unit is determined. Other units are created by multiplying the basic unit by powers of ten. In the field of computers, for example, the basic unit of memory is the **byte**. Larger amounts of computer memory are designated by combining the “byte” unit with the prefixes kilo-, mega-, giga-, and so on.

All of the prefixes for numbers greater than the basic unit come from Greek or Latin. The list below shows the prefixes, their multiple, the power of ten, and their root. Prefixes for massive numbers make it easy to compare them.

Prefix	Multiple	Power of Ten	Root
yotta	septillion	10^{24}	Greek: octo—"eight"
zetta	sextillion	10^{21}	Latin: septem—"seven"
exa	quintillion	10^{18}	Greek: hex—"six"
peta	quadrillion	10^{15}	Greek:: penta—"five"
tera	trillion	10^{12}	Greek: teras—"monster"
giga	billion	10^{9}	Greek: gigas—"giant"
mega	million	10^{6}	Greek: megas—"great"
kilo	thousand	10^{3}	Greek: chilioi—"thousand"
hecto	hundred	10^{2}	Greek: hekaton—"hundred"
deka or deca	ten	10^{1}	Greek: deka—"ten"

Using Massive Numbers

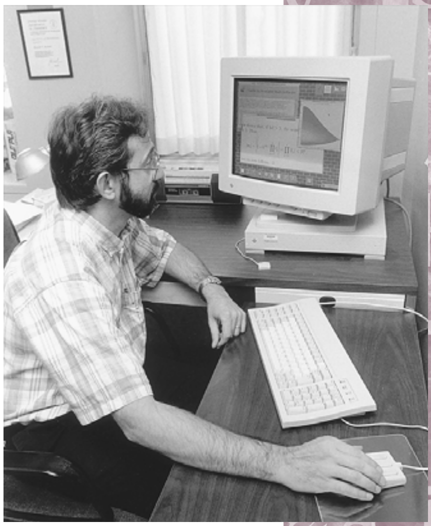
A computer stores information on its hard drive. Space on a personal computer (PC) hard drive, as well as computer memory, is measured in bytes. In the early days of personal computers, hard drives were measured in megabytes of space, and memory was measured in kilobytes. Eventually, the one-gigabyte hard drive was developed. As software programs for computers became more complicated, the files created took up more space on a computer's hard drive. Therefore, increasingly larger hard drives became necessary to store all of the information. PCs sold in 2000 were available with 80 gigabytes.

Speed in computer microprocessors is measured in terms of frequency, or the number of computations completed per second. The metric unit of frequency is the hertz (Hz). A microprocessor with a speed of 1Hz can do one computation per second.

Imagine that a store is selling two different computers for the same price. One of them has a 500-megahertz microprocessor, and the other has a one-gigahertz microprocessor. Which computer is the better value?

Look first at the prefixes "mega-" and "giga-." "Mega" means million and "giga" means billion. Five hundred megahertz is equal to 500 million hertz and one gigahertz is equal to one billion hertz. It is apparent that the computer with the one-gigahertz microprocessor is a better value since one billion hertz is twice as many as 500 million hertz. This means the computer with the one-gigahertz microprocessor is twice as fast as the one with the 500-megahertz microprocessor.

In the early 1990s some of the fastest microprocessors were only 64-megahertz. The first one-gigahertz microprocessors became available in personal computers in 2000. Computers will continue to get faster and more powerful in the future. It is possible that the speed of microprocessors will eventually be measured in terahertz and petahertz.



Prefixes representing massive numbers are commonly used in computer technology to describe the amount of memory and the speed of microprocessors.

MASSIVE ABBREVIATIONS

Abbreviations such as K (kilo), M (mega), and G (giga) help simplify terms representing massive numbers. For example, a 5G hard drive means that the hard drive has 5 gigabytes of space. A 600M microprocessor operates at a speed of 600 megahertz.

It is also helpful to understand massive numbers when looking at Internet service providers. Most important to consider is the speed at which information can be downloaded. Speed, which is measured in bytes per second, tells you how fast you can receive information over the Internet. The speed at which information can be uploaded, or sent, is generally much slower with Internet connections because people spend more time downloading than uploading.

In 2000, several types of Internet connections were available to home Internet users. Dial-up connections, where the user dials a local phone number to access the Internet, are the slowest, capable of downloading a maximum of 56 Kbps (kilobytes per second). Cable modems that are constantly connected to the Internet when the computer is on are much faster, with the ability to download up to 5 Mbps (megabytes per second). Just as microprocessors have become increasingly faster, Internet connections too will only become faster in the future. SEE ALSO Measurement, Metric System of.

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Numbers, Negative *See Inverses.*

Numbers, Prime *See Primes, Puzzles of.*

Numbers, Random *See Randomness.*

Numbers, Rational

integer a positive whole number, its negative counterpart, or zero

A number in the form of a ratio a/b, where a and b are **integers**, and b is not equal to 0, is called a rational number. The rational numbers are a subset of the real numbers, and every rational number can be expressed as a fraction or as a decimal form that either terminates or repeats. Conversely, every decimal expansion that either terminates or repeats represents a rational number.

Rational numbers can be written in several different forms using equivalent fractions. For example, $1 = \frac{2}{2}$, $\frac{1}{4} = \frac{2}{8}$, and $\frac{1}{6} = \frac{3}{18}$. There are an infinite number of ways to write 1, $\frac{1}{4}$, or $\frac{1}{6}$ by multiplying both the numerator and denominator by the same nonzero integer. Therefore, there are an infinite number of ways to write every rational number in terms of its equivalent fraction.

The following example shows how to find the ratio of integers that represents a repeating decimal.

$$x = 7.353535\ldots$$

$$100x = 735.353535\ldots$$

$$\text{Subtract } x = 7.353535\ldots$$

$$\text{So } 99x = 728.0$$

$$\text{And } x = 728/99.$$

One way to compare two rational numbers is to convert them into a decimal form. Dividing the numerator by the denominator results in the decimal equivalent. If the division has no remainder, then the decimal is called a terminating decimal. For example, $\frac{1}{2} = 0.5$, $\frac{1}{20} = 0.05$, and $\frac{2}{5} = 0.4$.

Although some decimals do not terminate, they do repeat because at some point a digit, or group of digits, repeats in a regular fashion. Examples of repeating decimals are $\frac{1}{3} = 0.333\ldots$, $\frac{1}{7} = 0.142857142857\ldots$, and $\frac{1}{12} = 0.08333\ldots$. A bar written over the digits or group of digits that repeat shows that the decimal is repeating: $\frac{1}{3} = 0.\overline{3}$, $\frac{1}{7} = 0.\overline{142857}$, and $\frac{1}{12} = 0.08\overline{3}$.

Properties of Rational Numbers

Rational numbers satisfy the following properties.

1. Given two rational numbers x and y, there are three possibilities: x is equal to y, or x is less than y, or x is greater than y.
2. The sum of two rational numbers is another rational number.
3. The product of two rational numbers is another rational number.
4. Except for 0, the reciprocal (multiplicative inverse) of every rational number is also a rational number.

SEE ALSO Integers; Numbers, Irrational; Numbers, Real; Numbers, Whole.

Rafiq Ladhani

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Numbers, Real

A real number line is a familiar way to picture various sets of numbers. For example, the divisions marked on a number line show the integers, which are the counting numbers {1, 2, 3,. . .}, with their opposites {−1, −2, −3,. . .}, with the number 0, which divides the positive numbers on the line from the negative numbers.

But what other numbers are on a real number line? One could make marks for all the fractions, such as $\frac{1}{2}$, $\frac{2}{3}$, $7\frac{1}{9}$, $-\frac{3}{5}$, $-\frac{45}{97}$, and so forth, as well as marks for all the decimal fractions, such as 0.1, −0.01, 0.0000001, and so on. Any number that can be written as the ratio of two integers (such as $\frac{1}{9}$, $\frac{6}{10000}$, $\frac{5}{1}$, and so on), where the divisor is not 0, is called a rational number, and all the rational numbers are on a real number line.

Are there any other kinds of numbers on a real number line in addition to the integers and rational numbers? What about $\sqrt{2}$, which is approximately 1.4142? Because the decimal equivalent for $\sqrt{2}$ never ends and never repeats, it is known as an irrational number. The set of real numbers consists of the integers and the rational numbers as well as the irrational numbers. Every real number corresponds to exactly one point on a real number line, and every point on a number line corresponds to exactly one real number.

Are there any “unreal” numbers? That is, are there any numbers that are not on a real number line? The set of real numbers is infinitely large, and one might think that it contains all numbers, but that is not so. For example, the solution to the equation $x^2 = -1$ does not lie on a real number line. The solution to that equation lies on another number line called an imaginary number line, which is usually drawn at right angles to a real number line.

There are numbers, called complex numbers, that are the sum of a real number and an imaginary number and are not found on either a real or an imaginary number line. These complex numbers are found in an area called the complex plane. So both imaginary and complex numbers are “unreal,” so to speak, because they do not lie on a real number line.

The set of real numbers has several interesting properties. For example, when any two real numbers are added, subtracted, multiplied, or divided (excluding division by zero), the result is always a real number. Therefore, the set of real numbers is called “closed” for these four operations.

Similarly, the real numbers have the commutative, associative, and distributive properties. The real numbers also have an identity element for addition (0) and for multiplication (1) and inverse elements for all four operations. These properties, taken all together, are called the field properties, and the real numbers thus make up a field, mathematically speaking. SEE ALSO Field Properties; Integers; Number Sets; Numbers, Complex; Numbers, Irrational; Numbers, Rational; Numbers, Whole; Number System, Real.

Lucia McKay

Numbers, Tyranny of

Numbers from opinion polls are frequently discussed in the news. Political candidates throw around numbers to justify the programs they wish to initiate. If consumers want to make a purchase, they almost always check the price. Whether buying a car, following a recipe, or decorating a home, humans use mathematics every day.

Math can help people shop wisely, buy the right insurance, remodel a home within a budget, understand population growth, or even bet on the horse with the best chance of winning the race. If a scientist or an academic researcher wants to prove a hypothesis, she or he almost always uses the statistical analysis of numbers. Qualitative research is held in a much lower position than quantitative research. Why is this phenomenon of numbers so prevalent in our society? Where did it come from?

A Look Back

The origin of our decimal number system can be traced to ancient Egyptian, Babylonian, and Sumerian societies that existed more than 5,000 years ago. The bulk of the credit for the base-10 system goes to the Hindu-Arabic mathematicians of the eighth to eleventh centuries C.E. Other societies in various parts of the world independently developed sophisticated systems of arithmetic.

The requirements of more demanding measurements, analytical quantifications, and complex calculations provided the impetus for the transfer from a symbolic system to the modern number system. Societies of all cultures and time periods have known that if humans are well versed in this language of numbers, it can help them make important decisions and perform everyday tasks.

The Prevalence of Mathematics

Researchers are demonstrating that all cultures have math and use it, like language, as a system to make meaning of the world. Numeracy is a set of cultural practices that reflect the particular values of the social, cultural, and historical context of a society. From the mental math of bazaar merchants, to the navigational skills of South Pacific Islanders, to the astronomical calculations of ancient Mayans, a huge variety of mathematical techniques and ideas have been developed worldwide.

Some mathematical practices are used in all parts of the world. People have been using these same principles for thousands of years, across countries and continents. These principles include counting, measuring, locating, designing, playing, and explaining. But there are cultural differences among these activities.

Academic mathematics may be similar in many societies because of imposed factors such as competitive economics and political ethics of the dominant culture. Whether one is sailing a boat off the coast of Japan or building a house in Peru, mathematics is necessary.

How can math be so universal? First, human beings did not invent math concepts; they discovered them. Also, the language of math is numbers, not English or German or Russian. Virtually every known society has had some method of dealing with numbers.

Primitive systems of mathematics may be as simple as having a name for "one" and a name for "more than one." Another elementary representation of numbers is the tally. It originally consisted of single strokes that were put in a one-to-one correspondence with the items being counted. Later these were combined into groups of five or more. These tally marks are also important today because items are still counted.

Numbers are important in many aspects of society. Number skills are needed to function in everyday life, including in the home, in the work-

The use of currency is one of the most common applications of numbers and mathematics.

place, and in the community. Although it is not always recognized, numbers are used in most everyday situations: cooking, shopping, financial transactions, crafts, interpreting information in the media, traveling, taking medications, using VCRs and microwave ovens.

Different people need different types of math skills, and their needs change in response to changes in their lives, such as buying a new house, learning a new hobby, or getting a new job. Consumer education typically uses mathematics to teach about credit, budgeting, and money management.

Beyond daily living skills, numbers are now being defined as the knowledge that empowers people. Thus, numbers have economic, social, and political consequences for individuals, organizations, and society. Low levels of numeracy limit access to education, training, and jobs. On-the-job lack of knowledge about numbers can hinder performance and productivity. Lack of this knowledge can also cause over-dependence on experts and professionals.

Inability to interpret numerical information can be costly. It can limit full participation as a citizen and make one vulnerable to political or economic manipulation. Sometimes the math of a particular group restricts access to professions. The attitude of a dominant group can become the norm under which others are measured. The ones who are attuned to a certain mathematical way of thinking succeed where it is used. Those who think in other ways may be considered lacking in ability.

Mathematics underlies every facet of science and technology from computer games, cellular phones, and the Internet to medical diagnostic tests, the design of new prescription drugs, and minimally invasive surgery. A "simple" differential equation modeling a very ordinary pendulum can reveal that the pendulum exhibits extraordinarily complicated and unstable behavior. Because pendulums are the basic subunit in some types of robots, mathematical understanding of their behavior can contribute to the design of better robots in industry.

One nonintuitive phenomenon that is illustrated by the fact that pendulums exhibit both simple predictable behavior, and complicated, unstable behavior is that chaos and controllability go hand in hand. This can be further illustrated in the history of airplane design. The first airplanes were designed to be extremely stable because nineteenth-century engineers identified control with stability. In World War I, however, pilots found that this stability made it impossible for them to dodge enemy fire. Thus making airplanes less stable made them safer because they were easier to maneuver. Math was used to change this stability factor and, therefore, saved many lives.

People who are empowered with numerical skills can participate fully in civic life and can skeptically interpret advertising and statistics. Knowledge of numbers is one way a society positions people. It can give a person a voice and more control over life circumstances. Lack of this knowledge can serve as a barrier in one's economic, political, and social life. Numeracy skills are needed to function in our technological society and workplace. Math is universally important, from counting pennies to the development of the newest technological advances. SEE ALSO Mathematics, Very Old; Number System, Real; Numbers and Writing.

Max Brandenberger

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Numbers, Whole

The whole number set contains the counting numbers 1, 2, 3,. . . and the number 0. In mathematics, the whole number set is the most basic number set. Whole numbers are part of the real number set, which contains other number sets, such as the integers and the rational numbers.

Using the basic mathematical definition of a set as a collection of objects that share a well-defined property, the whole number set is expressed as $\mathbf{W} = \{0, 1, 2, 3, \ldots\}$. Starting from 0, each element x of the whole number set is generated by adding one to its predecessor, the number before x, which is $x - 1$. The use of the ellipsis (. . .) within the braces signifies that the number of elements in the set is not finite (that is, infinite).

Except 0, every whole number x has exactly one immediate predecessor—the number that comes before x. Every whole number y has exactly one immediate successor—the number that comes after y.

An interesting characteristic of the whole number set is that there is no largest whole number. Suppose b is the largest whole number, then by definition, $b + 1$ is also a whole number. But $b + 1$ is larger than b. This method shows that a larger whole number can always be found.

If 0 is removed from the set **W,** the resultant is the positive integer set **P.** Thus, $\mathbf{P} = \{1, 2, 3, \ldots\}$. Some mathematicians also call **P** natural num-

Young students first experience whole numbers because they represent things that can be counted. As the photograph shows, learning to add numbers correctly takes practice.

bers or counting numbers. SEE ALSO Integers; Numbers, Rational; Numbers, Real.

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Numerator *See Fractions.*

Nutritionist

Nutritionists plan food and nutrition programs, and supervise the preparation and serving of meals. They help prevent and treat illnesses by promoting healthy eating habits, scientifically evaluating clients' diets, and offering advice on weight loss, cholesterol reductions, or other diet-related concerns. Nutritionists can be teachers, researchers, health care workers, or managers. They might also direct experiments to find alternative food or diet recommendations.

A nutritionist should be able to read and write recipes and solve mathematics and science problems. Due to the variety of roles a nutritionist can have, their math knowledge should be varied. Menu planning and recipe development require basic arithmetic. Those who work as management nutritionists, overseeing large-scale meal planning and preparation in health care facilities, will also need to know basic geometry. This includes percentages, ratios and proportions, and volume. Nutritionists who work in research conduct scientific tests, and should have a solid understanding of

Nutritionists use mathematics when calculating nutrition requirements, designing meal plans, and developing recipes.

algebra, geometry, and calculus. SEE ALSO Percent; Ratio, Rate, and Proportion.

Marilyn Schwader

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Odds *See Probability, Theoretical.*

O

Ozone Hole

The so-called ozone hole sometimes is confused with the problem of global warming. Even though there is a connection between the two environmental issues, because ozone contributes to the greenhouse effect, the ozone hole is a separate issue. This article briefly addresses how ozone depletion is measured.

Ozone in Earth's Atmosphere

Ozone is a colorless, gaseous form of oxygen found in the Earth's atmosphere, primarily in the upper region known as the stratosphere, where it is naturally produced and destroyed. The chemical element oxygen normally forms a molecule containing two atoms (O_2). But in the presence of ultraviolet light or an electrical spark in the air, oxygen can form a molecule containing three atoms (O_3). The molecule of three oxygen atoms is called ozone.

Within the stratosphere is a layer between 20 and 40 kilometers (km) above Earth's surface that is known as the ozone layer. Here ozone takes up a greater proportion of the atmospheric column than at any other height. In the stratosphere, the concentration of ozone is 1,000 times greater than in the lower region of Earth's atmosphere known as the troposphere. Ozone in the stratosphere is beneficial because it protects Earth's inhabitants from the Sun's harmful ultraviolet radiation.

OZONE CLOSE TO EARTH

In Earth's lower atmospheric layer known as the troposphere, the concentration of ozone is usually between 0.02 and 0.03 parts per million (ppm). Under smog conditions, the impurities in the air can act as catalysts and allow sunlight to form ozone. In the troposphere, ozone is harmful and can damage lung tissue and plants.

Measuring Ozone Levels

Scientists assess ozone by calculating how much there would be if all the ozone over a particular spot on Earth were compressed to a standard atmosphere of pressure—that is, the average pressure of air at sea level. On average, this would result in a column of ozone no more than 3 millimeters (mm) thick.

The unit of measure used to represent the amount of ozone above a particular position on the surface is the Dobson unit (DU), with one unit

representing 0.01 mm of ozone compressed to one standard atmosphere. Therefore, there is typically 300 DU in a column of the normal atmosphere.

G. M. B. Dobson was a British physicist who initiated the first regular monitoring of atmospheric ozone using spectrographic instruments in the 1920s. He was able to derive the vertical distribution of ozone from a series of measurements of the relative intensities of two particular wavelengths of light scattered in the **zenith** sky. One wavelength is more strongly absorbed by ozone than the other wavelength. As the Sun's **zenith angle** varies, a reversal occurs in the variation of the ratio of the intensities. Dobson compared the intensities of these two wavelengths with an instrument he constructed using a photomultiplier and an optical wedge; this instrument is now known as a Dobson spectrophotometer.

zenith the point on the celestial sphere vertically above a given position

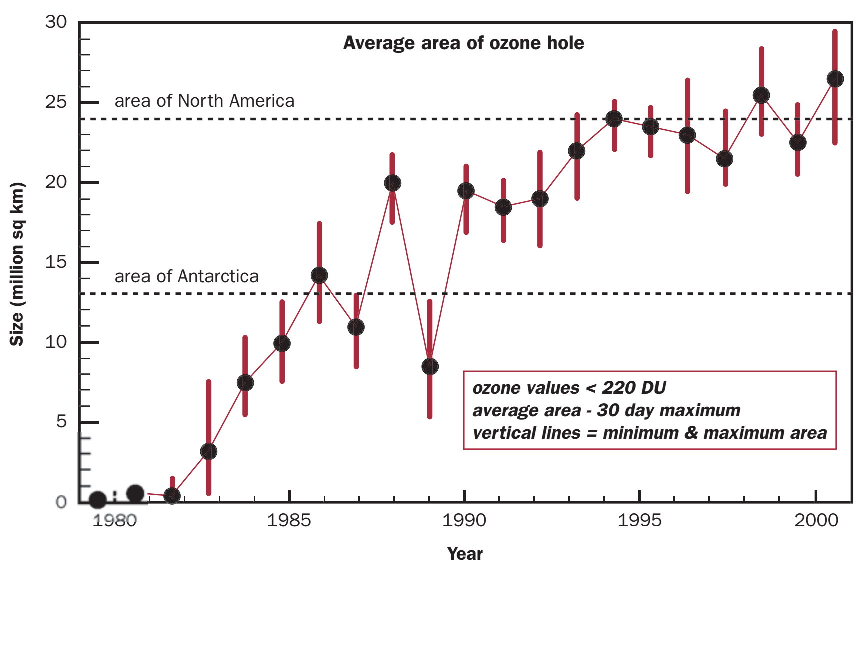
zenith angle from an observer's viewpoint, the angle between the line of sight to a celestial body (such as the Sun) and the line from the observer to the zenith point

Dobson found that the ozone in the atmosphere is far from uniformly spread. The lowest concentrations, around 250 DU, were consistently found at the equator, although the polar winters resulted in periods where their concentrations might fall below the equatorial level. The highest concentrations were found in higher latitudes, where the variation fluctuated from as high as 460 DU to 290 DU in the upper latitudes of the Northern Hemisphere and between 400 DU and 300 DU in the Southern Hemisphere.

The International Geophysical Year* of 1957–1958 witnessed the World Meteorological Organization (WMO) take responsibility for establishing uniform and high quality ozone measurement world wide. The WMO subsequently established 160 ground-based ozone observation stations.

***The International Geophysical Year (July, 1957 through December, 1958) consists of eighteen months of a period of maximum sunspot activity. It was designated for cooperative study of the solar-terrestrial environment by the scientists of sixty-seven nations.**

From the 1920s to the 1970s ozone was measured from the ground. Since the late 1970s scientists have used satellites, aircraft, and balloons to measure ozone levels from above Earth. The National Aeronautics and Space Administration (NASA) has also launched many scientific studies to investigate ozone. The figure below is one example of the results of this data monitoring.



Recording Low Ozone Levels

In the 1970s a research group with the British Antarctic Survey (BAS) was monitoring the atmosphere above Antarctica when the scientists first noticed a loss of ozone in the lower stratosphere. At first they believed their instruments to be faulty, and new instruments were sent to ensure that the readings were accurate.

By 1985 the BAS was reporting a dramatic decline of 50 percent in springtime ozone levels above Halley Bay Station when compared to the previous decade. At the most affected altitude, 14 to 19 km above the surface, more than 99 percent was lost. This was an unsettling discovery because NASA had been monitoring ozone levels globally since 1979 with the Total Ozone Mapping Spectrometer (TOMS) aboard the Nimbus 7 satellite.

The standard TOMS data-processing procedure was to automatically neglect ozone levels below a fixed value of 180 DU, considering such data to be unreliable. Hence, the Antarctic springtime data had been ignored. Only after the British survey team's report were the TOMS data reprocessed; the ozone depletion was verified and the geographical extent of the hole was determined. This lowering of the amount of ozone over the Antarctic became known as the ozone hole.

Phillip Nissen

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P

Pascal, Blaise

French Mathematician and Philosopher
1623–1662

Most students in a mathematics class at some point wonder why. Not "Why does this process or theory work?" but rather "Why do I have to learn this?" Blaise Pascal, a mathematician during the mid-seventeenth century, is certainly one of the people to "blame." Yet Pascal made many contributions to mathematics and society as a whole.

Education for children of the 1600s was very different from modern education. There had been a time during Pascal's youth when his father had forbidden him to study mathematics and physics. Pascal's father even went so far as to lock up all the books to keep them away from his son. It seemed as though his father was embarrassed or worried that he would get in trouble for doing mathematics.

Pascal showed that he was capable of learning mathematics, such as infinitesimal calculus, on his own. This made his father realize that Pascal had a special talent for the subject. Thankfully, his father then allowed him access to all books, and Pascal blossomed in his studies. He even wrote an original treatise on **conic sections** at the age of 16.

Blaise Pascal's work on binomial coefficients would lead Sir Isaac Newton (1643–1727) to his discovery of the general binomial theorem for fractional and negative powers.

Pascal not only had a natural ability to understand mathematics, but he also was an inventor. His father was a bookkeeper, which meant doing computations by hand. It was quite easy to make a simple mistake while adding and subtracting long columns of numbers. Pascal ultimately created a device that could do addition and subtraction of multiple numbers. Pascal's adding and subtracting device is considered the first calculator—more than 300 years before the modern calculator became more compact and much more sophisticated in its capabilities.

Pascal was also partly responsible for discovering the field of mathematics known as **probability theory**. He had a friend who was a professional gambler. Pascal's friend could not understand why he was losing when he felt he had a good chance of winning. With the help of another French mathematician, Pierre de Fermat (1601–1665), Pascal studied the probability of dice-throwing and other such games of chance. Ultimately this work on probability led to a treatise on binomial coefficients, the table of which is now known as Pascal's triangle.

conic section the curve generated by an imaginary plane slicing through an imaginary cone

probability theory the branch of mathematics that deals with quantities having random distributions

Pascal died at the age of 39, but he left behind quite a legacy of knowledge in geometry, probability, physics, calculus, and a number of inventions, including the syringe and the hydraulic press. After his death, his sister found boxes and drawers full of scraps of paper with Pascal's writing on them. She published the papers posthumously and titled the collection *Pensées* (1670), meaning thoughts. Every student should be familiar with at least one quotation from the work: "When we read too fast or too slowly, we understand nothing." SEE ALSO Fermat, Pierre de; Mathematical Devices, Mechanical.

Elizabeth Sweeney

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Patterns

Patterns in mathematics may be either numerical or visual. Three common numerical patterns (also referred to as sequences) are arithmetic, geometric, and exponential.

Common Numerical Sequences

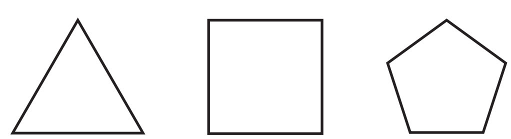
In an arithmetic sequence, a common difference exists between a term and its previous term. For example, the sequence {1, 2, 3, 4, 5} has a difference of 1 between every term, making it arithmetic.

Geometric sequences have a common ratio, that is, a multiplying number, between every term. For example, the sequence $\{3, -6, 12, -24\}$ is geometric because it has a common ratio of -2.

Finally, an exponential sequence has a base, or a number, that is raised by an increasing power. For instance, the sequence $\{1, 2, 4, 8, 16\} = \{2^0, 2^1, 2^2, 2^3, 2^4\}$ is exponential with 2 as the base.

Visual Sequences

Visual sequences may consist of geometric objects, a form often used in standardized tests. Usually, the sequences can be equated to a numeric pattern. For example, the number of sides in each figure can numerically state the visual sequence shown below. The arithmetic sequence {3, 4, 5} explains this visual sequence.



Another example of patterns from numbers is from the Fibonacci sequence {1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .}, in which each term is calculated by adding the previous two terms. This mathematical pattern also occurs, for example, in pinecones in which the number of petals in the right diagonal rows is eight and in the left diagonal rows thirteen.

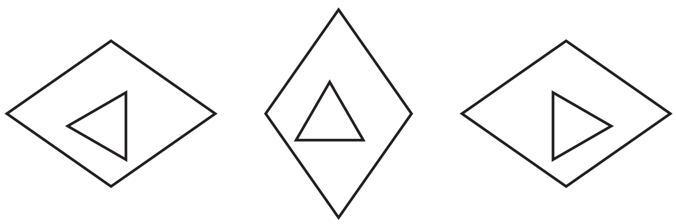
Predicting a Sequence

The answers to the questions outlined below help to determine the pattern and to predict the next (or any) term of a sequence.

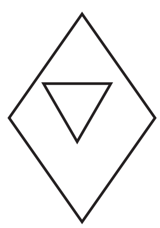
1. Is the sequence numeric or visual? What is the common difference, ratio, or base?
2. Is there more than one pattern? Is one term calculated by using the previous terms?
3. What is the next term in the pattern?

For example, the sequence {1, −1, 1, −1} can be identified as geometric because there is a common ratio of −1; hence, the next term is 1.

Another example, the visual sequence shown below, illustrates two patterns occurring at one time.



The long orientation of the diamond alternates between horizontal and vertical positions, whereas the triangle inside is rotating 90 degrees clockwise. Therefore, the sequence is predictable, and the next term should be as shown below.



In contrast, the sequence {1, 2, 5, 6, 11, 13} is neither arithmetic nor geometric because there is no common difference or ratio, and it is not exponential because no base is being raised by an increasing power. In addition, any combination of previous terms does not equal a following term. Therefore, the sequence has no pattern and is not predictable.

Patterns do not necessarily occur in a single sequence. One of the most important applications of patterns is in chaos theory. Natural systems, like weather, which are chaotic and display little pattern, can be modeled with computers by generating millions of different possibilities. When all the different scenarios are analyzed together, a pattern may occur that exhibits the most likely possibilities. This process of analysis and pattern recognition allows meteorologists to understand the chaotic behavior of weather and to make more accurate weather predictions. SEE ALSO Chaos; Nature; Sequences and Series.

Michael Ota

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Percent

The use of percent is commonplace, not only in calculating tips, tax, and loan interest, but also in everyday language. When someone says he gave 100 percent or 110 percent, what does he mean? Whether he realizes it or not, he is referring to the definition of percent.

Percent means "parts per one hundred," and is designated by the % symbol. Thus, 100% is equivalent to $\frac{100}{100}$, which is why 100% in common language can be interpreted as "totally," "completely," and "without any left behind." So, 110% is equivalent to $\frac{110}{100}$, which is why in everyday language, 110% means to go above and beyond.

Conversions between Percents, Fractions, and Decimals

Since a percent is parts per one hundred, converting percent to a fraction is the easiest conversion. Just remove the percent symbol, place the number over one hundred, and reduce, as shown in the first (left-most) portion of the table.

To convert a percent to a decimal, start the same way. Remove the percent symbol and place the number over one hundred. Since a fraction represents division, divide the number by one hundred to get decimal form. (This is the same as moving the decimal to the left two places.) See the second part of the table.

To go in the opposite direction and convert decimal to percent, do the opposite operation. Multiply by one hundred. This is the same as moving the decimal to the right two places. See the third part of the table.

Converting a fraction to a percent can sometimes be done in one step but may require two steps. To convert in one step, multiply the fraction by some number to get parts per one hundred. If parts per one hundred cannot be easily obtained, convert the fraction to a decimal (by dividing), and then convert the decimal to a percent (by multiplying by 100). See the fourth (right-most) portion of the table.

EXAMPLES OF CONVERSIONS

Percent to Fraction		Percent to Decimal		Decimal to Percent		Fraction to Percent	
Percent	Fraction	Percent	Decimal	Decimal	Percent	Fraction	Percent
45	$\frac{45}{100} = \frac{9}{20}$	60	$\frac{60}{100} = 0.6$	0.23	0.23 x 100 = 23	$\frac{1}{100}$	$\frac{1}{100} = 1$
220	$\frac{220}{100} = 2\frac{20}{100} = 2\frac{1}{5}$	115	$\frac{115}{100} = 1.15$	0.8	0.8 x 100 = 80	$\frac{1}{20}$	$\frac{1}{20} \bullet \frac{5}{5} = \frac{5}{100} = 5$
		3	$\frac{3}{100} = 0.03$	5.25	5.25 x 100 = 525	$1\frac{1}{4}$	$1\frac{1}{4} \bullet \frac{25}{25} = 1\frac{25}{100} = 125$
						$\frac{2}{22}$	$2 \div 22 = 0.090909\ldots \approx 9$

Solving Percent Word Problems

There are at least two methods for solving percent word problems. One is to set up a proportion with percent over one hundred equal to parts out of the whole. Another way is to write an equation by translating English into algebra. Most commonly, "of" means *multiply* and "is" means *equals*. Both methods work, and the preferred method depends on the individual's strengths. See the sample problems in the table below.

Smart Shopping

Understanding percent is especially important in planning large purchases. The formula for simple interest is $I = prt$, in which I is the amount of

SAMPLE PROBLEMS INVOLVING PERCENT

(a) Four friends ate out for $32.56. Calculate the tip at 15% of the bill.

Proportion Method: Let x be the amount of the tip. Remove the percent symbol and place 15 over 100. Set this equal to x over $32.56 and solve.

$$\frac{15}{100} = \frac{x}{32.56}$$

$488.4 = 100x$ — *Cross multiply.*

$\$4.88 = x$ — *Divide both sides by 100.*

Equation Method: Let x be the amount of the tip. Convert 15% to a decimal, translate "15% of the bill" into $0.15 \bullet 32.56 = x$, and solve.

$0.15 \bullet 32.56 = x$

$\$4.88 = x$ — *Multiply.*

(b) A pizza is cut into twelve slices, and six friends each have two slices. What percent of the whole pizza is two slices?

Proportion Method: Let x be the percent. Place x over 100. Set this equal to 2 over 12 and solve.

$$\frac{x}{100} = \frac{2}{12}$$

$\frac{x}{100} = \frac{1}{6}$ — *Reduce (optional).*

$6x = 100$ — *Cross multiply.*

$x = 16\frac{2}{3}\%$ — *Divide by six and insert the percent symbol.*

Equation Method: Let x be the percent. Translate "what percent of the whole pizza is 2?" into $x \bullet 12 = 2$ and solve.

$x \bullet 12 = 2$ — *Divide by 12 on both sides.*

$x = 0.1666\ldots$ — *Convert the decimal to a percent.*

$x = 16\frac{2}{3}\%$

(c) Mandy surveyed her friends and found that 20% of her friends prefer strawberry ice cream to chocolate or vanilla. If three of her friends like strawberry, how many friends did she survey?

Proportion Method: Let x be the number of friends surveyed. Place 20 over 100. Set this equal to 3 over x and solve.

$$\frac{20}{100} = \frac{3}{x}$$

$20x = 300$ — *Cross multiply.*

$x = 15$ — *Divide by twenty on both sides.*

Equation Method: Let x be the number of friends surveyed. Convert 20% to a decimal, set "20% of her friends" equal to "3 of her friends," and solve.

$0.2 \bullet x = 3$

$x = 15$ — *Divide both sides by 0.2.*

interest, p is the principal (the initial amount), r is the interest rate, and t is time in years.

Suppose Susan is planning to finance a $20,000 automobile over 4 years. The lender offers her an interest rate of 8%, and she takes it. Using the simple interest formula, the interest paid out over the 4 years is $I = 20{,}000 \times 0.08 \times 4 = \$6{,}400$.

Notice that Susan would have saved $6,400 if she had paid cash for the car. In addition to saving $6,400, that $6,400 could have been earning interest in a savings account. If the $6,400 had been placed in a savings account earning 2% interest over those same 4 years, the interest she would have earned is $I = 6{,}400 \times 0.02 \times 4 = \$512$. In those 4 years, she could have saved $\$6{,}400 + \$512 = \$6{,}912$!

Unfortunately, for many people it is difficult to save $20,000. Three ways Susan could have gotten a loan but still saved money are by (1) putting down a down payment, (2) shopping around for a loan with a lower interest rate, and (3) paying off the loan in less time.

Suppose Susan had put down $5,000 on the car and had taken out a loan for only $15,000. If she had shopped around to find an interest rate of 7.25% and had agreed to pay off the loan in 3 years, the amount of interest paid over the 3 years would have been $I = 15{,}000 \times 0.0725 \times 3 = \$3{,}262.50$.

By putting down a down payment, finding a lower interest rate, and paying the car off sooner, she would have saved over $3,000. Furthermore, she could have earned interest on the $3,000, making the savings even more. SEE ALSO Decimals; Fractions; Interest; Ratio, Rate, and Proportion.

Michelle R. Michael

Permutations and Combinations

The study of permutations and combinations is at the root of several topics in mathematics such as number theory, algebra, geometry, probability, statistics, discrete mathematics, graph theory, and many other specialties.

Permutations

A permutation is an ordered arrangement of objects. For instance, the fraction $\frac{3}{7}$ is a permutation of two objects, whereas the combination to open a lock—23 L, 5 R, and 17 L—is a permutation of three objects. The ordered arrangements of objects likely dates all the way to the beginning of organizing and recording information.

Sometimes one is interested in knowing the number of permutations that are available from a collection of objects. Suppose a club has two candidates for president: Bob (B) and Janice (J); three candidates for secretary: Katy (K), Rob (R), and Harry (H); and three candidates for parliamentarian: Abe (A), Calvin (C), and Mary (M). In how many different ways could these candidates be elected to the three offices? One way to find out is to simply list the possible permutations: {(B,K,A), (B,K,C), (B,K,M), (B,R,A), (B,R,C), (B,R,M), (B,H,A), (B,H,C), (B,H,M), (J,K,A), (J,K,C), (J,K,M),

(J,R,A), (J,R,C), (J,R,M), (J,H,A), (J,H,C), (J,H,M)}. Another way to find the number of permutations is to use the fundamental principle of counting. The fundamental principle of counting states that if an event A can occur in a ways, and is followed by an event B that can occur in b ways, then the event A followed by the event B can occur $a \times b$ ways. In our example, the office of president can be filled two ways, the office of secretary can be filled three ways, and the office of parliamentarian can be filled three ways. Therefore, the number of permutations, using the fundamental principle of counting, is $2 \times 3 \times 3$, or 18. This method of finding the number of permutations can be extended to any finite number of events.

There are various permutation situations that are worthwhile exploring. For instance, a company wanting to use the nine nonzero digits for a source of five cell identification (ID) cards where the nonzero digits could be repeated would have 9^5 possible ID cards, an exponential expression. A company of five employees wanting to assign each employee to an office has $5 \times 4 \times 3 \times 2 \times 1 = 5!$ possible choices, a factorial expression. One way to define factorial notation is: $0! = 1$ and $n! = n(n-1)!$ for $n \geq 1$. A related situation would be the possibility of having 100 employees with five offices and 95 workstations. There would be $100 \times 99 \times 98 \times 97 \times 96 = \frac{(100 \times 99 \times 98 \times 97 \times 96) \times 95!}{95!} = \frac{100!}{95!}$ possible choices for the five offices. This situation suggests a fundamental formula for determining the number of permutations of ordering r objects, without replacement, selected from n available objects, $P(n, r)$, where $0 \leq r \leq n$, to be $P(n, r) = \frac{n!}{(n-r)!}$.

Combinations

A combination is an unordered arrangement of objects. For instance, there may be six ordered arrangements of three lengths, (3 cm, 4 cm, 5 cm) but since each arrangement determines the same unique trigon ("triangle"), any collection of the three lengths {3 cm, 4 cm, 5 cm} will represent the others. An example of a combination is thus {3 cm, 4 cm, 5 cm}. An extended example would be to consider 100 distinct points on a circle and to inquire as to number of unique trigons, a combination, which would be determined with the **vertices**. Initially one would know that there are $\frac{100!}{(100-3)!}$ permutations, but 3! permutations represent a combination, so there are $\frac{1}{3!} \times \frac{100!}{(100-3)!} = \frac{100!}{3!(100-3)!}$ combinations. This situations suggests that the number of combinations with r objects, r-combinations, from a collection with n objects, $C(n, r)$, is, $C(n, r) = \frac{n!}{r!(n-r)!}$, and this formula suggests, $C(n, r) = \frac{P(n, r)}{r!}$. Notice that $C[n, (n-r)] = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} = \frac{n!}{r!(n-r)!} = C(n, r)$. Another application of combinations might be to suppose that a local pizza parlor has seven different toppings available, besides cheese, which is on every pizza. The number of different combinations available, since putting toppings on a pizza is not an ordered collection, is $C(7,0) + C(7,1) + C(7,2) + C(7,3) + C(7,4) + C(7,5)$

vertex the point on a triangle or polygon where two sides come together

$+ C(7,6) + C(7,7)$. For convenience of notation, $C(n,r)$ is often expressed as $\binom{n}{r}$. Thus, the fifth power of the sum of a and b, $(a + b)^5$, would be expressed as $\binom{5}{0}a^{5-0}b^0 + \binom{5}{1}a^{5-1}b^1 + \binom{5}{2}a^{5-2}b^2 + \binom{5}{3}a^{5-3}b^3 + \binom{5}{4}a^{5-4}b^4 + \binom{5}{5}a^{5-5}b^5$.

John J. Edgell, Jr.

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Pharmacist

A pharmacist must be able to measure dosage amounts correctly for patients so that someone does not receive too little or too much of the prescribed medicine.

A pharmacist dispenses drugs prescribed by doctors and health practitioners, and gives information to patients about medicines. Additionally, pharmacists advise physicians on the selection, dosages, interactions, and side effects of drugs. Pharmacists understand the use, composition, and effects of drugs. In a hospital or clinic, they may also make sterile solutions and help doctors and nurses plan and monitor drug regimens. In home health-care settings, they monitor drug therapies and prepare infusions, or solutions that are injected into patients. Many pharmacists specialize in specific drug therapy areas, such as psychiatric disorders or intravenous nutrition.

A pharmacist must have a solid grounding in mathematics. Some of the math is basic and includes fractions, decimals, and percentages. Pharmacists have to understand the metric system and convert measurements such as ounces into their metric equivalents. To read and understand pharmacological research studies, they also must have a working knowledge of algebra, calculus, and especially **statistics**.

statistics the branch of mathematics that analyzes and interprets sets of numerical data

Statistics is a crucial discipline for pharmacists. To measure the effectiveness of drugs, researchers and drug companies use statistical concepts such as population samples, significance levels, statistical errors, statistical power, and mean, standard deviation, and variance. For example, a pharmacist needs to be able to interpret a statement such as this: "The difference in outcomes between drug X and drug Y was significant with $p < 0.05$." This means that the probability of observing this difference in outcomes would be less than 5 percent if X were no better than Y. SEE ALSO Measurement, Metric System of; Statistical Analysis.

Michael J. O'Neal

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Photocopier

Have you ever noticed that when you place a picture on a photocopier turned one way the copy comes out flipped upside down? Have you ever tried to enlarge a copy of an object to fit into an special frame, and had to enlarge it several times before you got the size "just right?" Have you ever made a photocopy of a book cover or of your hands? Well, if you have witnessed or experienced any of these things, then you have been involved in some of the everyday mathematics that surrounds us.

Congruency, Similarity, and Nonsimilarity

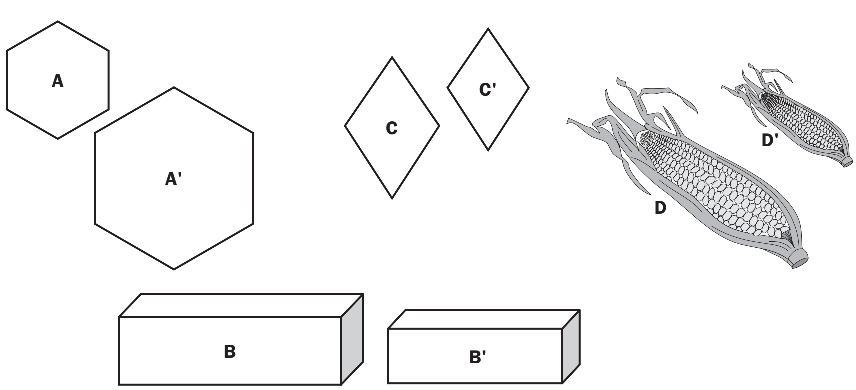
The photocopier can produce copies that have virtually the same size as the original item copied. The two objects—original and its copy—are said to be mathematically congruent to each other. Two 2-dimensional objects are congruent if the objects have the same shape and the same size. On the photocopier, the copy comes out as a reflection of the original (flipped upside down), so you may have to flip the copy to fit it on top of the original to see the perfect match in shape and size.

The photocopier also can produce copies of objects that are smaller (reduction) or larger (magnification) than the original object. Mathematically, we refer to the process of making this different-sized copy as a dilation. The copy has the same shape as the original, but not the same size. Certain corresponding measurements (such as the left side on an original and the left side on a copy) are multiples or fractions of the original. Objects that fit this description are referred to as mathematically similar shapes. The word "similar" is being used here in a more restricted way than you probably use it in everyday conversations.

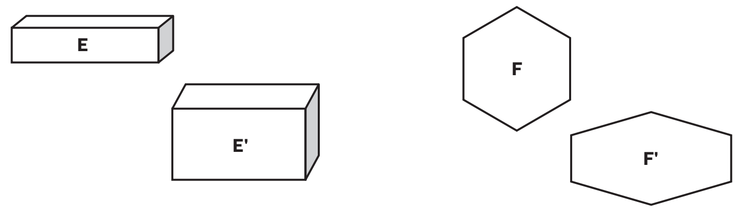
In the illustration below are four pairs of similar shapes. For example, A′ is the copy of A. Do these shapes—A and A′, B and B′, C and C′, and D and D′—seem similar?

In contrast, the illustration at the top of the next page shows two pairs of shapes that are almost but not quite similar. Shape E is almost similar to E′ and F is almost similar to F′. Yet the ratios of width to height are different in the two pairs, and cause the shapes to fail the test of being similar.

SHAPES THAT ARE SIMILAR



SHAPES THAT ARE NOT QUITE SIMILAR



Examples of Similarity

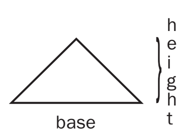
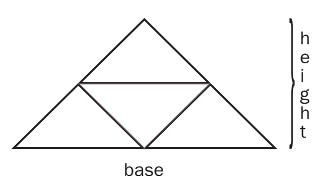
A few examples of similarity on a photocopier illustrate the mathematics behind reductions and enlargements.

Copy of a Triangle. Imagine your original shape is a triangle with base length that is 2 inches and height that is 1 inch. (See the single small triangle on the left in the table.) If you enlarge the triangle with a photocopier and measure the base length and height of the copy, you can compare those new lengths to the original values. You will find that you can either multiply or divide by one specific number to get the measurement of the copy based upon the measurement of the original.

In this example, the length of the base of the copy of the triangle is 4 inches, yet the original base length was 2 inches. What number might you multiply times 2 to get 4? So, if you multiply the same number, 2, times the height measure of the original triangle (1 inch), you will get 2 inches (2 × 1 inch), the height of the similar copy. Since multiplying the measurements of the original triangle by 2 will yield the measurements of the new triangle, we say we have used a scale factor of 2.

There are other ways of referring to this same scale factor of 2. We might say that the ratio of side lengths of the similar copy to the side lengths of the original is 2:1, or 2-to-1. When we use a photocopier, we use percentages to refer to the scale factor. In this example, the scale factor on the photocopier would be represented as 200%:100%, or we would simply choose the enlargement (or magnification) factor to be 200%.

COMPARISON OF AN ORIGINAL AND ITS COPY

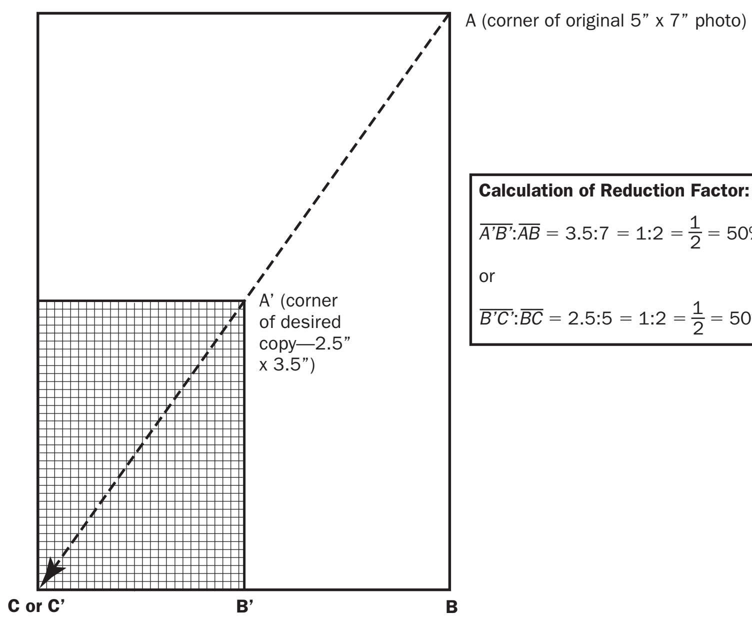
Original Triangle	Formula	Copy of Triangle (scale factor of 2)
2 inches	Base length (b)	4 inches
1 inch	Height (h)	2 inches
0.5(2)(1) = 1 sq. in.	Area = 0.5bh	0.5(4)(2) = 4 sq. in.
		

Determining the relationship between the area of the original shape and its similar copy is more challenging. You will need to give thought to the overall size of the photocopy, because you should make sure that the size of paper in the photocopier is large enough to hold the copy of the original shape. So, if you have doubled the base length and doubled the height of the original triangle, do you think the area will also be doubled?

There are at least two ways to find the answer, as shown in the table. Calculate the area with a formula or discover the area by tessellating copies of the smaller triangle until you cover the similar copy of the original triangle. ("Tessellating" means to assemble the smaller triangles so they adjoin one another with no gaps in between.)

Reducing a Photograph. Suppose one of your friends has given you a 5-by-7-inch photograph. Although you plan to place the photo into a frame, you would like to have a wallet-sized copy of this photo, too. The photo is not copyrighted to prohibit copying, so you find a color photocopier to make your copy. Your wallet will hold a 2.5-by-3.5-inch photo. Which setting will you use on the photocopier to make your copy?

There are at least two ways to find the answer without guessing at the reduction. If the original has a width of 5 inches, and your copy requires a width of 2.5", the ratio is 5:2.5 which equals 2:1. Since you are making a reduction, you must reverse the order to 1:2. Hence, you will need to use 50% as the reduction factor on the photocopier.



Calculation of Reduction Factor:

$\overline{A'B'}:\overline{AB} = 3.5:7 = 1:2 = \frac{1}{2} = 50\%$

or

$\overline{B'C'}:\overline{BC} = 2.5:5 = 1:2 = \frac{1}{2} = 50\%$

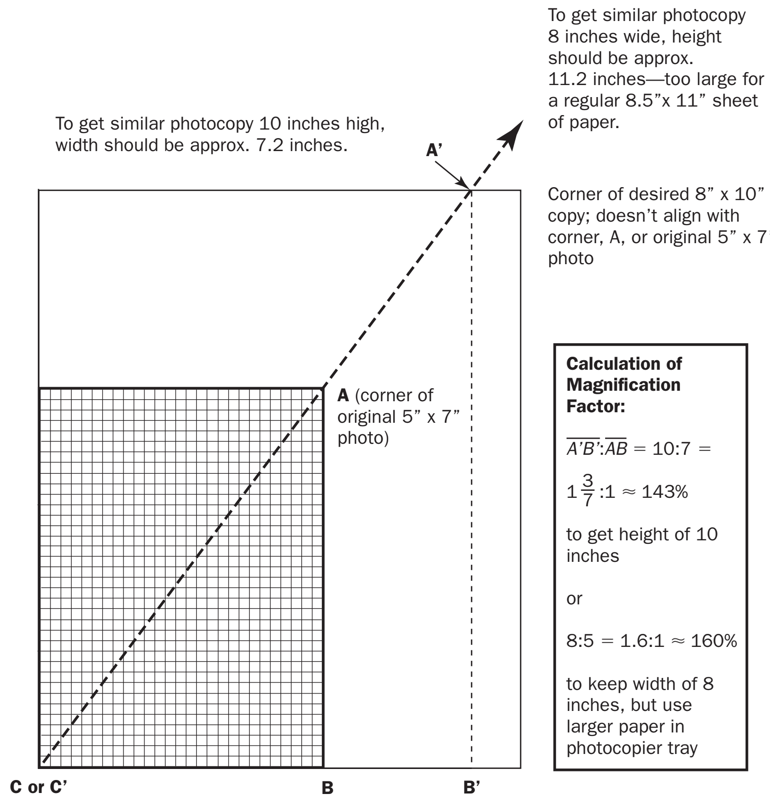
The 50% setting works for the width of the photo, but will it work for the length of the photo as well? If we want a similar copy, we realize that the scale factor for the length of the photo must be the same as the scale factor of the width. Thus, the photocopier setting should be correct at 50%. Will a 50% reduction from 7 inches in length yield a new length of 3.5 inches? Yes, 50% of 7 is 3.5.

Enlarging a Photograph. Now suppose you decide to also enlarge the 5-by-7-inch photo to make it fit into an 8-by-10-inch frame. Can you figure out the setting for the photocopier?

First, try changing the 5-inch width to 8 inches. So the scale factor is eight-fifths, or 1.6. On the photocopier, you would choose 160%. This same scale factor should give the desired length of 10 inches on the photocopy.

Will a magnification of 160% from 7 inches in length yield a new length of 10 inches? Not quite, because 160% of 7 is 11.2 inches. That poses a problem if you are using 8.5-by-11-inch paper in the photocopier—your copy will be 0.2 inches too long! So what do you do? You could switch to a larger paper tray in the photocopier. However, your copy will be too large to fit into the 8-by-10 photo frame. You can also make a decision to cut off part of the length of the photo, or use a smaller width and try for a photocopy that will be similar to the original. If you choose the latter option, you will decide on a slightly smaller magnification setting on the photocopier.

Remember there is more than one way to solve a problem, so the drawing below shows another approach you may try when you make measurements to determine a proper setting for a reduction or magnification on the photocopier. SEE ALSO Congruency, Equality, and Similarity; Percent; Scale Drawings and Models; Ratio, Rate, and Proportion.



Calculation of Magnification Factor:

$\overline{A'B'}:\overline{AB} = 10:7 =$

$1\frac{3}{7}:1 \approx 143\%$

to get height of 10 inches

or

$8:5 = 1.6:1 \approx 160\%$

to keep width of 8 inches, but use larger paper in photocopier tray

Iris DeLoach Johnson

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Photographer

A photographer is an artist and a part-time mathematician. As an artist, the photographer, who often learns the craft at a two- or four-year college, usually as part of an art program, takes pictures in places as varied as natural settings, wedding reception halls, studios, and industrial environments. As a mathematician, the photographer must understand the mathematics of cameras, lighting and optics, and print processing.

The mathematics of the camera involves shutter speed and f-stops. Shutter speed measures the amount of time the shutter is open, exposing the film to light. The numbers of the shutter speed represent the denominator of the time the shutter is open, so that a shutter speed of "500" means that the shutter is open 1/500th of a second. Such a fast shutter speed helps photographers capture action shots or take photos in bright environments. A low shutter speed, like "60" (or 1/60th of a second), slows the shutter to allow more light to enter the camera.

The f-stop is how wide the lens opens (that is, how much light is allowed to reach the film) when taking a picture. The f-stop measures the size of the lens's opening (also called aperture) relative to the lens's focal length. So, a 50 millimeter lens with a 12.5 millimeter opening has a maximum f-stop of 4 (or f/4). To achieve a certain pictorial effect, the right combination of these settings is needed. Photographers also use mathematical formulas to increase or decrease the amount of light and to determine the focal point of a lens. If the combination is not calculated correctly, photos can be overexposed (too much light is let in and the image is washed out) or underexposed (not enough light was let in and the image is too dark).

To help the photographer make precise determinations, most cameras have built-in light meters. Such devices measure the amount of light and indicate the optimum settings for the camera's f-stop and shutter speed.

In each photograph he develops, a photographer combines two different disciplines, art and mathematics, to create the finished product.

Although these features are automatic on many cameras, some photographers prefer to use manual cameras and adjust the settings themselves.

When developing prints, photographers not only must get the chemical composition of the developing fluids correct, they must also measure accurately the temperature of the solutions. Additionally, photographers must use a mathematical scale to determine how long to leave the print in the solution, depending on the type of film.

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Pi

Pi is represented by the symbol π (the sixteenth letter of the Greek alphabet) and appears in a number of measurement formulas. These formulas include $c = 2\pi r$ and $A = \pi r^2$, where c is a circle's circumference or the distance around its outer edge, r is its radius or the distance from the center of the circle to its outer edge, and A is the circle's area. What is pi and why does π appear in these formulas?

What Is Pi?

What do the following items have in common with each other and all other circular figures: a quarter, approximately 1 inch across; a compact disc, 12 centimeters wide; a circular patio pool, spanning about 23 feet; and Earth's equator, roughly 8,000 miles in **diameter**. These estimated diameters are summarized in the table below, along with the estimated circumferences of the circular items.

diameter the chord formed by an arc of one-half of a circle

In each case in the table, the circumference is about three times the diameter. The table also lists more precise measurements for the dimensions for our four examples. To determine the ratio of circumference (c) to its diameter (d), divide c by d. The resulting ratio of c:d is, in each case, a decimal slightly larger than 3.14. In fact, the circumference of *any* circle is about 3.14 times larger than its diameter. The relation, then, between the diameter and circumference of a circle can be summarized algebraically as $c \approx 3.14 \times d$. One may recognize that 3.14 is the approximate (rounded off) value of π. Thus, the exact relation between the diameter of a circle and its circumference can be summarized by the equation $c = \pi d$.

So, if one knows the diameter of a circle, it is possible to determine the distance around the circle without measuring it. For instance, if a car tire is 25 inches in diameter, its circumference can be calculated by multiplying

DIMENSIONS FOR FOUR COMMON CIRCULAR OBJECTS

	Diameter		Circumference	
Item	**Estimated**	**Precise**	**Estimated**	**Precise**
Quarter	1 in	15/16 or 0.9375 in	3 in	2.95 in
CD	12 cm	12.0 cm	38 cm	37.7 cm
Pool	23 ft	22.9 ft	72 ft	72.0 ft
Earth's equator	8,000 mi	7,927 mi	25,000 mi	24,901 mi

this measure by 3.14, yielding 78.5 inches. Conversely, if a tire is about 78.5 inches around, then the diameter must be about 3.14 times smaller, or 25.0 inches.

What then is pi? Recall from the previous discussion that if the circumference of any circle is divided by its diameter, the result is always 3.14. . . or π. This fixed relation can be summarized by the equation $\frac{c}{d} = \pi$. So, π is a constant; namely, the constant quotient or result of dividing a circle's circumference by its diameter. Put in more formal terms, the constant pi is the ratio of a circle's circumference to its distance ($\pi = \frac{c}{d}$ or c:d).

A History of Pi

The person or persons who discovered the constant relationship between a circle's circumference and diameter are unknown. What history does show is that people's understanding of pi has developed gradually, and that gauging its value has involved a range of methods, from educated guesses based on measurements to using highly theoretical mathematics.

The number represented by pi appears in the earliest historical records of mathematics. Babylonian clay tablets (1800 B.C.E.–1650 B.C.E.) mention how a circle's area could be determined from its circumference by using a constant, which in effect is equivalent to $3\frac{1}{8}$ or 3.125. The Rhind papyrus (c. 1650 B.C.E.) includes mention of a pi-like constant, equivalent to $4(\frac{8}{9})^2$ or about 3.16, in the context of geometrical problems that involved finding a square equal in area to a given circle. A biblical account of the construction of King Solomon's temple (c. 1000 B.C.E.) suggests that the ancient Hebrews were aware that $c = 3 \bullet d$: "And he made a molten sea, ten cubits from one brim to the other. It was round all about. . .and a line of thirty cubits did compass it about" (I Kings 7:23). Archimedes of Syracuse (287 B.C.E.–212 B.C.E.) explicitly discussed the ratio of a circle's circumference and diameter and may have been the first to provide a theoretical calculation of pi. By using a series of **polygons** inscribed in a circle and another circumscribing it, he was able to determine pi was less than $\frac{22}{7}$ but greater than $\frac{223}{71}$ or about 3.1418.

polygon a geometric figure bounded by line segments

Until the sixteenth century, it was commonly thought that pi, which embodies a fundamental truth about the most perfect of geometric forms (the circle) had to be a special number. Efforts by mathematicians to determine the value of pi since have led to the conclusion that it does not have an exact value—that its decimal value extends on infinitely without repeating a pattern (3.14159265. . .). As Carl Sagan noted in his novel *Contact* (1985), mathematicians can use calculus to prove formulas for π that would permit calculation of its value. As of 1999, computers have been used to do such calculations to at least 200 billion places without a repeating pattern.

A TEENAGER'S QUEST

After five months of human time and one-and-a-half years of computer time, Colin Percival discovered the five trillionth binary digit of pi: 0. This 1998 accomplishment was significant because, for the first time, the calculations were distributed among twenty-five computers around the world.

Percival, who was 17 years old and a high school senior at the time, had concurrently been attending Simon Fraser University in Canada since he was 13.

Characteristics of Pi

As an infinitely nonrepeating decimal, pi is not a unique number. Most people are familiar with rational numbers, which can be represented by a ratio or common fraction. Some rational numbers are represented by terminating decimals. For example, $-\frac{1}{2}$, $\frac{3}{8}$, $\frac{6}{4}$, and 5 (which can be put in the form $\frac{5}{1}$) have decimal equivalents of -0.5, 0.375, 1.5, and 5.0, respectively. Other rational numbers have as a decimal equivalent an infinitely repeating deci-

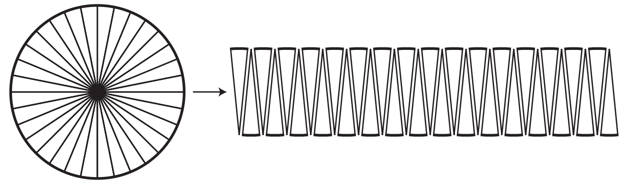
mal (i.e., a set of digits that repeats forever). For example, $-\frac{2}{3}$, $\frac{5}{6}$, $\frac{8}{7}$ or $2\frac{1}{11}$ are represented by 0.6666. . . $(0.\overline{6})$, 0.8333. . . $(0.8\overline{3})$, 1.142857142857 . . .$(1.\overline{142857}$ and 2.0909. . . $(2.\overline{09})$ respectively.

Unlike rational numbers, pi cannot be represented by a common fraction, although $\frac{22}{7}$, which is equal to the infinitely repeating decimal $3.\overline{142857}$, is a reasonable approximation of its value for many everyday uses. In other words, π is an irrational number. Although it is probably the best known number in this category, there are, in fact, an infinite number of other irrational numbers (e.g., $\sqrt{3}$, which can represent the diagonal of a square with a side of 3 linear units).

Why Does π Appear in Measurement Formulas Involving Circles?

Consider the formula for calculating the circumference of a circle. Recall that we established that $c = \pi \bullet d$. The diameter of a circle can be thought of as two radii stuck together to form a straight line, $d = 2r$. Substituting $2r$ for d in the equation gives us $c = \pi \bullet 2r$, which can be rewritten as the familiar formula for the circumference of a circle $c = 2\pi r$.

The formula for calculating the area can be derived by considering how a circle can be transformed into a shape for which we already know the area formula. As suggested by the following figure, a circle can, in theory, be divided up into an infinite number of infinitely small "pie wedges" and these pie wedges can then be put "head to toe." The result of rearranging the infinitely small wedges in this manner is a rectangle, the area of which is length (l) times width (w). Because the circumference of the circle forms the top and bottom of the rectangle, a length of the rectangle is one-half the circumference of the original circle ($l = \frac{c}{2}$). The radius of the original circle forms the width of the rectangle, $w = r$. Substituting $\frac{c}{2}$ for l and r for w in the area formula for a rectangle ($A = l \bullet w$) yields $A = \frac{c}{2} \bullet r$. Recall that $c = \pi d$ and that $d = 2r$, $c = \pi \bullet 2r$ or $2\pi r$. Substituting $2\pi r$ for c in the formula $A = \frac{c}{2} \bullet r$, produces the equation $A = \frac{2\pi r}{2} \bullet r$, which can be simplified by canceling the 2's in the numerator and denominator. This equation can then be rewritten as the familiar formula for the area of a circle: $A = \pi r^2$.



Note that rearranging the equation $A = \pi r^2$ can yield $\pi = \frac{A}{r^2}$. In other words, pi can also be thought of as the ratio of a circle's area and the square of its radius. SEE ALSO Decimals; Fractions; Numbers, Irrational; Numbers, Rational.

Arthur J. Baroody

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Kanada Laboratory Homepage. Dr. Yasumasa Kanada, University of Tokyo. <http://www.super-computing.org>

Poles, Magnetic and Geographic

Many people do not realize that there are several different "north poles." The most familiar is the north geographic pole. The north magnetic pole is not the same as the north geographic pole. It is the point on Earth's surface that a compass needle points to. Other interesting north poles include the instantaneous north pole, the north pole of balance, and the geomagnetic north pole. The mathematics of measuring angles and degrees, as well as distances, are essential tools in locating these various poles.

Polar Exploration

The north geographic pole lies in the Arctic Ocean. It is the north pole important in map making, since it is the point where all of Earth's lines of longitude come together. The geographic south pole is in Antarctica. The expedition led by American explorer Robert E. Peary is generally considered to be the first to reach the north pole. The expedition included Peary, his assistant Matthew Henson and four Inuit: Ootah, Egingwah, Ooqueah, and Seegloo. They made the trip by dogsled in 1909.

In 1926, American explorers Richard E. Byrd and Floyd Bennett became the first persons to fly over the north pole. The submarine *U.S.S. Nautilus* became the first ship to reach the pole when it passed under the ice in 1958. Since then, many expeditions have traveled over the Arctic ice and various research stations have been established on the ice.

The position of the north magnetic pole was determined about the same time. Magnetic observations made by explorers showed that the magnetic north pole and the geographic pole were not in the same place. By the early nineteenth century, the accumulated observations proved that the magnetic north pole must be somewhere in Arctic Canada.

The first expedition to the area of the north magnetic pole was on a different mission. In 1829, the explorer Sir John Ross set out on a voyage to discover the fabled (and non-existent) Northwest Passage. His ship became locked in the ice off the northwest coast of Boothia Peninsula (in far northern Canada). During the four years the ship was trapped, James Clark Ross (Sir John's nephew) explored the Boothia coast and made a series of magnetic observations. These observations convinced him that the north magnetic pole was close by. So in the spring of 1831 he attempted to locate it. On June 1, 1831, at Cape Adelaide on the west coast of Boothia Peninsula, he measured a dip of 89° 59'. A dip of 90° would occur at the north magnetic pole, so he was within a few kilometers of the spot at the most.

In 1903, the Norwegian explorer Roald Amundsen made another attempt to reach the north magnetic pole. While the expedition had a sec-

ondary goal of trying (yet again) to find a northwest passage, his primary goal was to set up a temporary magnetic observatory in the Arctic and to make a second determination of the position of the north magnetic pole. According to Amundsen's observations, the pole was several kilometers north of where the younger Ross had located it.

Shortly after World War I, Canadian scientists measured a compass dip of 89° 56' at Allen Lake on Prince of Wales Island. This, in conjunction with other observations made in the vicinity, confirmed that the pole was moving. It had moved 250 km (kilometers) northwest since the time of Amundsen's initial observations. Additional observations by Canadian scientists were made in 1962, 1973, 1984, and 1994. These observations have shown that the average position of the north magnetic pole is moving in a general northwestward direction at around 10 km per year. It also seems to be accelerating its northward motion.

Suppose you followed a compass north until you reached the north magnetic pole. How would you know you were there? At some point the compass would begin to move erratically. If you measured the vertical direction of the magnetic field it would be pointing almost straight down. The point where the magnetic field is exactly straight down (90°) is magnetic north. This pole can change position more quickly than any of the other poles. It can move many kilometers in a few years. It even has a daily motion following a roughly elliptical path around its average position. At times, it wanders as far as 80 km from its average position.

Today, the north magnetic pole is located near Ellef Ringnes Island in northern Canada. The magnetic south pole lies in Antarctica, near Vostok. It also appears to be moving, but, curiously, not in the same direction or at the same rate as the north magnetic pole.

Other Poles

Earth rotates on its axis (an imaginary line through the Earth) once every day. However, Earth wobbles a bit as it rotates, so this axis is not fixed. The instantaneous north pole lies at the point where Earth's axis passes through the surface at any moment. This point is close to the north geographic pole, but it is not exactly the same. The path followed by the instantaneous north pole is an irregular circle called the Chandler Circle whose diameter varies from a few centimeters to a few meters. It takes about 14 months for the instantaneous north pole to complete one circle.

The average of the positions of the instantaneous north pole is called the north pole of balance. It lies at the center of the Chandler Circle. The pole of balance marks Earth's actual geographic north pole, which is not exactly the same as the map makers' geographic north pole. Even this point is gradually moving toward North America at around 15 centimeters per year. Consequently, the latitude and longitude of every point on Earth is constantly changing.

The geomagnetic north pole is defined by the magnetic lines of force that loop into space outside of Earth. The geomagnetic field is sort of the average of the local magnetic field. The geomagnetic north pole lies near Etah, Greenland, north of the town of Thule. In the upper atmosphere, Earth's magnetic field points down toward Earth at this point.

Magnetic Declination

Since the geographic north pole and the magnetic north pole are different, a magnetic compass usually does not point to geographic north. The difference in degrees between magnetic north and geographic north is called the magnetic declination. Since Earth does not have a uniform magnetic field, the magnetic declination at any point on Earth's surface is the result of a complex interaction between Earth's various internal and external magnetic fields. The magnetic declination also changes over time as the magnetic north pole moves around and as local variations in Earth's magnetic field change.

Navigating with a magnetic compass requires adjusting the compass by adding or subtracting the magnetic declination from the compass reading. In the western United States, the compass needle will point a few degrees east of geographic north, and in the eastern United States the needle will point a few degrees west of geographic north. So to get the correct compass reading, the magnetic declination must be added or subtracted from the compass reading. Many modern compasses can be adjusted for magnetic declination by turning a small screw. Then the compass may be read directly, without having to add or subtract the declination.

Magnetic Anomalies

In addition to corrections that must be made for the differing positions of the north geographic and magnetic poles, corrections must also be made for local magnetic anomalies. These are local variations in the magnetic field that cause a compass to be off as much as three or four degrees from what would be expected from the position of the north magnetic pole.

Causes of the Magnetic Field

Seismic data have shown us that Earth has a molten core. Because of the overall high density of Earth, much higher than surface rocks, the core is most likely nickel and iron. Shortly after Earth first formed, it got hot enough to melt all the way through. The heavy materials, such as nickel and iron, sank to the center. Since then, Earth has been gradually cooling off. However, the core is still molten. The molten core is due to leftover heat from Earth's formation. At the center of the molten outer core, there is a solid inner core of nickel and iron. Earth's solid inner core actually rotates a tiny bit faster than the rest of the planet, making one additional rotation every 400 years or so.

Although Earth's magnetic field resembles the magnetic field that would be generated by a huge bar magnet, a bar magnet is not the source. It is not even a permanent feature of the planet. The circulation of electrical currents in the hot liquid metal in the outer core generates the magnetic field. Earth's rotation contributes to this. Venus probably has a core similar to Earth, but Venus probably does not rotate very rapidly and, consequently, it does not have a strong magnetic field. The theory that explains planetary magnetic fields in terms of rotating, conducting material in the core is known as dynamo theory. Both rapid rotation and a conducting liquid core are necessary.

Moving Poles

If the Earth acts as a large permanent bar magnet, the magnetic pole would not move, at least as rapidly as it does. In nature, processes are seldom sim-

ple. The flow of electric currents in the core is continually changing, so the magnetic field produced by those currents also changes. This means that at the surface of the Earth, both the strength and direction of the magnetic field will vary over the years. As the strength and direction of the electrical currents in the outer core changes the magnetic field of Earth changes. So the position of the north magnetic pole slowly moves across the Arctic.

Magnetic Reversals

Earth's magnetic field has regularly reversed its orientation many times over the last few million years. Such reversals seem to be part of the way in which planetary magnetic fields are generated. The current rapid and accelerating movement toward the northwest of the north magnetic pole may be in some way related to an impending magnetic reversal in Earth's core. SEE ALSO Flight, Measurements of; Navigation.

Elliot Richmond

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Polls And Polling

Opinion polling has a long history in American politics. On July 24, 1824, a Harrisburg, Pennsylvania newspaper reported on a "straw" poll taken "without discrimination of parties" which indicated that Andrew Jackson was the popular choice for U.S. President over John Quincy Adams. As it turned out, Jackson did receive the most electoral votes. However, Jackson did not have at least 50 percent of the electoral votes, so the election was decided by the U.S. House of Representatives. The methods used by the Harrisburg newspaper to take the poll are not known. Modern statistical theories on opinion polling had not yet been invented.

Scientific methods for measuring public opinion had to wait for advances in the mathematical topics of **probability theory** and **statistics**, and the technique of **sampling**. These advances came in the late 1800s and early 1900s.

probability theory the branch of mathematics that deals with quantities having random distributions

statistics the branch of mathematics that analyzes and interprets sets of numerical data

sampling selecting a subset of a group or population in such a way that valid conclusions can be made about the whole set or population

Random Sampling

In 1935, George Gallup founded the American Institute of Public Opinion to apply the latest techniques in mathematical statistics and sampling theory to take public opinion polls on various vital issues. The first major test of Gallup's scientific sampling techniques was the presidential election of 1936 between Franklin D. Roosevelt and Alfred E. Landon. This election provided a great contrast between the old and new methods of opinion polling. The *Literary Digest* magazine, one of the most respected publications of the time, conducted a massive survey of more than two million Americans, which predicted that Landon would win the presidency. Roo-

sevelt won the election, as predicted by the new Gallup poll, which was based upon a sample less than one one-thousandth the size of the *Digest's*.

Analysis of the sampling technique used by the *Literary Digest's* pollsters revealed that they had chosen the names of people to be surveyed from telephone directories and automobile registration lists, a common practice at that time. By 1936, however, Roosevelt's New Deal to end the Great Depression had helped millions of poorer families get public works jobs and welfare assistance. Vast numbers of these lower income people migrated from the Republican Party to the Democratic Party to show appreciation for Roosevelt's programs. But wealthier Americans who did not benefit as much from the Democratic policies were more likely to remain Republicans or switch to the Republican Party. Thus, using telephone and automobile ownership to select the sample to be polled biased the sample in the direction of the people who were wealthy enough to own cars and phones and who were, therefore, more likely to vote for the Republican (Landon) than for the Democrat (Roosevelt).

Telephone polling is an unbiased way of sampling only if members of the sample population all are equally likely to own and use a telephone. Conducting a telephone poll in a geographic region or a country where only a small percentage of the population uses telephones would yield biased results.

Gallup's organization, on the other hand, relied less on the size of the sample and more on the randomness of the selection process. This technique, known as random sampling, makes it far more likely that the sample will have proportional representation of various subgroups according to their proportions in the entire population.

Random sampling is based upon the mathematical theory of probability. There are two key principles involved. The first is called "equal probability of selection" and says that if every member of a population has an equal probability of being selected to the sample, then the sample will be representative of the entire population. The second principle is in the form of a mathematical equation that allows the polling organization to determine the size of a random sample necessary to give a specified degree of accuracy to the results of the poll.

From the initial poll in 1936 to the mid-1980s, Gallup's interviews were conducted in person at randomly selected residential addresses throughout the country. By 1985 the number of households with telephones was approaching 95 percent, which meant that it was no longer necessary to avoid randomly selected telephone numbers as a method for conducting interviews. But one remaining problem is that about 30 percent of households have unlisted telephone numbers.

To get around the problem of unlisted numbers, Gallup uses a technique called random digit dialing. A computer is given a list of all telephone exchanges in the country and then randomly generates lists of numbers from those exchanges. In essence, the computer creates a list of all possible phone numbers in the United States and then randomly generates a subset of that list for the Gallup interviewers to call.

Sample Size

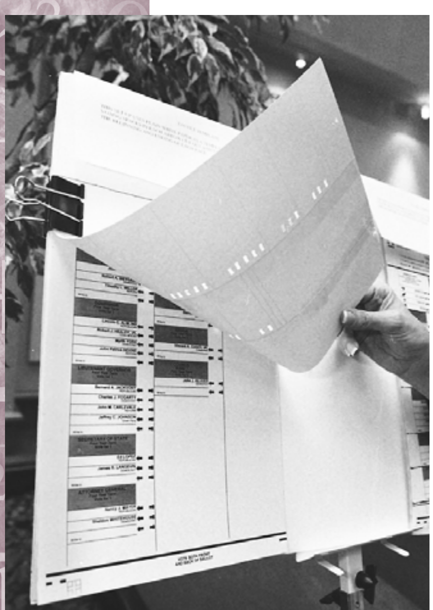
Once a random sample is assured, the next question becomes: How many people should be called? The answer depends on how much possible error one is willing to tolerate in the results of the poll. Of course, ideally, we want no error at all. Unfortunately, this can only be accomplished by interviewing every voting-age American, which, of course, is an impractical

task. So we are back to the question of how much error is acceptable given the amount of money and time available to the organization commissioning the poll. Once this acceptable level of error is established, the answer is purely mathematical.

Although the mathematics is advanced, it is straightforward and unambiguous. It says that if you are willing to settle for poll results that will be within plus or minus 5 percent of the actual election results, then interview 385 people. If you would rather be within ± 3 percent of the actual results, then survey 1,068 people. If ± 2 percent sounds better, call 2,401 people. For an error of ± 1 percent, you will need to make over 9,600 phone calls.

In practice, most organizations are willing to settle for poll results that are within ± 3 percent of actual population results due to the costs involved in the polling process. Notice that to go from 3 percent to 2 percent requires more than doubling the sample size, which will also more than double the cost of the poll. To get results that are within ± 1 percent of the true election results would increase the cost of the polling by nine-fold over the ± 3 percent level.

It should also be stated that the mathematics does not really guarantee that when 1,068 people are interviewed, you will always get results within ± 3 percent of the true population values. It only guarantees that it will happen 95 percent of the time. That is, if one hundred different polls of 1,068 people each were carried out, in ninety-five of those polls we would get our ± 3 percent accuracy. In the other five polls, we would not. Statisticians call this the 95-percent confidence level for the results of the poll. You can increase the confidence level to 98 percent or 99 percent or even 99.9 percent, but, again, it will come at huge increases in the sample sizes and, therefore, the cost of the poll. It is standard practice to use a 95-percent confidence level for public opinion polling.



Secret ballots ensure voter privacy. Shown here is a voting ballot for the blind, as used in the 2000 U.S. Presidential election.

The Exit Poll

Another type of poll that has come under great scrutiny in recent years is the "exit" poll commissioned by the television networks and other news media, and carried out at polling sites in certain "key" precincts across the nation on election day. The idea is that by doing random sampling of voters just after they have voted, the news media will be able to "project" the winner as soon as the polls close. This process has long been criticized, particularly in presidential elections, because it has been contended that projecting the results of the election in the eastern time zone before the polls have closed in the rest of the country may cause voters in the west to stay away from the polls.

Whether or not this has actually ever occurred, no one really knows, but in the presidential election of 2000, events happened that brought calls for congressional investigations of the entire process of exit polling. Prior to Election Day, nearly every major poll indicated that the race between Democrat Al Gore and Republican George W. Bush was going to be one of the closest in history. On the evening of the election, the Associated Press and all the major television networks reported that results of exit polling in key precincts in the state of Florida showed that Gore would win the state's electoral votes.

Later that same evening, the media began to backtrack on its earlier call, reporting that something had gone wrong in the exit polling and that they were putting Florida's electoral votes back into the undecided column. Within the next 2 hours, the networks determined that they once again had sufficient information to declare the winner in Florida, only this time it was George W. Bush. This pushed Bush's total above the 270 needed for election and the networks declared George W. Bush the 46th President of the United States.

This, however, was not the last word. The media would have to do yet another flip-flop as late returns from Florida showed Gore dramatically closing the gap. Florida was once again returned to the undecided column and remained there until a controversial series of Florida recounts, court cases, and an ultimate U.S. Supreme Court decision would declare Bush the winner on December 12, 2000.

What went so terribly wrong in the exit polling? Most of the scrutiny has been focussed on Voter News Service (VNS), the polling organization that all the major networks used in their projections, and on the media themselves for rushing to judgement based on too little data in an election that was ultimately decided by fewer than 500 votes out of six-million cast in the state of Florida. If the VNS "key" precincts truly showed either candidate winning by more than the margin of error in the exit polls, then something was wrong in the selection of those precincts.

In an election this close, those polls should have been showing the race too close to call, especially early in the evening when the media projected Gore the winner in Florida. The television networks' reliance on a single source (VNS) for their exit polling and their desire to be the first to call the race are certainly areas for investigation. SEE ALSO Census; Data Collection and Interpretation; Predictions.

Stephen Robinson

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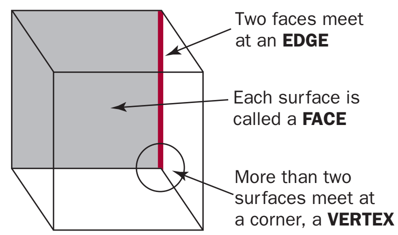
Polyhedrons

A polyhedron is a closed, three-dimensional solid bounded entirely by at least four polygons, no two of which are in the same plane. Polygons are flat, two-dimensional figures (planes) bounded by straight sides. A square and a triangle are two examples of polygons.

The number of sides of each polygon is the major feature distinguishing polyhedrons from one another. Some common polygons are the triangle (with three sides), the quadrilateral (with four sides), the pentagon (with five sides), the hexagon (with six sides), the heptagon (with seven sides), and the octagon (with eight sides).

A regular polygon, like the square, is one that contains equal interior angles and equal side lengths. A polygon is considered irregular if its interior angles are not equal or if the lengths of its sides are not equal.

Each of the polygons of a polyhedron is called a face. A straight side that intersects two faces is called an edge. A point where three or more edges meet is called a vertex. The illustration below indicates these features for a cube, which is a well-known polyhedron comprised of six square faces.

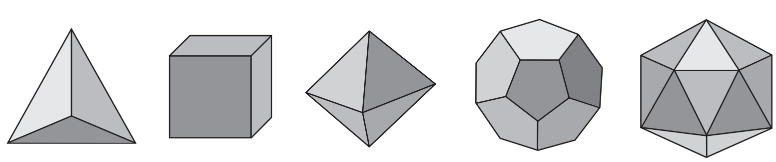


The relationship between the number of vertices (v), faces (f), and edges (e) is given by the equation $v + f - e = 2$. For example, the cube has 8 vertices, 6 faces, and 12 edges, which gives $8 + 6 - 12 = 2$. The value of $v + f - e$ for a polyhedron is called the Euler characteristic of the polyhedron's surface, named after the Swiss mathematician Leonhard Euler (1707–1783). Using the Euler characteristic and knowing two of the three variables, one can calculate the third variable.

Platonic and Archimedean Solids

There are many groupings of polyhedrons classified by certain characteristics—too many to discuss here. One common group is known as the Platonic solids, so-called because its five members appeared in the writings of Greek philosopher Plato. The Platonic solids are within the larger grouping known as regular polyhedrons, in which the polygons of each are regular and congruent (that is, all polygons are identical in size and shape and all edges are identical in length), and are characterized by the same number of polygons meeting at each vertex.

The illustration below depicts the five Platonic solids (from left to right): tetrahedron, cube, octahedron, dodecahedron, and icosahedron.



The tetrahedron consists of four triangular faces, and is represented as {3, 3}, in which the first 3 indicates that each face consists of three sides and the second 3 indicates that three faces meet at each vertex. The cube, sometimes called a hexahedron, has six square faces, and is represented as {4, 3}.

The octahedron contains eight equilateral triangles, and is constructed by placing two identical square-based pyramids base to base. The octahedron is represented as {3, 4}. The dodecahedron consists of five sides to each face, and three pentagons meeting at each of the polyhedron' twenty vertices. It is represented by {5, 3}. The icosahedron is made by placing five equilateral triangles around each vertex. It contains congruent equilateral triangles for its twenty faces and twelve vertices, and is described as {3, 5}.

Archimedean Solids. Another common group of polyhedrons is the Archimedean solids, in which two or more different types of polygons appear. Each face is a regular polygon, and around every vertex the same polygons appear in the same sequence. For example, a truncated dodecahedron is made of the pentagon-pentagon-triangle sequence.

Nets

A polyhedron can be "opened up" along some of its edges until its surface is spread out like a rug. The resulting map, similar to a dressmaker's pattern, is called a net. A net contains all faces of a polyhedron, some of them separated by angular gaps. Because a net is a flat pattern that can then be folded along the edges and taped together to regenerate the polyhedron of origin, a net therefore enables the easy construction of basic polyhedrons out of paper. The construction of polyhedron models can help make concepts in geometry easier to learn. SEE ALSO Nets.

William Arthur Atkins (with Philip Edward Koth)

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Population Mathematics

According to the U.S. Census Bureau, the world's population reached 6 billion people on June 19, 1999. The Bureau operates a population "clock" that shows the population of the world changing second by second. How is this possible? How can anyone possibly know the exact population of the world at any given second? Of course, the answer is that no one can know the world population with the degree of accuracy implied by the population clock.

And yet accurate projections of future population growth are indispensable to world leaders who must plan and set policies for health care, education, housing, energy, transportation, food and water delivery, environmental protection, disaster preparedness, and national defense. The collection and continual updating of data from the nations of the world and the use of sophisticated mathematical models provide projections that are as accurate as is currently possible.

Making Projections

The 6 billion mentioned in the opening paragraph is an estimate created by a mathematical model using data from every country in the world. The data used to make the current estimate are actually about 2 years old. So the

number on the clock is really a mathematical projection from the 2-year-old data. The following example is greatly oversimplified, but it illustrates the principle involved in such projections.

Suppose that you had high-quality data showing that the population of a certain city was one million at the beginning of the year 2000 and that this city's population had historically grown at about 2 percent per year. If you assume that there are no extraordinary events that would change this growth rate, you could calculate as follows. Two percent of 1 million is 20,000, so at the beginning of 2001, you would project the city's population to be 1,020,000. Then 2 percent of 1,020,000 is 20,400, which would give 1,040,400 at the beginning of 2002, and so on.

To make your projections as accurate as possible, you would want to continue to collect data about this city and watch for changes that might effect the growth rate. For example, suppose that the city's largest employer decided to move its business to another city, causing many people to lose their jobs and move to other locations to find work. This could negatively impact the city's growth rate and necessitate a modification in the mathematical model. The keepers of the world population clock must do a similar sort of continual updating of data and assumptions, only on a much larger scale.

Types of Models. The mathematical model used in the simple example above was an exponential model, one of the most basic models used in the study of population growth. Exponential functions can be effective models for populations that grow essentially unchecked in a straightforward way for relatively short periods of time. However, human population growth over time is constrained by environmental, social, political, and economic factors. To account for such limitations, more sophisticated mathematical models must be developed.

Mathematics helps us know what is happening with the population of the world and the possible consequences of the population patterns.

A fairly simple alternative to the exponential model is the logistic model, which projects short-term growth as exponential-like, but which includes a factor that models a slowing of the growth rate as the population moves closer to its upper limit, or carrying capacity. The logistic model can be effective with relatively isolated animal and human populations, but, by itself, it is not sophisticated enough to handle the kind of complex dynamics that occur in a system with a number of different populations, each requiring its own assumptions about its rate of growth.

This is the situation facing the scientists who make projections about the world population. Their models must be a great deal more complex to account for the multitude of variables that go into modeling the growth of the entire population of the world. In fact, such models are so complex that they take the form of sophisticated computer programs that essentially simulate the population growth for all of the countries of the world and then aggregate the results to get an estimate of the total population.

Projecting Population Growth

The first step in making projections for population growth is to obtain a base population for each country. This is usually done by using data from the most recent census. It is not sufficient just to know how many people are in a country. The population must also be broken down by gender and age, since different genders and different age groups have varying life expectancies and varying rates of producing offspring. It is of utmost importance to make sure that the counting of young children is as accurate as possible, because a serious miscount of this age group will have a ripple effect over generations. The three major components that go into making projections from some base population are fertility, mortality, and international migration.

Fertility. Fertility refers to the frequency of childbearing of women in different age groups. This can vary widely from country to country or even within the same country. Some societies encourage childbearing for women in their teens, and others have strong proscriptions against it. Poorer nations tend not to provide easy access to birth-control measures, whereas wealthier nations do. Some countries, worried about the future consequences of ballooning populations, penalize parents for having too many children. Others may be dominated by religions that place a high value on childbearing and regard contraception as unacceptable. All of these factors must be taken into account when making estimates of the fertility rate of a given country.

Mortality. Mortality refers to the life-expectancy of a given age group. Here, again, this can vary dramatically in different countries. Wealthy nations tend to have good healthcare for their citizens, while developing nations do not. This results in lower infant mortality and longer life spans for the citizens of the economically developed countries. Lack of research and development of new drugs and inadequate levels of education for medical personnel in the developing nations lead to higher susceptibility to diseases and epidemics among the populations of those countries.

An example of this is the devastation being inflicted on certain African nations by the AIDS epidemic. If left unchecked, this will have serious reper-

cussions on the populations of those countries for generations to come. The U.S. Census Bureau maintains an HIV/AIDS Surveillance Database which contains data for each country giving the percent of that country's population who are HIV positive.

These data are collected from more than 3,800 scientific publications on HIV/AIDS. The database is updated twice a year and provides the criteria for selecting countries whose population models will include an AIDS mortality component for use in projections. As of spring, 2001, twenty-six countries have such AIDS mortality components in their projections. Twenty-one of these countries are in Africa.

For each of these countries a special mathematical modeling tool is used to project the effect of a high impact, medium impact, and low impact AIDS epidemic on future population growth. This model uses nonlinear partial differential equations to simulate the epidemics. The three scenarios generated by the model are then combined with actual data points to interpolate a "most likely" projection of the effect of HIV/AIDS on the population growth.

For many of these African nations, it is estimated that the AIDS epidemic will peak around 2010, and that population projections for these countries will not return to their pre-AIDS normalcy until near the middle of the twenty-first century.

International Migration. The third major component in projecting population growth is international migration. This is the component that is subject to the highest relative error in the modeling process. The factors that influence migration among countries include changing economic conditions, political unrest, natural disasters, and other extreme and unfavorable conditions in the original homeland. These types of events tend to be highly unpredictable, making the modeler's development of accurate assumptions difficult.

Fortunately, while migration projections are the least accurate of the three components of population growth, the effect of such errors is generally much less than for errors in fertility or mortality. In the future, however, migration may become the deciding factor in whether populations in the more developed nations continue to grow or begin to decline. This is because fertility rates in these countries have been declining since the seventies, whereas in the less developed nations fertility rates have remained steady or have declined at a much slower rate. Ninety-six percent of the world's population increase now occurs in the less developed regions of Africa, Asia, and Latin America. By the year 2020, according to projections, it will be virtually 100 percent. SEE ALSO Census; Computer Simulations; Predictions.

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Population of Pets

Using mathematics to estimate pet populations has occurred for many years. The famous sequence of numbers, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, . . . , in which each new number is the sum of the previous two, was given as the answer to the following problem, stated by Leonardo Pisano Fibonacci in his *Liber Abaci:*

> What is the number of pairs of rabbits at the beginning of each month if a single pair of newly born rabbits is put into an enclosure at the beginning of January and if each pair breeds a new pair at the beginning of the second month following birth and an additional pair at the beginning of each month thereafter?

The first twelve numbers of the Fibonacci series give the answer for the first year. The sum of these first twelve is 376. At least in these theoretical circumstances, one pair of rabbits results in close to 400 rabbits after 1 year.

However, the theoretical math of this problem does not represent the actual situation. It is true that a rabbit can have a litter about every 31 days, so that part of the problem is correct. However, rabbits must usually be from 3 to 6 months old before they start to breed successfully, so the problem, which gives them only 2 months from birth before they begin breeding, is not completely realistic.

The Impact of Estimating Pet Populations

The mathematics of animal populations is a matter of interest to people in many fields. Those who work to rescue threatened and endangered species, such as the whooping crane, condors, and Kemp's Ridley turtles, among others, use population mathematics to assess their work and predict future levels of animal populations.

Cats can have their first litters when they are as young as 6 months old. Many cat owners have their pets spayed or neutered to keep the cat population under control, resulting in fewer unwanted pets euthanized at animal shelters.

The mathematics of pet populations is of interest because of the effort to minimize the number of animals that must be euthanized (or put to sleep) in animal shelters all over the United States. One California county, Santa Clara, decided that its policies and procedures for population control of unwanted animals should be based on research, rather than on pure speculation. After conducting a study, it was determined that four times more cats than dogs were euthanized in the county's shelters. Citizens also learned that 86 percent of owned cats are already spayed or neutered. This revealed that their educational efforts emphasizing the importance of spaying or neutering pets had been effective. This is also true on a national scale: since the 1980s, there has been a tremendous drop in animal euthanasia.

The survey found that some cats had litters because the owners did not realize that cats can reproduce when they are as young as 6 months. In Santa Clara, unowned, or feral, cats made up a minimum of 46 percent of the known cat population. The report showed that 10 percent of all households in the county fed stray cats.

By gathering mathematical data, organizing and analyzing them, and using graphs and tables to report the findings as useful information, those who conducted this survey came up with a specific plan to address the goal of minimizing unwanted cats. The recommendations suggested a program of education, adoption, and TTVAR, which means Trap, Test, Vaccinate, Alter, and Release. The data support such a program and predict that it would be much more effective than a new set of harsh and unenforceable regulations. SEE ALSO Fibonacci, Leonardo Pisano; Population Mathematics.

Lucia McKay

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Postulates, Theorems, and Proofs

Postulates and theorems are the building blocks for proof and deduction in any mathematical system, such as geometry, algebra, or trigonometry. By using postulates to prove theorems, which can then prove further theorems, mathematicians have built entire systems of mathematics.

axiom a statement regarded as self-evident; accepted without proof

Postulates, or **axioms**, are the most basic assumptions with which a reasonable person would agree. An example of an axiom is "parallel lines do not intersect." Postulates must be consistent, meaning that one may not contradict another. They are also independent, meaning not one of them can be proved by some combination of the others. There may also be a few undefined terms and definitions.

Postulates or axioms can then be used to prove propositions or statements, known as theorems. In doing so, mathematicians must strictly follow agreed-upon rules of argument known as the "logic" of the system. A theorem is not considered true unless it has been rigorously proved by valid arguments that have strictly followed this logic.

Deductive reasoning is a method by which mathematicians prove a theorem within the pre-defined system. Deduction begins by using some combination of the undefined terms, definitions, and postulates to prove a first theorem. Once that theorem has been proved by a valid argument, it may then be used to prove other theorems that follow it in the logical development of the system.

Euclid's Deductions

paradigm an example, pattern, or way of thinking in some subject area

Perhaps the oldest and most famous deductive system, as well as a **paradigm** for later deductive systems, is found in a work called the *Elements* by the ancient Greek mathematician Euclid (c. 300 B.C.E.). The *Elements* is a

massive thirteen-volume work that uses deduction to summarize most of the mathematics known in Euclid's time.

Euclid stated five postulates, equivalent to the following, from which to prove theorems that, in turn, proved other theorems. He thereby built his well-known system of geometry:

1. It is possible to draw a straight line from any point to any point.
2. It is possible to extend a finite straight line continuously in a straight line.
3. It is possible to draw a circle with any center and distance (radius).
4. All right angles are equal to one another.
5. Given a line and a point not on the line, there is exactly one line parallel to the given line.

Starting with these five postulates and some "common assumptions," Euclid proceeded rigorously to prove more than 450 propositions (theorems), including some of the most important theorems in mathematics. The *Elements* is one of the most influential treatises on mathematics ever written because of its unrelenting reliance on deductive proof. Its "postulate-theorem-proof" paradigm has reappeared in the works of some of the greatest mathematicians of all time.

Changing Postulates

What are considered "self-evident truths" may change from one generation to another. Until the nineteenth century, it was believed that the postulates of Euclidean geometry reflected reality as it existed in the physical world. However, by replacing Euclid's fifth postulate with another postulate—"Given a line and a point not on the line, there are at least two lines parallel to the given line"—the Russian mathematician Nikolai Ivanovich Lobachevski (1793–1856) produced a completely consistent geometry that models the space of Albert Einstein's theory of relativity. Thus the modern pure mathematician does not regard postulates as "true" or "false" but is only concerned with whether they are consistent and independent. SEE ALSO CONSISTENCY; EUCLID AND HIS CONTRIBUTIONS; PROOF.

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Powers and Exponents

An exponent is a number that indicates how many times a certain number, say b, is multiplied by itself. The expression $2 \times 2 \times 2 \times 2$ can be written as 2^4, where 4 is the exponent and 2 is called the base. An exponent, which is also called a power, is written as a superscript of the base number. A base b raised to a power n is written as b^n.

Exponent is a simple but powerful idea and can be used to create shortcuts in problems. To multiply 2^4 by 2^6, for instance, simply add the powers to find the product 2^{4+6}, or 2^{10}.

What if 2^4 is multiplied by 3^6? The bases, 2 and 3, are different. In this case, the product cannot be found by adding the powers. The following are the three basic rules of exponents. Using these three laws, more properties of exponents can be found.

1. When multiplying two numbers with the same base, add the exponents: $b^n \times b^m = b^{n+m}$

   Example: $3^3 \times 3^8 = 3^{3+8} = 3^{11}$

2. When dividing two numbers with the same base, subtract the exponents: $\frac{b^n}{b^m} = b^{n-m}$

   Example: $\frac{6^7}{6^2} = 6^{7-2} = 6^5$

3. When raising a power to a power, multiply the exponents: $(b^n)^m = b^{n \times m}$

   Example: $(4^3)^2 = 4^{3\times 2} = 4^6$.

Zero Exponent

An interesting rule involving exponents is that a number raised to zero power, say b^0, is equal to 1. This surprising result follows directly from Rule 2. Recall, a number divided by itself is 1.

$$\frac{b^n}{b^n} = 1$$

Apply Rule 2.

$$\frac{b^n}{b^n} = b^{n-n} = b^0$$

Therefore, $b^0 = 1$. This means that any number raised to 0 is 1. Hence, 5^0, 2^0, and 31^0 are all equal to 1.

Negative Exponent

What does a negative exponent mean? Here is another rule that also follows from Rule 2.

$$b^{-1} = \frac{1}{b}$$

Using $b^0 = 1$ and $b^1 = b$, $\frac{1}{b}$ can be expressed as follows.

$$\frac{1}{b} = \frac{b^0}{b^1}$$

Apply Rule 2 to the right-hand side.

$$\frac{b^0}{b^1} = b^{0-1} = b^{-1}$$

Therefore, $b^{-1} = \frac{1}{b}$.

$$\text{So, } \frac{1}{5} = 5^{-1}$$

Fractional Exponent

A base raised to a fractional power, say $b^{1/2}$, is another way to express the square root of b.

$$b^{1/2} = \sqrt{b}$$

Therefore, $9^{1/2} = \sqrt{9} = 3$. Similarly, $b^{1/3}$ is another way to express the cube root of b.

$$b^{1/3} = \sqrt[3]{b}$$

Therefore, $8^{1/3} = \sqrt[3]{8} = 2$. In a general case, the nth root of b is $b^{1/n}$.

$$b^{1/n} = \sqrt[n]{b}$$

Combining Rule 1 and the fractional exponent rule results in the following exponent property.

$$b^{m/n} = (\sqrt[n]{b})^m$$

For instance, $4^{5/2} = (\sqrt[2]{4})^5$. The number within the parenthesis, square root of 4, is 2.

$$4^{5/2} = (2)^5$$

$$4^{5/2} = 2 \times 2 \times 2 \times 2 \times 2$$

$$4^{5/2} = 32$$

SEE ALSO Radical Sign.

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Predictions

One of the primary roles of the applied mathematician is to create mathematical models to be used in predicting the outcomes of future events. These models take the form of equations or systems of equations, inequalities or systems of inequalities, computer simulations, or any combination of these that the mathematician finds appropriate.

Mathematics can sometimes give accurate predictions about future events as with Newtonian mechanics, whereas in other cases, such as predicting public opinion and estimating animal populations, the predictions given by mathematics are probabilistic. In cases where the underlying dynamical system is chaotic, such as the weather, mathematical models can be used to describe the long-term behavior of the system, but such systems by their very nature defy specific predictions due to their sensitivity to initial conditions.

Predictive Models and their Validity

The predictive validity of a mathematical model can only be judged by how well it actually forecasts future events. Models that fail to give predictions in accordance with empirical observation are either abandoned or modified. The creation of a predictive mathematical model is part science, part mathematics, and part art.

The science and mathematics of model building are often straightforward once the conditions and assumptions about the events being modeled are set. It is in the area of making correct assumptions that a bit of art is re-

quired. On the one hand, the mathematician wants to keep the model as simple as possible, and make only the assumptions that are absolutely necessary for the process being modeled. On the other hand, leaving out some critical assumption can make the model useless in predicting future events. Thus the mathematician must balance the desire for simplicity in the model with the need to consider all relevant assumptions about the real-world processes being modeled.

In some situations, the predictive ability of a mathematical model is quite good. A classic example is Newtonian mechanics as applied to the motion of planets in the solar system. Using Newton's laws, astronomers are able to predict the exact minute a lunar or solar eclipse will occur hundreds of years in the future. Such is the predictive power of Newton's model that humans can send a spacecraft to orbit around or land on distant planets.

In other situations, such as weather forecasting, mathematical models have proved disappointing in their ability to accurately predict conditions very far in the future. In theory, one ought to be able to use Newtonian mechanics to predict the motion of particles in the atmosphere and, therefore, to predict the weather, but, in fact, when this is attempted, predictions are not so good.

The Role of Chaos. It is ironic that while it is possible to launch a spacecraft from Cape Canaveral and have it reach one of Saturn's moons at a precise time years from now, humans cannot predict with much certainty whether tomorrow's weather at the Cape will allow the launch to proceed.

The reason that the motion of celestial bodies can be predicted with a high level of accuracy, but weather conditions cannot, is related to a phenomenon known as sensitivity to initial conditions. In systems that exhibit sensitivity to initial conditions, even the smallest error in one's knowledge of the initial conditions of the system can lead to large errors in predicting the outcome of that system. Such systems are said to be "chaotic," not in the usual English language meaning of the word, but according to a precise mathematical definition. On the timescale of human existence, the solar system is not chaotic, and is therefore predictable. The weather is chaotic and hence highly unpredictable.

Statistical Models

Systems that fall somewhere between the high predictability of the solar system and the low predictability of the weather can sometimes be modeled using the mathematics of probability and statistics. In making predictions based on probabilities, it is important to understand from the outset that the predictions should not be viewed as certain. Statistical theory, though, makes it possible to quantify the degree of uncertainty present in predictions.

Polling. If an opinion poll of 1,000 randomly selected people was taken to predict the sentiments of the entire country on a certain issue, then the results of that poll will typically be reported as having a margin of error of about plus or minus 3 percentage points with 95 percent confidence. This means that, if the poll showed 65 percent of the sample in favor of a certain proposition, you could be 95 percent confident that between 62 percent and 68 percent of the entire population favored the proposition. It

would be inappropriate to say that 65 percent of the population favored the proposition, because the statistical theory does not allow the making of this type of precise prediction.

Sampling and Estimation. It is often necessary for wildlife agencies to predict the number of a certain species of animal living in a particular habitat. Since it is usually impossible to do an actual count of wild animals, some method of estimation is necessary. Often a technique known as "capture-recapture" is used.

The "capture-recapture" method involves capturing a number of the animals and tagging or marking them in some way and then releasing them back into the habitat. After a period of time sufficient to allow the tagged animals to mix back into the population of untagged animals, a second capture or "recapture" is carried out, and the number of tagged animals in the recapture group is counted. Then, using a proportion, an estimate of the number of animals in the entire population can be made.

To illustrate, suppose that 50 deer are captured, tagged, and released back into the habitat. Several weeks later, 80 deer are captured, 16 of which are tagged. Then, letting N be the number of deer in the entire population, set up the proportion $16/80 = 50/N$ and solve for N to get 250 as our estimate of the entire herd.

Now, as with the opinion polling example, you cannot conclude that there are exactly 250 deer in the entire herd. In fact, because the sample size is only 80 compared with 1,000 in the opinion poll, you get an even wider margin of error. In this case, the mathematical statistics says that you can be 95 percent confident that the true number of deer in the herd lies between 213 and 303. SEE ALSO CHAOS; DATA COLLECTION AND INTERPRETATION; ENDANGERED SPECIES, MEASURING; POLLS AND POLLING; PROBABILITY AND THE LAW OF LARGE NUMBERS; STATISTICAL ANALYSIS.

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Primes, Puzzles of

Many number puzzles involve prime numbers. Understanding the distribution of prime numbers is hence an important part of understanding the number system.

Generating Primes

A prime number is a number evenly divisible only by itself and by one. Seventeen is a prime number because there is no positive **integer** that can be divided into seventeen without a remainder except for one and 17. The smaller primes are well known: 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, and so on. (Some mathematicians leave out 1.)

integer a positive whole number, its negative counterpart, or zero

A straightforward method for generating primes was proposed by the Greek mathematician Eratosthenes (276 B.C.E.–194 B.C.E.). To begin this procedure, write down all the integers as high as you wish to go. Starting with 2, cross out every second number. Then, starting with 3, cross out every third number. Find the next uncrossed number, 5, and then cross out every fifth number. The next uncrossed number is 7, so cross out every seventh number. This procedure can be repeated indefinitely.

The scheme just described is sometimes called the sieve of Eratosthenes. The procedure will produce every prime number up to whatever value one chooses, but the process is time-consuming and labor-intensive. Different methods are used to generate larger prime numbers.

Predicting Primes

There are many unsolved questions about prime numbers. One of the most famous of all unsolved mathematical problems is predicting the occurrence of prime numbers. If the primes are numbered consecutively, is there a relation between a prime and its number? No one has ever discovered one. If the nth prime is known, can the $(n + 1)$th prime always be calculated? No one has figured out how.

One direct method of determining if a number is prime is by attempting to factor it. Factoring a number involves dividing it by all the prime numbers smaller than the number's square root. For example, suppose we wish to determine if 221 is a prime number. The square root of 221 is less than 15 (since $15^2 = 225$), so we must test each prime number smaller than 15. The number is odd, so 2 is ruled out as a factor. Likewise, 3 can be ruled out since the digits do not add up to a multiple of 3. The number does not end in 0 or 5, so 5 cannot be a factor. Long division shows that 7 and 11 are not factors. However, 13 is a factor, so 221 is not prime.

The process just described was easy to use for determining if 221 is a prime number, but it would be a very difficult procedure to use for a very large number.

Considering the Largest Prime

Is there a largest prime number? Euclid proved there was not, so at least that question about primes has been answered. Euclid's proof is fairly easy to follow. Assume that there is a largest prime number and call it p. Let $N = 2 \times 3 \times 5 \times 7 \times \ldots \times p$. Thus, N is a number divisible by all the prime numbers up to p. $N + 1$ is clearly not divisible by any of the prime numbers up to and including p, because division by p or any smaller prime would leave a remainder. So there are only two possibilities for $N + 1$:

1. $N + 1$ is divisible by some prime number larger than p.
2. $N + 1$ is prime.

Either of these possibilities contradicts the original assumption that p is the largest prime number. When an assumption leads to a contradiction, the assumption must be false. Thus Euclid proved there is no largest prime number.

STUDENTS, COMPUTERS, AND LARGE PRIME NUMBERS

In 1998, college student Roland Clarkson discovered the largest prime number known at that time. It was so large it could fill several books. In fact, college students are finding some of the largest prime numbers today by using networks of cooperating personal computers and software downloadable from the Internet.

Prime Twin Pairs

One of the first things noticeable about tables of primes is that there are many instances of pairs of prime numbers with the form n and $n + 2$. Examples are 11 and 13, 17 and 19, 29 and 31, 101 and 103, 881 and 883, and so on. These are sometimes called prime twin pairs. No one has ever determined if this is an interesting property of numbers or just a curious coincidence. The instances of pairs of prime numbers decrease for ever larger numbers. This leads to the question: Is there a largest prime twin pair? Like most questions about primes, this one has yet to be answered. No such simple proof as Euclid's exists for prime twin pairs.

Distribution of Primes

Another pattern that mathematicians try to understand is the distribution of primes. While prime numbers generally get less frequent as they get larger, there is so much variation in this behavior that it cannot be used to predict the next prime. The interval between primes also tends to get larger with larger numbers, but with so much variation that prediction is difficult. For example, between 100,001 and 100,100, there are six primes. In the next 100-number interval—100,101 to 100,200—there are nine prime numbers. Between 299,901 and 300,000 there are also nine prime numbers.

Approximating Primes

While there is no formula that can be used to predict the occurrence of primes, there are several formulas that come close. One of the best is a formula developed by the mathematician Georg Riemann: $\int_2^x \frac{dt}{\ln t} - \frac{1}{2}\int_2^{x^{1/2}} \frac{dt}{\ln t}$. This formula is exactly correct nineteen times between 1 and 9,000,000 and quite close the rest of the time. While the formula can be used to suggest a candidate large prime, the number must still be tested to see if it is actually prime.

Goldbach's Conjecture

An interesting unsolved problem concerning primes is known as Goldbach's Conjecture, which states that every even number greater than 2 can be written as the sum of two primes. Thus $12 = 5 + 7$, $18 = 7 + 11$, and so on.

Goldbach's Conjecture was first stated in 1742 in a letter written by Prussian historian and mathematician Christian Goldbach (1690–1764) to the Swiss mathematician Leonhard Euler (1707–1783). Although Euler spent much time trying to prove it, he never succeeded. For the next 250 years, other mathematicians would struggle in similar fashion.

While it seems obvious and deceptively simple, as of this writing no one has proved Goldbach's Conjecture. Supercomputers have proved Goldbach's Conjecture for every even number up to 400,000,000,000,000, and most mathematicians believe it to be true, yet the search for an abstract proof goes on. In 2000 a British publishing company offered a \$1 million prize for proof of Goldbach's Conjecture by the year 2002 if it is accepted and published by 2004.

Palindromic Primes

One of the more curious prime number puzzles that mathematicians have examined is the occurrence of palindromic primes. A palindrome is a word or phrase that reads the same forward or backward. Some examples of palindromic primes are 101, 131, 151, 181, 313, 353, 727, 757, 787, 79997, 91019, 1818181, 7878787, 7272727, and 3535353. There is an infinite number of palindromic primes. SEE ALSO Number Sets.

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Probability and the Law of Large Numbers

Theoretical and experimental probabilities are linked by the Law of Large Numbers. This law states that if an experiment is repeated numerous times, the relative frequency, or experimental probability, of an outcome will tend to be close to the theoretical probability of that outcome. Here the relative frequency is the quotient of the number of times an outcome occurs divided by the number of times the experiment was performed.

The Law of Large Numbers is more than just a general principle. The Swiss mathematician Jakob Bernoulli (1659–1705) was the first to recognize the connection between long-run proportion and probability. In 1705, the year of his death, he provided a mathematical proof of the Law of Large Numbers in his book *Ars Conjectandi* ("The Art of Conjecturing"). This principle also plays a key role in the understanding of sampling distributions, enabling pollsters and researchers to make predictions based on **statistics**.

statistics the branch of mathematics that analyzes and interprets sets of numerical data

Demonstrating the Law of Large Numbers

When a fair die is tossed, the likelihood that the number on the top face of the die will be 2 is $\frac{1}{6}$ because only one of the six numbers on the die is a 2. So the theoretical probability of rolling a two is $\frac{1}{6}$ or about 17 percent. If a die is tossed six times, a 2 may be rolled more than once or not at all; hence, the percentage of times that a 2 is rolled will vary from the theoretical probability of $\frac{1}{6}$. However, if the die is tossed 600 times, the relative frequency should approximate the theoretical probability. Hence, the number of times the result is 2 after 600 tosses should be fairly close to $600 \times \frac{1}{6}$, or 100.

Suppose each of 100 people rolls a fair die 600 times while keeping track of the percentage of times a 2 was rolled. There most likely would be variations in their resulting relative frequencies. Still, the vast majority of the relative frequencies would be close to $\frac{1}{6}$. If each die were rolled many more times, each of the individual results would tend to be even closer to $\frac{1}{6}$.

Misconceptions about Probability

If a coin is flipped once and it lands heads up, does that mean it will land tails up next time? Certainly not. The Law of Large Numbers does not ap-

ply to any individual flip of the coin, but rather to the long-run behavior. If the coin landed heads up nine times in a row, it cannot be assured that the next flip will show tails. The probability that the next flip of the coin will be heads is still 50 percent. Even if many more heads than tails have been rolled initially, it should not be expected that heads will appear less often in the future.

Gambling houses and insurance companies use the Law of Large Numbers in order to determine a reasonable return that will encourage customers to participate while assuring the company of a nice profit and protecting the company from serious loss. Because gambling houses and insurance companies do a large amount of business, the experimental probability of wins or claims closely resembles the theoretical probability. However, an individual gambler or insured customer does not represent a large number of experiments, and individual results will often appear more or less than the average.

Gamblers who have been losing at roulette, or noticing that others have been losing recently at a particular slot machine, should not expect any increased likelihood of winning in the near future. The Law of Large Numbers does not imply that future winnings will occur to compensate for the earlier losses. In the same way, a gambler who is on a lucky streak should not expect a string of losses to balance out the wins.

Instead, flipping a fair coin and rolling a standard die are considered to be random, independent events. The result of one flip of a coin does not affect the likelihood of heads on subsequent flips. On each flip, the coin is just as likely to land heads up as tails up. It is unaffected by any tosses that have already occurred. SEE ALSO Predictions; Probability, Experimental; Probability, Theoretical.

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KERRICH'S COIN TOSSES

Mathematician John Kerrich tossed a coin 10,000 times while interned in a prison camp in Denmark during World War I. At various stages of the experiment, the relative frequency would climb or fall below the theoretical probability of 0.5, but as the number of tosses increased, the relative frequency tended to vary less and stay near 0.5, or 50 percent.

For example, Kerrich recorded the number of heads within each set of ten flips. In his first ten flips, the coin landed heads-up four times for a relative frequency of 0.4. In the next ten flips, he observed six heads, bringing his overall relative frequency to 0.5. After 30 tosses, the proportion of heads was 0.567. After 200 tosses, it was 0.502. With this small number of tosses, the proportion of heads fluctuates.

But after 10,000 tosses, Kerrich counted a total of 5,067 heads for a relative frequency of 0.5067, which is fairly close to the theoretical probability.

Probability, Experimental

The experimental probability of an event is defined as the ratio of the number of times the event occurs to the total number of trials performed. As a consequence of this definition, experimental probability is not a fixed value but varies depending on the particular set of trials on which it is based.

As with theoretical probability, the possible values of the experimental probability range from 0 to 1. An event would have an experimental probability of 0 if, during the trials performed, that outcome never occurred. But unlike theoretical probability, experimental probability bases its value solely on the results of the experiment, rather than assuming that certain events are equally likely to occur.

The theoretical probability of rolling "snake eyes" is 1 in 36, but the experimental probability will depend on the number of trials.

Conducting Experiments

Performing experiments for the purposes of determining experimental probabilities can provide powerful insights into the nature of probability. Rolling dice, tossing coins, drawing marbles from a hat, and selecting playing cards from a deck are just a few of the common probability experiments that can be performed.

Consider, for example, the toss of a die. A die is a cube, each face of which displays from 1 to 6 dots. In tossing a die one hundred times, the following results were attained:

2, 4, 4, 5, 2, 3, 1, 5, 1, 3, 2, 1, 3, 3, 4, 5, 1, 5, 5, 4, 3, 6, 5, 6, 6, 5, 6, 6, 2, 6, 4, 4, 4, 3, 3, 2, 2, 4, 3, 4, 1, 2, 2, 5, 5, 6, 4, 1, 1, 6, 4, 2, 6, 6, 5, 1, 3, 4, 6, 1, 5, 4, 3, 1, 6, 4, 3, 5, 6, 2, 4, 6, 1, 5, 5, 3, 3, 6, 4, 4, 6, 2, 6, 1, 3, 1, 4, 5, 2, 3, 3, 4, 6, 3, 6, 3, 3, 3, 3, 5.

In order to calculate the experimental probability of each value occurring, tally the results.

1	2	3	4	5	6
13	12	21	19	16	19

Based on these results, the experimental probability of tossing a 1 is $\frac{13}{100}$, or 0.13, and so forth. Now consider some important questions. Will the same values for these experimental probabilities occur if the same die is tossed another one hundred times? Is tossing a 3 on this die more likely than tossing a 2 because the experimental probability of tossing a 3 is $\frac{21}{100}$, whereas the experimental probability of tossing a 2 is only $\frac{12}{21}$?

The answer to the first question is rather obvious. Although it is possible for the same experimental probabilities to recur, it is extremely unlikely.

The second question is more interesting and, in fact, not enough information is known. The difficulty arises in not knowing whether the die being tossed is fair or not.

Is a Die Fair?

A fair die is one for which the probability of tossing any of the six possible outcomes is equal, with each having a probability of $\frac{1}{6}$, or approximately 0.167, of occurring. Dice manufacturers work hard to produce fair dice because games of chance are based on the premise that the six outcomes are equally likely. The experimental probabilities presented earlier were not exactly equal to $\frac{1}{6}$, and so more investigation is necessary in order to determine if these particular die are fair.

Although there is no reason to believe the die being tossed is unfairly weighted on one side or another, this is certainly possible. If the die is not fair and the probability of tossing a 3 is somewhat greater than the probability of tossing the other values, this could be verified by tossing the die a much greater number of times. For example, if this die were tossed 10,000 times and we found the experimental probability of tossing a 3 to be around 0.21, we would be reasonably satisfied that the die is not fair. Conversely, if the experimental probability using 10,000 trials were about 0.16 or 0.17, we would assume the die is fair.

Assessing Experimental Results

In some circumstances it is impossible to calculate the theoretical probability that an event will occur. In these instances, data are gathered, an experimental probability is determined, and decisions are formed based on this information.

Consider the following scenario. A friend suggests playing a game of chance involving cutting a deck of cards. One person will be given the choice of calling "picture card" or "not picture card" before cutting the deck. If that person correctly guesses the category of the card selected, he or she wins the game, but if the guess is incorrect, he or she loses the game. After each game, the card is returned to the deck, and the deck is shuffled.

Given that there are only twelve picture cards in a standard deck of fifty-two cards, a smart strategy would be to guess "not picture card" every time. Using this strategy, the results of the first twenty games played are as follows: win, win, loss, win, win, loss, loss, loss, win, loss, win, win, loss, win, loss, win, win, loss, win, loss. The results of the first five games are not unexpected, but the loss of games 6, 7, and 8 may make the player feel unlucky. After completing twenty games, eleven were won and nine were lost. Although more games were won than lost, one may still be quite surprised by the results.

Because there are more than three times as many non-picture cards as picture cards, the player may have expected to win closer to fifteen or sixteen games rather than only eleven. Thus, the experimental probability of winning by guessing non-picture card based on this series of games is $\frac{11}{20}$, or 0.55. The theoretical probability, however, is $\frac{40}{52}$, or approximately 0.77. The discrepancy between these values should lead the player to take a closer look at the deck of cards.

Upon inspecting the deck, the player finds that the friend had been using a pinochle deck, which is composed of forty-eight cards: eight nines, eight tens, eight jacks, eight queens, eight kings, and eight aces. Because twenty-four of these cards are picture cards (jacks, queens, and kings) and twenty-four are not picture cards (nines, tens, and aces), the probability of getting a non-picture card is $\frac{1}{2}$. The probability of getting a picture card is also $\frac{1}{2}$. Given this new information, the results attained seem more reasonable. SEE ALSO GAMES; GAMING; PROBABILITY AND THE LAW OF LARGE NUMBERS; PROBABILITY, THEORETICAL.

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Probability, Theoretical

The theoretical probability of an event is based on the assumption that each of a number of possible outcomes is equally likely. The theoretical probability of an event can be defined as the ratio of the number of favorable outcomes to the total number of outcomes in the sample space. For example, if a die is tossed, the probability of getting a 4 is $\frac{1}{6}$ because 4 is one of six

possible outcomes. Similarly, the probability of getting a number less than 5 is $\frac{4}{6}$, because either 1, 2, 3, or 4 is favorable.

The possible values of the theoretical probability of an event range from 0 to 1. If none of the possible outcomes is favorable to an event, the theoretical probability is 0. For example, the probability that the number attained on a roll of a die is greater than 8 is 0, because such an event is not a possible outcome. Similarly, the probability that the number attained is positive equals 1, because every possible outcome is favorable to this event.

LET'S FLIP FOR IT

The concept of theoretical probability is often used when the goal is to make a fair selection. The National Football League, for example, uses the flip of a coin to determine which of the two teams will have their choice of kicking or receiving when the game begins. In this way, neither team is given an advantage since the theoretical probability of heads is $\frac{1}{2}$ and the theoretical probability of tails is $\frac{1}{2}$.

Understanding Equal Likelihood

The assumption of equally likely outcomes, although critical to the determination of theoretical probability, can also be somewhat misleading. For example, consider the person who, when asked what he thinks his chances are of winning the lottery, responds, "Fifty percent, because there are only two possibilities: either I will win or I will lose." Clearly in this situation, winning and losing are *not* equally likely!

Consider the sum that results when two dice are tossed. The possible outcomes for this experiment are 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. These outcomes are not equally likely because they are based on a two-part experiment rather than a simple experiment. Yet the principle of equally likely outcomes can be applied to determine the appropriate individual theoretical probabilities.

To simplify the analysis, assume that one die is blue and the other red. The number attained on one die will not affect the number attained on the other. Thus, if a 1 is tossed on the blue die, the red die can show any of the numbers from 1 to 6. Similarly, if the blue die is 2, the red die can be any number from 1 to 6. Continuing in this fashion, one can see that the sample space for this experiment consists of thirty-six possible outcomes, any one of which is equally likely. Using the information from the matrix below, it can be determined that the probability of the sum of the dice being 10 is $\frac{3}{36}$, or $\frac{1}{12}$, since a sum of 10 occurs in three of the thirty-six possible outcomes.

The window washer who thinks he has a 50-50 chance of falling misunderstands theoretical probability. In reality, the chance of falling and the chance of not falling are *not* equally likely, because he has taken safety measures to reduce the chance of injury.

Blue / Red	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Games Involving Probability

Analyzing theoretical probabilities provides the basis for determining the best strategy to use in a wide range of games. It is important to keep in mind, however, that when games involving probabilistic events are played, the actual results will vary from the theoretical probabilities. In the short run, these differences may be quite dramatic.

For example, in the game of Monopoly, a player who lands in jail can either pay a fine or attempt to get out by rolling a pair of dice. If doubles are rolled (the same number appearing on each die), the player gets out of jail. Based on the matrix above, the theoretical probability of rolling doubles is $\frac{6}{36}$, or $\frac{1}{6}$. But anyone who has played Monopoly knows that sometimes a player may roll the dice ten or more times before gaining freedom, whereas on other occasions a player gets out on the first roll.

In some situations, the determination of theoretical probability must be adjusted based on known information. In the seven-card stud version of poker, for example, each player receives seven cards, four of which are dealt face-up. Suppose a player's first six cards are the 3 of hearts, 6 of diamonds, 7 of diamonds, 9 of spades, jack of diamonds, and queen of diamonds. If the next card is a diamond, the player will have a flush (five cards of the same suit) and will most likely win, otherwise the player will almost certainly lose.

If the decision to remain in the game or drop out is based on the assumption that the probability of getting a diamond is 1/4 (because thirteen of the fifty-two cards in the deck are diamonds and $\frac{13}{52}$ equals $\frac{1}{4}$), the reasoning will be seriously flawed. Instead, the player must consider all of the cards that have been seen.

If six other players are in the game, their twenty-four face-up cards will be known in addition to the six cards held by the player. If none of the other twenty-four cards seen is a diamond, the player's probability of getting the flush is $\frac{9}{22}$, because it is equally likely that any of the twenty-two unseen cards will be dealt and nine of them are diamonds. But if nine of the twenty-four face-up cards of the other players are diamonds, the probability of getting the flush is 0, since all thirteen diamonds have already been dealt. Depending on the composition of the cards already seen, the probability of getting the flush may be significantly less than or greater than $\frac{1}{4}$. SEE ALSO GAMES; GAMING; PROBABILITY AND THE LAW OF LARGE NUMBERS; PROBABILITY, EXPERIMENTAL.

Robert J. Quinn

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Problem Solving, Multiple Approaches to

Most of the mathematical problems that one encounters have already been solved and documented. Thus, the most important problem-solving tools are references. A reference can be anything—a book, a person, or a past experience—that aids in understanding and solving a problem. References enable one to solve problems independently when no other source of help is available.

Basic Strategies

The following are some general problem-solving strategies: break the problem into smaller parts, find a new perspective, work backward, guess. In all of these strategies, simple logic is used.

Logic. Mathematics is internally consistent. If at any point a false statement is generated, like $0 = 1$, it is at once apparent that an error or a false assumption has been made. A false statement *can* be generated intentionally, as in a proof by contradiction.

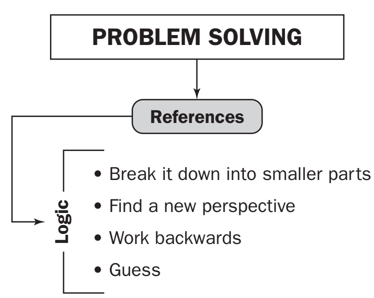
Mutually exclusive dichotomies and trichotomies categorize the universe. Perhaps the most well-known dichotomy is true versus false. If x is true, then x is not false, and vice versa. The Trichotomy Axiom is a mathematical truism: If a and b are numbers, then either $a = b$, $a > b$, or $a < b$. Dividing all possible solutions into mutually exclusive categories can quickly eliminate incorrect solutions.

The Whole Is Equal to the Sum of Its Parts. When one is overwhelmed by a problem, it is a good strategy to break the problem down into smaller parts. As a general rule, the whole is equal to the sum of its parts, so, if all the pieces of a problem are solved, then the entire problem is solved.

Sometimes, the whole is *greater* than the sum of its parts. For example, "proper completed need in parts, may be the to assembled order once," makes no sense, but "once completed, the parts may need to be assembled in the proper order," does. After solving all of the pieces of a big problem, it must be determined that the pieces fit together into a form that makes sense.

Change of Perspective. Mathematical equations and concepts often have different forms that may be better suited to specific situations, and the internal consistency of mathematics guarantees that different forms of the same thing are equally valid. Here are 3 ways to represent 8: 2^3, $\frac{16}{2}$, $\sqrt{64}$.

Manipulatives and pictures can also provide a different perspective. When large amounts of information are presented, a visual or tangible connection can aid organization and understanding. For example:



Work Backward. In mathematical proofs in which the solution is known and the problem is proving the solution, working backward determines prerequisites for the solution. A series of questions may need to be asked of the task: What must be true (or false) in order that the solution is true (or false)? Do these prerequisites have prerequisites? And so on. In this manner, one continues working backward until the correct problem–solving strategy is determined.

When in Doubt. . . If one is confronted with a completely unfamiliar problem, guessing may well be the best strategy. Trial and error are an essential part of science and often the only way to proceed when charting new territory. A methodical approach, including meticulous recording of data

and a careful search for patterns, makes guesses more informative and more accurate. SEE ALSO Proof.

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Proof

What do the statements 2 + 2 = 4 and "the sky is blue" have in common? One might say that they are both true. The statement about the sky can be confirmed by going outside and observing the color of the sky. How, then, can one confirm the truth of the statement 2 + 2 = 4? A statement in mathematics is considered true, or valid, on the basis of whether or not the statement can be *proved* within its mathematical system.

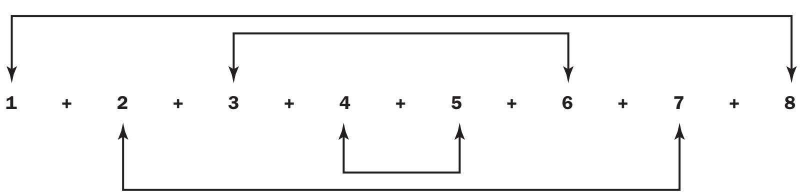
What does it mean to *prove* something mathematically? A mathematical proof is a convincing argument that is made up of logical steps, each of which is a valid deduction from a beginning statement that is known to be true. The reasons used to validate each step can be definitions or assumptions or statements that have been previously proved.

As an example of a proof, look at finding the sum of the first n whole numbers.

> The sum of the first three whole numbers is 1 + 2 + 3, so $n = 3$, and the sum is 6.
>
> The sum for $n = 4$ is 1 + 2 + 3 + 4, and this sum is 10.

Is there a pattern here? Can the sum be found (without doing all the addition) for $n = 8$?

In 1787, a teacher gave this problem to a 10-year-old boy, Carl Friedrich Gauss. Gauss pointed out the following pattern:



What is the sum of each of the number pairs indicated by the arrows? The sum is four pairs that sum to 9, and there are eight numbers in all.

What would the **mean** of these numbers be? The mean is (4 × 9) divided by 8. The mean of the whole set is $\frac{9}{2}$, which is also the mean of the first and last numbers in the set. The sum of eight numbers, whose mean is $\frac{9}{2}$, is $8\left(\frac{9}{2}\right)$, or 4 × 9, which is 36. Maybe the sum of the first n whole numbers can be found by finding the mean of the first and last whole numbers, and then multiplying that mean by n.

mean the arithmetic average of a set of data

Try the following pattern to see if it works for other values of n. What happens if an odd number is chosen for n?

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$$

Is the sum the same as the mean of 9 and 1 ($\frac{10}{2}$ or 5) multiplied by n (which is 9 in this case)? Is the sum 5×9, or 45?

Is the sum of the first n whole numbers always equal to the mean of the first and last multiplied by n? This seems to be true, but in mathematics even a huge number of examples is not enough to prove the truth of the statement. Therefore, a proof is needed.

The first number in the sum is always 1. The last number in the sum is always n. The mean of n and 1 is $\frac{(n+1)}{2}$ according to the definition of mean. In algebra, n multiplied by this mean looks like $n \times \frac{(n+1)}{2}$. A proof must show that the sum of the first n whole numbers is always $n \times \frac{(n+1)}{2}$.

First, consider n as an even number. In that case, there is an even number of pairs, and each pair has a sum of $n + 1$, regardless of the size of n. The truth of this pattern does not depend on the size of n, as long as n is an even number so $\frac{n}{2}$ pairs can be made, each of which adds to $(n + 1)$.

$$1 + 2 + 3 + 4 + \ldots + (n-3) + (n-2) + (n-1) + n$$

There are $\frac{n}{2}$ pairs, each of which adds to $(n + 1)$, so the total is $(\frac{n}{2})(n+1)$ or $n \times \frac{(n+1)}{2}$. This means that the pattern is proved true as long as n is an even number.

Next, consider the case where n is an odd number:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$$

The circled number in the middle will always be the mean of the first and last numbers. In algebra, the middle number will be $\frac{(1+n)}{2}$. So the middle number adds one more mean to the $(n - 1)$ means that were made by the paired numbers.

So, again, the total sum is n multiplied by the mean of the first and last numbers, or $n \times \frac{(n+1)}{2}$. These two cases, for n, an even number, and n, an odd number, together make up the proof.

There are several forms of mathematical proofs. The one just given is a direct proof. Indirect reasoning, or proof by contradiction, can also be used. A third kind of proof is called mathematical induction.

Although many examples do not prove a statement, one counter-example is enough to disprove a statement. For example, is it true that $y + y = y \times y$? Try substituting values of 0 and then 2 for y. Although the statement is true for 0 and 2, it is not true in general. One counter-example is $y = 1$, since $1 + 1$ is not equal to 1×1 because 2 is not equal to 1.

Here is a well-known proof that $0 = 1$. Try to find the flaw, or mistake, in this proof.

1. Assume that $x = 0$.	Assumption
2. $x(x - 1) = 0$	Multiplying each side by $(x - 1)$
3. $(x - 1) = 0$	Dividing each side by x
4. $x = 1$	Adding 1 to each side
5. $0 = 1$	Substitute $x = 0$, the original assumption

All the steps except one are valid. In Step 3, the proof divided each side by x. The reason for this is that, if $a = b$, then $\frac{a}{c} = \frac{b}{c}$, if c is not equal to 0. But the original assumption said that x was equal to 0, so Step 3 involved dividing by 0, which is undefined. Allowing division by 0 can lead to proving all sorts of untruths! SEE ALSO Induction.

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Puzzles, Number

The study of number puzzles has fascinated mathematicians and others for thousands of years. These studies reveal much about the way our number system is constructed. Studies of number puzzles also often produce new methods of mathematical thinking and new computer programming techniques. These classic number puzzles are explained in the following paragraphs.

Triangles

Many people know that $3^2 + 4^2 = 5^2$. There are many other combinations of three whole numbers that have the same property, such as $5^2 + 12^2 = 13^2$ and $7^2 + 24^2 = 25^2$. These numbers are called Pythagorean triples because of the property of right triangles known as the Pythagorean Theorem (after the Greek mathematician Pythagoras).

Since the time of the early Greeks, mathematicians have known of a way of generating any number of Pythagorean triples. If m and n are any two **integers**, then a triangle with sides X, Y, and Z can be "generated" by these formulas:

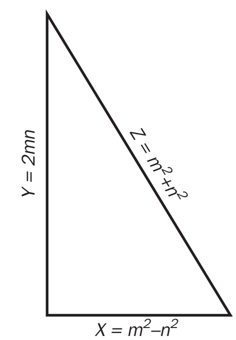
integer a positive whole number, its negative counterpart, or zero

$$m^2 - n^2 = X$$

$$2mn = Y$$

and

$m^2 + n^2 = Z$. The illustration below gives a graphical depiction.



It is obvious that there are infinitely many different sets of Pythagorean triples that can be generated using this formula.

Fermat's Last Theorem

The equation $x^2 + y^2 = z^2$ has many different solutions which can be generated by the technique in the paragraph above. What about the equation $x^3 + y^3 = z^3$? Or the equation $x^4 + y^4 = z^4$? Do these equations have integral solutions or even rational solutions? (A rational number is any number that can be written as $\frac{a}{b}$ where a and b are integers and $b \neq 0$.) In general, does the equation $x^n + y^n = z^n$ have rational solutions for values of n greater than 2? The surprising answer is no.

The statement that "$x^n + y^n = z^n$ has no rational solutions for values of n greater than 2" is referred to as Fermat's Last Theorem because the famous mathematician Pierre de Fermat wrote in the margin of a book, "I have discovered a truly marvelous proof of this, which, however, the margin is not large enough to contain."

Fermat's reputation as a mathematician was so great that everyone assumed he had found a proof. However, no one has ever been able to rediscover it. The proof of Fermat's Last Theorem, which was published by Andrew J. Wiles in the *Annals of Mathematics* in 1995, is based on other theorems that would not have been available to Fermat.

So, did Fermat actually have a proof? Probably not. Although he claimed to have found a proof of the theorem, he spent much time and effort proving the cases $n = 4$ and $n = 5$. Had he derived a proof to his theorem earlier, there would have been no need for him to study these specific cases.

Beyond Infinity

Can anything be bigger than infinity? At first glance, the question seems ridiculous. What is normally meant by infinity, ∞, is the limit of a mathematical operation. If $x = \frac{12}{n}$ and n is allowed to get progressively smaller and smaller, approaching zero, we say that x approaches ∞. Since n never reaches zero, x never reaches ∞. So ∞ is a "potential" infinity, approached but never reached.

There are also actual infinities. Suppose you went to a football game and looked around and saw that every seat was taken. You know the stadium holds 50,000 people, so you could say with certainty that there were 50,000 people at the game, even though you had not counted them. The people and the seats had a one-to-one correspondence.

In the same way, any set of numbers that can be placed in one-to-one correspondence with the set of whole numbers has the same size as the set of whole numbers. That means that the set of even numbers is the same size as the set of whole numbers. This is one of the curious properties of **transfinite** numbers discovered by the mathematician Georg Cantor. He also proved that the set of rational numbers can be counted (placed in one-to-one correspondence with the whole numbers) and is also the same size.

transfinite surpassing the finite

However, the set of real numbers cannot be counted. Real numbers contain numbers such as pi (π) and $\sqrt{2}$ that cannot be written as the ratio of two whole numbers. There is no way to place the set of real numbers into one-to-one correspondence with the whole numbers because there will always be numbers left over. Hence, the set of real numbers is larger than the set of rational numbers. In other words, the size of the set of real numbers is a larger infinity! Cantor discovered an entire class of transfinite numbers of ever-larger size. SEE ALSO Fermat's Last Theorem; Infinity; Numbers, Rational; Numbers, Real; Numbers, Whole.

Elliot Richmond

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Pythagoras

Mathematician and Philosopher
c. 582 B.C.E.–c. 500 B.C.E.

Considered a mathematician, but foremost a philosopher, Pythagoras was a very important figure in mathematics, astronomy, musical theory, and in the world's history. However, little in the way of reliable record is known about his life and accomplishments. The accounts of Pythagoras inventing the musical scale, performing miracles, and announcing prophecies are probably only legend, and appear to have little historical foundation. Scholars generally agree only upon the main events in his life, and usually combine together discoveries by Pythagoras with those by his band of loyal followers.

Although the events of Pythagoras's life have been obscured by countless legends spread by fervent devotees, his contributions to philosophy, mathematics, physics, and astronomy are without question.

Pythagoras established in what is now the southeastern coast of Italy a philosophical, political, and religious society whose members believed that the world could be explained using mathematics as based upon whole numbers and their ratios. Their motto was "All is number." Even the words philosophy (or "love of wisdom") and mathematics (or "that which is learned") is believed to have been first used (and defined) by the Pythagoreans.

MAGIC OVER MATHEMATICS

During the time of Pythagoras, most people either believed that the world could only be explained by magic or that it could not be explained at all. Thus, many people did not attempt to understand mathematics.

Many Pythagorean beliefs (such as secrecy, vegetarianism, periods of food abstinence and silence, refusal to eat beans, refusal to wear animal skins, celibacy, self-examination, immortality, and reincarnation) were directed as "rules of life." The main focus of Pythagorean thought was ethics, developed primarily within philosophy, mathematics, music, and gymnastics. The beliefs of the society were that reality is mathematical; philosophy is used for spiritual purification; the soul is divine; and certain symbols possess mystical significance. Both men and women were permitted to become members. In fact, several female Pythagoreans became noted philosophers.✶

✶Aesara of Lucania was a Pythagorean philosopher known for her theory of the tripart soul, which she believed consisted of the mind, spiritedness, and desire.

How Pythagoreans Conceptualized Numbers

Pythagoreans believed that all relationships could be reduced to numbers in order to account for geometrical properties. This generalization originated from the observation that whenever the ratios of lengths of strings were whole numbers, harmonious tones were produced when these strings were vibrated.

The society studied properties of numbers that are familiar to modern mathematicians, such as even and odd numbers, prime and square numbers. From this viewpoint, the Pythagoreans developed the concept of number, which became their dominant principle of all proportion, order, and harmony in the universe.

The society also believed in such numerical properties as masculine or feminine, perfect or incomplete, and beautiful or ugly. These opposites, they believed, were found everywhere in nature, and the combination of them brought about the harmony of the world.

The primary belief of Pythagoreans in the sole existence of whole numbers was later challenged by their own findings, which proved the existence of "incommensurables," known today as **irrational numbers**. What is commonly called the "first crisis in mathematics" caused a scandal within the society, so serious that some members tried to suppress the knowledge of the existence of incommensurables.

irrational number a real number that cannot be written as fractions of the form a/b, where a and b are both integers and b is not 0

The Pythagorean philosophy was dominated by the ideal that numbers were not only symbols of reality, but also were the final substance of real things, known as "number mysticism." They held, for example, that one is the point, two the line, three the surface, and four the solid. Seven was considered the destiny that dominates human life because infancy ends there, and also because the number was associated with the seven wandering stars. Moreover, Pythagoreans believed that maturity began at age 14, marriage occurred in the twenty-first year, and 70 years was the normal life span. Ten was identified as the "perfect number" because it was the sum of one, two, three, and four.

Pythagorean Contributions to Mathematics

The formalization of mathematics with the use of **axiomatic systems** was the most profound contribution that the Pythagorean society made to mathematics. Pythagoreans developed this significant concept by showing that arbitrary laws of empirical geometry could be proved as logical conclusions from a small number of axioms, or postulates. Typical of the developed axioms was "A straight line is the shortest distance between two points."

axiomatic system a system of logic based on certain axioms and definitions that are accepted as true without proof

From these axioms, a number of **theorems** about the properties of points, lines, angles, curves, and planes could be logically deduced. These theorems include the famous Pythagorean theorem, which states that "the square of the **hypotenuse** of a right-angled triangle is equal to the sum of the squares of the other two sides." Another theorem states that the sum of the interior angles of any triangle is equal to the sum of two right angles.

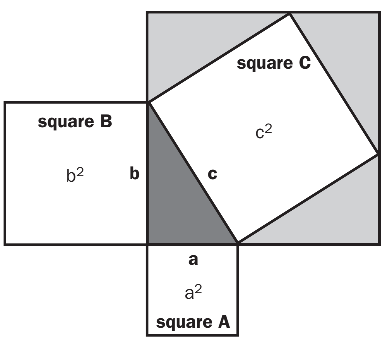
theorem a statement in mathematics that can be demonstrated to be true given that certain assumptions and definitions (called axioms) are accepted as true

hypotenuse the long side of a right triangle; the side opposite the right angle

The Pythagorean Theorem

The Pythagoreans knew that any triangle whose sides were in the ratio 3:4:5 was a right-angled triangle. Their desire to find the mathematical harmonies of all things led them to prove the geometric theorem, today named for Pythagoras. The earlier Egyptians stated this theorem as an empirical relationship and, as far as is known today, the Pythagoreans were the first to prove it.

The Pythagorean (hypotenuse) theorem states that the square of the hypotenuse of a right-angle triangle (c) is equal to the sum of the squares of the other two sides (a and b), shown as $c^2 = a^2 + b^2$. The numbers 3, 4, and 5 are called Pythagorean numbers ($5^2 = 3^2 + 4^2$, or $25 = 9 + 16$). However, the Pythagoreans did not consider the square on the hypotenuse to be that number (c) multiplied by itself (c^2). Instead, it was conceptualized as a geometrical square (C) constructed on the side of the hypotenuse, and that the sum of the areas of the two squares (A and B) is equal to the area of the third square (C), as shown below.



Astronomy and the Pythagoreans

In astronomy, the Pythagoreans produced important advances in ancient scientific thought. They were the first to consider the Earth as a sphere revolving with the other planets and the Sun around a universal "central fire." Ten planets were believed to exist in order to produce the "magical" number of 10. This arrangement was explained as the harmonious arrangement of bodies in a complete sphere of reality based on a numerical pattern, calling it a "harmony of sphere." The Pythagoreans also recognized that the orbit of the Moon was **inclined** to the equator of the Earth, and were one of the first to accept that Venus was both the evening star and the morning star.

inclined sloping, slanting, or leaning

Even though much of the Pythagorean doctrine consisted of numerology and number mysticism, their influence in developing the idea that nature could be understood through mathematics and science was extremely important for studying and understanding the world in which we live.

SEE ALSO Numbers: Abundant, Deficient, Perfect, and Amicable; Numbers, Forbidden and Superstitious; Numbers, Irrational; Numbers, Rational; Numbers, Whole; Triangle.

William Arthur Atkins (with Philip Edward Koth)

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Quadratic Formula and Equations

Q

A quadratic equation is an equation of the second degree, meaning that for an equation in x, the greatest exponent on x is 2. Quadratics most commonly refer to vertically oriented parabolas—that is, parabolas that open upward or downward. The graph of a vertically oriented parabola has the shape of a rounded "*v*," and the bottom-most (or top-most) point is called the vertex.

The equation for a parabola is usually written in either standard or vertex form; however, the standard form is more commonly used to solve for the x-intercepts, or roots. The standard form is $y = ax^2 + bx + c$ for any real numbers a, b, c where $a \neq 0$. The vertex form is $y - k = a(x - h)^2$ with vertex (h, k) and where $a \neq 0$.

Because x-intercepts are the points at which the graph crosses the x-axis, the solutions are always found by substituting 0 for y. The roots are often useful in solving real world problems, and there are three common ways to find the roots: factoring, using the quadratic formula, and completing the square.

Solving Quadratic Equations by Factoring

Not all quadratics can easily be factored, but if they can, the quickest way to solve them is to factor and use the zero product property. The zero product property basically states that if the product of two numbers is 0, then at least one of the numbers multiplied must be a 0. In other words, for any real numbers a and b, if $ab = 0$, then either $a = 0$ or $b = 0$.

Consider a swimmer who starts at one end of a pool, swims down to pick up a ring at the bottom of the middle of the pool, and then surfaces at the other end of the pool with the ring. The equation $y = x^2 - 6x + 9$ can be used to model the path of the swimmer, where y is the water level in the pool measured in feet, and x is the time in seconds since the swimmer started.

The equations below show how to solve for the roots of the equation to find the number of seconds it took the swimmer to reach the ring at the bottom of the pool: namely, by substituting 0 for y, factoring, and using the zero product property.

$$0 = x^2 - 6x + 9 \qquad \textit{Substitute 0 for } y.$$

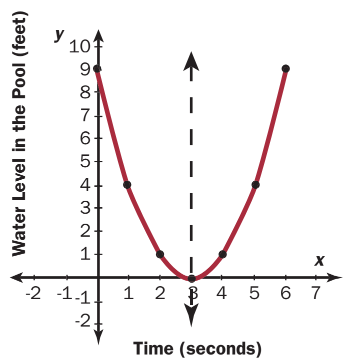
$$0 = [x - 3](x - 3) \qquad \textit{Factor } x^2 - 6x + 9.$$

$$\text{Either } [x - 3] = 0 \quad \text{or} \quad (x - 3) = 0 \qquad \textit{Use the zero product property.}$$

$$\text{So } x = 3 \quad \text{and} \quad x = 3 \qquad \textit{Solve each equation.}$$

In this example, the quadratic has only one repeated root, $x = 3$. This root is the time at which the swimmer reached the bottom of the pool. This quadratic can be graphed by substituting values into the equation to make a table of points, then graphing the realistic portion of the parabola, as shown below.

The graph below illustrates that the parabola has a vertical line of symmetry that passes through (3, 0). The equation for the line of symmetry of this parabola is $x = 3$. The graph of the parabola continues infinitely; however, to model the path of the swimmer, only the points from 0 to 6 seconds are graphed. This is because the swimmer starts at $x = 0$ seconds, swims down for 3 seconds to get the ring, and then swims up for 3 seconds.



Solving Quadratic Equations Using the Quadratic Formula

Because not all quadratic equations can be factored, other methods for finding roots are needed. One other method of finding roots for a quadratic is to use the quadratic formula. In the formula, the plus or minus sign means to solve the formula twice—once with a plus, and once with a minus. In other words, given a quadratic equation in standard form, $y = ax^2 + bx + c$, the solutions can be found by $x = \dfrac{[-b \pm \sqrt{(b^2 - 4ac)}]}{2a}$.

Consider that a delayed space shuttle leaves Earth about 20 minutes after the scheduled departure. At 6 miles out, the shuttle turns around and returns to Earth. The distance of the shuttle from Earth can be described by the equation $y - 6 = -0.1(x - 30)^2$, where x is the number of minutes the shuttle is in flight.

The equations below show how to find the total number of minutes the shuttle was off the ground. To find the roots of the equation, first solve for standard form and substitute 0 for y, as shown in Step One. The resulting trinomial cannot easily be factored into two binomials, so the quadratic equation must be used to solve for the roots, as shown in Step Two.

Step One

$$
\begin{aligned}
y - 6 &= -0.1(x - 30)^2 && \\
y - 6 &= -0.1(x^2 - 60x + 900) && \textit{Square the binomial.} \\
y - 6 &= -0.1x^2 + 6x - 90 && \textit{Multiply through by } -0.1. \\
y &= -0.1x^2 + 6x - 84 && \textit{Add six to both sides.} \\
0 &= -0.1x^2 + 6x - 84 && \textit{Substitute zero for } y.
\end{aligned}
$$

Step Two

$a = -0.1,\ b = 6,\ c = -84$

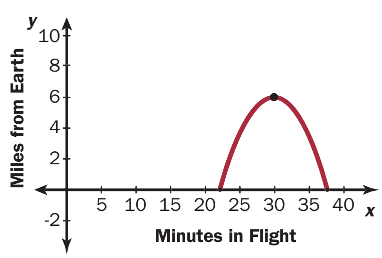
$$x = \frac{[-6 \pm \sqrt{(6^2 - 4(-0.1)(-84))}]}{(2)(-0.1)}$$

$$x = \frac{[-6 \pm \sqrt{(36 - 33.6)}]}{(-0.2)}$$

$$x \approx \frac{(-6 + 1.55)}{(-0.2)} \quad \text{and} \quad x \approx \frac{(-6 - 1.55)}{(-0.2)}$$

$$x \approx 22.25 \quad \text{and} \quad x \approx 37.75$$

To graph the parabola, plot and connect the two roots and the vertex. (The equation was originally given in vertex form.) If needed, more points can be found by substituting values for x into the equation. To graph the realistic portion of the parabola, graph only the portion in Quadrant I (see below).



The original problem said that the shuttle was delayed by about 20 minutes. This is the first intercept, $x \approx 22.25$ minutes. The vertex represents the point at which the shuttle was 6 miles from Earth. The second intercept, $x \approx 37.75$ represents the time at which the shuttle returned to Earth. To find the total number of minutes the shuttle was in flight, subtract its liftoff and landing times. The shuttle was in flight for about $37.75 - 22.25 = 15.5$ minutes.

Solving Quadratic Equations by Completing the Square

Some equations for parabolas may be solved more easily by completing the square. Completing the square forces the trinomial to be a perfect square by replacing the constant term with $(\frac{b}{2})^2$. In addition to finding roots, completing the square is also used for transforming an equation in standard form to vertex form. Furthermore, this method can be extended for use with the

conic sections the curves generated by an imaginary plane slicing through an imaginary cone

other **conic sections**. The quadratic formula can be derived by completing the square in $y = ax^2 + bx + c$.

The equations below show how to solve for vertex form of $y = x^2 - 6x + 7$ and find its roots by completing the square. The strategy is to move the constant term opposite the trinomial and replace it with $(\frac{b}{2})^2$. Then the new trinomial is written as the square of a binomial.

In Step Two, the vertex form is $y + 2 = (x - 3)^2$, and the vertex is $(3, -2)$. To find the roots, substitute 0 for y and solve for x. The two roots are $x = 3 + \sqrt{2}$ and $x = 3 - \sqrt{2}$. To graph this parabola, the vertex and approximations for the roots can be plotted and connected.

Step One

y	$= x^2 - 6x + 7$	
$y - 7$	$= x^2 - 6x$	*Subtract seven from both sides.*
$y - 7 + (3)^2$	$= x^2 - 6x + (3)^2$	*Add* $\left(\frac{6}{2}\right)^2$ *to both sides.*
$y + 2$	$= (x - 3)^2$	*Rewrite the trinomial as the square of a binomial.*

Step Two

$0 + 2$	$= (x - 3)^2$	*Substitute 0 for y.*
2	$= (x - 3)^2$	
$\pm\sqrt{2}$	$= \sqrt{(x - 3)^2}$	*Square root both sides.*
$\pm\sqrt{2}$	$= x - 3$	
$\pm\sqrt{2} + 3$	$= x$	*Add three to both sides.*

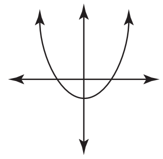
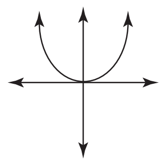
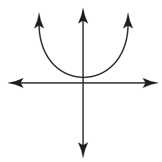
Features on the Graphs of Parabolas

The vertex can be found directly from vertex form, and it can also be found from standard form. From standard form, use $-b/2a$ to find the x-coordinate of the vertex and then substitute the result into the equation to find the y-coordinate of the vertex.

Equation	Vertex	Example
Vertex Form	(h, k)	Equation: $y - 1 = 7(x + 2)^2$
$y - k = a(x - h)^2$		$h = -2$, $k = 1$
		Vertex $(-2, 1)$
Standard Form	$(-b/2a, k)$	Equation: $y = x^2 - 8x + 9$
$y = ax^2 + bx + c$		$h = -(-8)/(2)(1) = 4$
		$k = 4^2 - 8(4) + 9 = -7$
		Vertex $(4, -7)$

The discriminant can be used to determine if the graph crosses the x-axis, and if so how many times. The discriminant is the expression under the radical in the quadratic formula, $b^2 - 4ac$. A square root usually yields two solutions, unless it is the square root of zero. The table summarizes the number and types of solutions that can occur and how they affect the appearance of the graph.

USING THE DISCRIMINANT TO DETERMINE THE NUMBER AND KINDS OF ROOTS

Possible Cases	Number and Kinds of Roots	Orientation of the Graph to the x-axis	Sample Graph
$b^2 - 4ac > 0$	2 real roots	The graph intersects the x-axis twice	
$b^2 - 4ac = 0$	1 real root	The vertex is the only point that intersects the x-axis	
$b^2 - 4ac < 0$	2 complex roots	The graph never intersects the x-axis	

The value of a in a quadratic equation also affects the placement of the graph on the plane. If a is positive, the graph opens upwards; if it is negative the graph opens downward. If a is greater than one, the graph will be narrow, and if a is a fraction between 0 and 1, the graph will be wide. This bit of information is especially useful because the value of a affects other types of graphs in the same ways as it does parabolas.

Quadratics That Are Not Functions

All conic sections are quadratics because they have equations of the second degree. However, only the vertically oriented parabolas that have been summarized in this article are functions.

Graphing calculators and computers perform functions by taking an input and giving an output. Hence, most graphing tools are only equipped to graph equations of functions. To graph a horizontally oriented parabola on a calculator, the graph must be broken into pieces that are functions. Then the equations for each piece are graphed on the same plane to create the appearance of one graph. SEE ALSO Conic Sections; Functions and Equations; Graphs and Effects of Parameter Changes.

Michelle R. Michael

Quilting

A quilt consists of two layers of fabric—the quilt top and the backing—with a third layer of soft insulating material between them, called the batting. Hand or machine stitching holds the three layers together. A quilt can be "whole cloth" (which means the top is one piece of material) but the more colorful and popular forms are patchwork and appliqué.

In patchwork, small patches are sewn together to form larger patterns, while in appliqué the motifs are stitched onto a background fabric. Patch-

work quilts are more likely to be arrangements of geometric figures, such as squares and triangles. Appliqué is more frequently used for freeform or representational designs. In either style, the quiltmaker first completes the top by piecing, then quilts together the sandwich of two layers of fabric with batting in between by stitching through all three layers.

Quilts were made even in the earliest civilizations—for example, in ancient Egypt and Central America. In Europe, the Crusaders brought the idea back from the Holy Land around the twelfth century. However, it was in America that the patchwork quilt blossomed into a distinctive and ubiquitous folk art form.

On the frontier, fabric was a valuable commodity, and quilts were a practical way to use leftover scraps and worn-out clothing. Pioneer women soon discovered the artistic possibilities of quiltmaking, and it became one of the few outlets for creativity and beauty in a difficult life. Quilts also became a valuable cultural record. To those who often did not have access to written media, quilts were a way of chronicling important life events: births, marriages, or even something so humble as a treasured friendship.

Over the generations, the practical importance of quilts diminished. Today, anyone can go to the store and buy a factory-made comforter. But over the last 30 years quilting has enjoyed a tremendous renaissance as a hobby and as a form of artistic and social expression. Fabrics and tools are now available that would have astounded previous generations of stitchers.

Patchwork Mathematics

Many traditional quilt designs are mathematically based, even if their inventors had no formal training in mathematics. A common geometric motif is the square. In the nineteenth century, many girls' first sewing project was a "nine-patch," a simple arrangement of nine squares in a grid. Another common figure is the isosceles right triangle, obtained by cutting a square along the diagonal. Hundreds of traditional blocks consist of these figures, arranged in different sizes and orientations to form stars, animals, buildings, or whatever the quilter's imagination suggested.

A traditional full-sized quilt is often composed of many copies of one basic block. Sometimes two blocks are alternated in checkerboard fashion, as shown in Figure 1. In both quilts, the individual blocks are difficult to discern once they have been incorporated into the completed quilt.

Quilters have learned that the skillful use of contrasting fabrics can lead the eye to see larger "secondary patterns," camouflaging the underlying grid. The traditional pattern called Jacob's Ladder illustrates this technique. As Figure 1 shows, the quilter can use the same geometric pattern yet can make the quilt look completely different by varying the colors. In the quilt on the left, the dominant color creates an illusion of two rectangles with broad borders, positioned diagonally across the quilt and intersecting one another at right angles. In the quilt on the right, the rectangles are barely visible because the color is muted. Instead, the dominant feature on the right is the series of long gray ladders ascending diagonally across the quilt.

Circular symmetry is another common geometric theme. Many quilts, instead of using a repeated pattern of square blocks, feature an elaborate

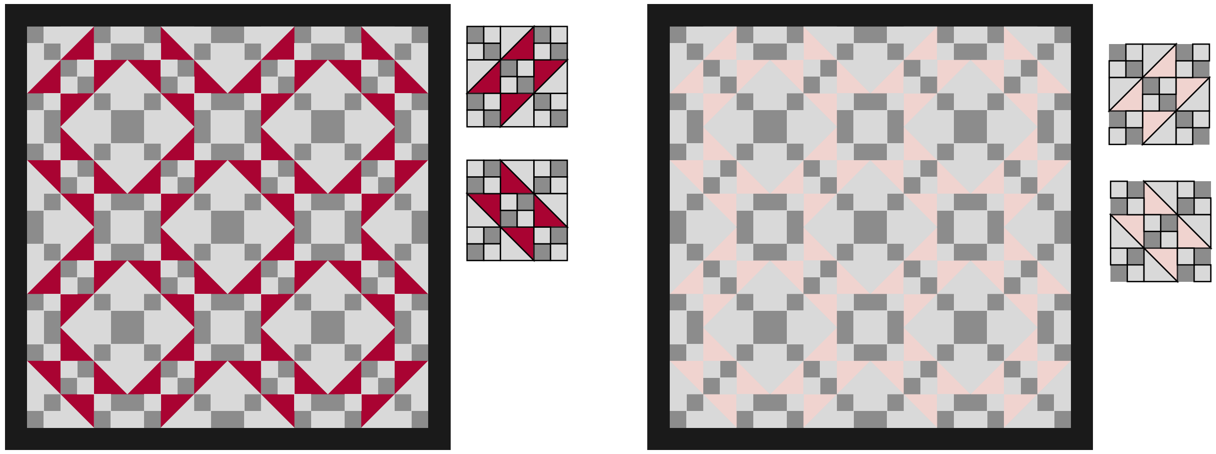
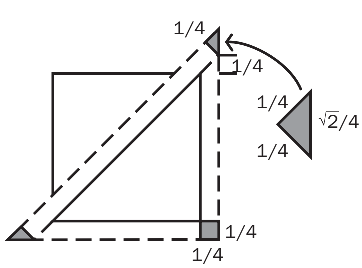


Figure 1. Although the two sets of small blocks have identical geometric grids, their different orientations and the different combinations of colors highlight different patterns. From this identical underlying grid, the quilter can produce two distinctive quilts: in this case, one in which "Jacob's Ladder" is evident (right) and one in which the long ladders are not apparent (left).

symmetric "medallion" in the center, such as an 8-pointed Lone Star or a 16-pointed Mariner's Compass. The quilt shown on the cover of this encyclopedia shows an 8-pointed star.

Measurements and Tools. In order to make the pieces of a quilt fit together correctly, quilters depend on precise measurements. If the diagonal lines in Jacob's Ladder were off by even a quarter inch, the sense of continuity from one block to the next would be ruined. The finished quilt can also develop unsightly ripples or bulges if the patches are not measured and cut with accuracy.

A little mathematics comes in handy, too. Because it is impossible to sew two patches together edge-to-edge, quilters cut out each patch a little larger than it will appear in the final quilt. To make a 1-inch square, quilters know they need to cut out a $1\frac{1}{2}$ inch square, adding in a $\frac{1}{4}$-inch seam allowance (a standard amount) on each side. But it is trickier to figure out the correct cutting size for an isosceles right triangle. Adding $\frac{1}{2}$ inch to the length of the short side is not enough, because this leaves no seam allowance on the long side, as shown in the figure below. The extra length needed, $\frac{\sqrt{2}}{4}$ inches, can be computed from the Pythagorean Theorem, making a total of $\frac{1}{4} + \frac{1}{4} + \frac{\sqrt{2}}{4} \approx \frac{7}{8}$ inches to add to the short sides of the triangle. Most quilters know the $\frac{7}{8}$-inch "rule of thumb," even if they do not know that they owe it to Pythagoras.



Modern quilters have access to an array of specialized tools for design and execution. The rotary cutter, a device somewhat like a pizza slicer, can cut more swiftly and accurately than scissors. When used with special see-through rulers that have grids and oblique angles marked on them, the

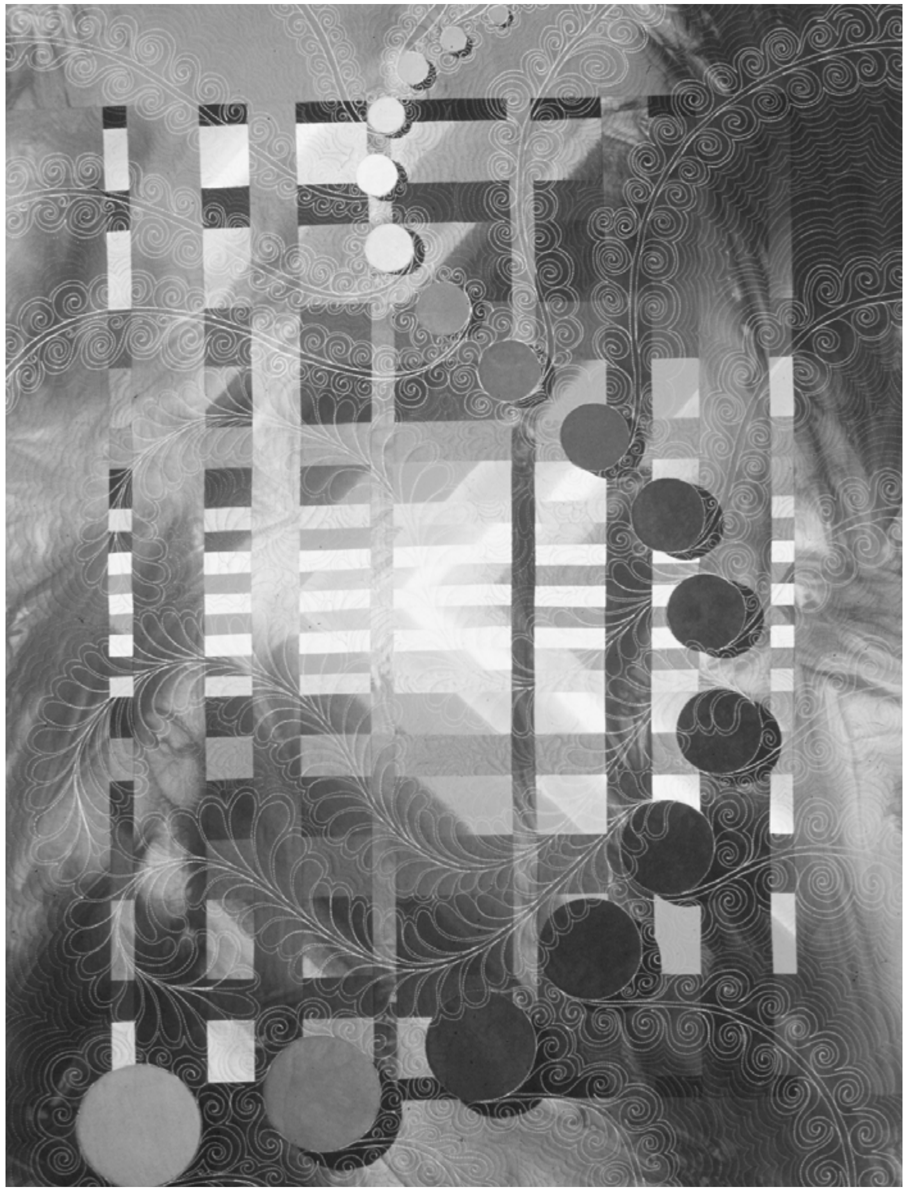


Figure 2. The underlying design for this modern quilt is based on orderly progressions of scale found in nature and observed by thirteenth-century Italian mathematician Leonardo Pisano Fibonacci. The circles arc across the surface of the lattice in a diagonal compound curve.

rotary cutter also allows quilters to skip the tedious step of marking fabric with a pencil. Other measuring devices associated with mathematics include protractors, compasses, and both rectilinear and isometric (triangular) graph paper. In recent years, some quilt designers have begun to use computer programs to design, preview, and calculate yardage.

Mathematics in Design. In addition to using mathematics as a tool, a few quilters have begun to use mathematics as their inspiration. In Caryl Bryer Fallert's *Fibonacci Series #3*, the long rectangles have side lengths that form a Fibonacci progression: 1, 2, 3, 5, 8, and so on. (See Figure 2.) Although not shown here, Jane LeValley Kerns's prizewinning *Fractal* ingeniously ties together the mathematical concept of a **fractal** (in this case, a cube made of smaller cubes made of smaller cubes) with a well-known optical illusion called the Necker cube. Other quilts with mathematical themes have been based on space-filling curves, logarithmic spirals, and nonperiodic **tessellations**. The mathematical motifs provide an inherent sense of rhythm and

fractal a type of geometric figure possessing the properties of self-similarity (any part resembles a larger or smaller part at any scale) and a measure that increases without bound as the unit of measure approaches zero

tessellation a mosaic of tiles or other objects composed of identical repeated elements with no gaps

balance that partners well with the color and tactile satisfaction of the quilting medium. SEE ALSO Fractals; Fibonacci, Leonardo Pisano; Geometry, Tools of; Patterns; Pythagoras; Tessellations; Transformations.

Dana Mackenzie

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Radian *See Circles, Measurement of.*

R

Radical Sign

The radical sign $\sqrt{}$ may be one of the stranger-looking mathematical symbols, but its strange looks should not fool anyone. The radical sign has a straightforward meaning.

The sign that is used when finding the square root or cube root, $\sqrt{}$, is called the radical sign. A radical sign with no index number written in the notch indicates a square root. A square root can also be written with an index of 2 ($\sqrt[2]{}$), but usually the 2 is understood rather than expressly written. A radical sign with an index of 3 is written as $\sqrt[3]{}$, which indicates a cube root.

For example, to calculate the area of a square (in which all sides are equal), multiply the length of one side s by itself (squaring), so the area is s^2. If the area of a square is known but not the length of the side, the **inverse** of squaring will find the length of the side. The inverse operation of squaring is finding the square root. For example, if the area of a square is 16 ft^2 (square feet), finding the square root is in effect asking "what number multiplied by itself, or squared, gives a result of 16?" The square root of 16 ft^2 is $\sqrt{16ft^2}$ or 4 ft, since $4^2 = 16$.

Similarly, suppose a cube-shaped box has a volume of 125 cm^3 (cubic centimeters). How can one find the length of a side, s, without measuring the box? To find the volume of a cube, s is multiplied by itself three times or raised to the third power, also called cubing the side. So the volume of a cube with side length s is s^3. The inverse operation of cubing is finding the cube root. Because $5^3 = 125$, the cube root of 125 cm^3 is $\sqrt[3]{125cm^3}$ or 5 cm.

One can also find the fourth root, fifth root, and so forth, by indicating which root by changing the index. So $\sqrt[4]{}$ means to find the fourth root, and $\sqrt[5]{}$ means to find the fifth root. Just as taking the square root is the inverse operation of squaring, and taking the cube root is the inverse operation of cubing, finding $\sqrt[4]{x}$ is the inverse of raising x to the fourth power, and so on.

Look at the expression $\sqrt[7]{x}$ (the seventh root of x) as an example. In this expression, the **integer** in front of the radical sign, 7, is known as the in-

inverse opposite; the mathematical function that expresses the independent variable of another function in terms of the dependent variable

integer a positive whole number, its negative counterpart, or zero

radicand the quantity under the radical sign; the argument of the square root function

dex. The number under the radical sign, in this case x, is called the **radicand**.

It is helpful to remember that $\sqrt{\ }$ is usually called "square" root instead of "second" root, and $\sqrt[3]{\ }$ is called the "cube" root instead of "third" root. If the index of a radical is greater than 3, one simply says fourth root, fifth root, and so on.

When solving the equation $x^2 = 4$, there are two solutions: 2 and -2. Does the expression $\sqrt{4}$ also have two solutions: 2 and -2? No. Although $(2)^2 = 4$ and $(-2)^2 = 4$, the mathematical expressions $\sqrt{4} = x$ and $x^2 = 4$ are not equivalent; that is, these two equations do not have the same solution set. The solution set for $\sqrt{4} = x$ is 2 and the solution set for $x^2 = 4$ is 2 and -2.

To verify this, use a calculator to find the square root of 4. On most calculators, the answer you get will be simply 2. Is the calculator wrong? The short answer to this question is no. When using the radical sign, the expression $\sqrt{4}$ is understood to mean *positive* square root of 4. When both solutions to a square root are wanted, the radical must have the symbol $\pm$ in front of it. So, $\sqrt{4} = 2$, and $\pm\sqrt{4} = 2$ and -2.

The answer to a problem when finding a square root also depends on the context of the problem. When using a square root to find the side length of a square table for which only the area is known, a negative value does not make sense. A table with an area of 16 ft^2 would not have a side length of -4 feet! When solving problems using the radical sign, write the radical sign alone if only the positive root is desired, and write $\pm$ in front of a radical sign when both roots are desired.

real number a number that has no imaginary part; a set composed of all the rational and irrational numbers

When working within the **real number** system (or the numbers that can be found on a real number line), finding roots that have an even number for the index—such as square roots, fourth roots, sixth roots, and so forth—the radicand must be greater than or equal to 0 in order to get an answer that is part of the real number system.

For example, consider the expression $\sqrt{-4}$. To solve this, one needs to find out which number, when multiplied by itself, equals -4. Consider this: $2 \times 2 = 4$, and $-2 \times (-2) = 4$. Clearly there is no number within the real number system that when multiplied by itself equals -4. Therefore, when simplifying a radical that has an even index and a radicand that is less than 0, the answer is undefined. However, this does not mean that it is impossible to find the square root of -4. This root does exist, but it is not defined within the real number system.

What about $\sqrt[3]{-27}$? Is this solution defined in the real number system? To simplify this radical, a number which when multiplied by itself three times equals 27 must be found. It is known that $3^3 = 27$ because $3 \times 3 \times 3 = 27$. What about $(-3)^3$? Because $(-3) \times (-3) \times (-3) = -27$, the cube root of -27 ($\sqrt[3]{-27}$) is -3. Therefore, it is possible to find the root of a radicand that is less than 0 when the index is an odd number (three, five, seven, and so on).

irrational number a real number that cannot be written as a fraction of the form a/b, where a and b are both integers and b is not 0

Radicals can also be used to express **irrational numbers**. An irrational number is one that cannot be expressed as a ratio of two integers, such as 1/2. An example of an irrational number is $\sqrt{2}$. In decimal form, $\sqrt{2}$ ex-

pands into 1.414213562. . .. Because this number cannot be written as a fraction, and because the decimal continues without repeating, it is often better to use the radical to express the number exactly as $\sqrt{2}$. SEE ALSO Numbers, Complex; Numbers, Irrational; Powers and Exponents.

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Radio Disc Jockey

Radio announcers who play music, or discs, are often called disc jockeys. They introduce records, commercials, news, and public announcements that are aired at a radio station.

While on the air, disc jockeys comment on the music, weather, news, and traffic. They may interview guests, take requests from listeners, and manage listener contests. They usually make up comments as they go, working without a formal written script.

Some disc jockeys get to choose the music that will be played during their shows. However, they usually must choose music from an approved play list put together by the station's program or music director. At smaller stations, disc jockeys may have off-air duties such as operating the control board, which is used to broadcast the programming, commercials, and public service announcements according to schedule.

A small number of radio disc jockeys reach celebrity status, such as Wolfman Jack, whose radio career spanned from the 1960s to his death in 1995.

Mathematics and Radio Work

Mathematics comes into play when preparing the station's schedule. A certain percentage of air time may be set aside for music, another percentage for commercials, and so on. Timing is critical at a radio station. The play times of music, commercials, and other programs must add up precisely so that there is no "dead air."

demographics statistical data about people—including age, income, and gender—that are often used in marketing

Mathematics is also used in the radio industry when conducting listener **demographic** studies. Statistics are used by the radio stations to determine their target audiences and to periodically assess the demographic makeup of their listeners.

Educational requirements for a radio disc jockey job vary. However, many stations prefer a college degree or vocational and technical training. In college, courses in communications and broadcasting are useful. Working at the college radio station is also valuable experience. SEE ALSO Music Recording Technician.

Denise Prendergast

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Randomness

When most people think of randomness, they generally think of a condition with an apparent absence of a regular plan, pattern, or purpose. The word random is derived from the Old French word *randon*, meaning haphazard. The mathematical meaning is not significantly different from the common usage. Mathematical randomness is exhibited when the next state of a process cannot be exactly determined from the previous state. Randomness involves uncertainty. The most common example of randomness is the tossing of a coin. From the result of a previous toss, one cannot predict with certainty that the result of the next coin toss will be either heads or tails.

Computers and Randomness

People performing statistical studies or requiring random numbers for other applications obtain them from a table, calculator, or computer. Random digits can be generated by repeatedly selecting from the set of numbers {0, 1, 2,. . ., 9}. One way of making the selection would be to number ten balls with these digits and then draw one ball at a time without looking, recording the number on the drawn ball and replacing the ball after each successive drawing. The recorded string of digits would be a set of random numbers. Extensive tables of random numbers have been generated in the past. For example, the RAND Corporation used an electronic roulette wheel to compile a book with a million random digits.

Today, rather than using tables, people requiring random numbers more frequently use a calculator or a computer. Computers and calculators have

programs that generate random numbers, but the numbers are really not random because they are based on complicated, but nonetheless deterministic, computational **algorithms**. These algorithms generate a sequence of what are called **pseudo-random numbers**.

Using computers to generate random numbers has altered the definition of randomness to involve the complexity of the algorithm used in the computations. It is not possible to achieve true randomness with a computer because there is always some underlying process that, with tremendous computational difficulty, could be duplicated to replicate the pseudo-random number. Physicists consider the emissions from atoms to be a truly random process, and therefore a source of generating random numbers. But the instruments used to detect the emissions introduce limitations on the actual randomness of numbers produced in that process. So at best, the many ways of generating a random number only approximate true randomness. With the advent of computers, mathematicians can define and develop methods to measure the randomness of a given number, but have yet to prove that a number sequence is truly random.

Randomness in Mathematics

Randomness has very important applications in many areas of mathematics. In statistics, the selection of a random sample is important to ensure that a study is conducted without bias. A simple random sample is obtained by numbering every member of the population of interest, and assigning each member a numerical label. The appropriate sample size is determined. The researcher then obtains the same quantity of random numbers as the sample size from a table of random numbers, a calculator, or a computer. The members of the population labeled with the corresponding random numbers are selected for study. In this way, every member of the population has an equal likelihood of being selected, eliminating any bias that may be introduced by other selection methods. Ensuring the randomness of the selection makes the results of the study more scientifically valid and more likely to be replicated.

There are many other applications of randomness in mathematics. Using solution methods involving **random walks**, applied mathematicians can obtain solutions for complex mathematical models that are the basis of modern physics. Albert Einstein, and later Norbert Weiner, used the method in the early twentieth century to describe the motion of microscopic particles suspended in a fluid. In the late 1940s, mathematicians Stanislaw Ulam and John von Neumann developed Monte Carlo methods, which apply random numbers to solve deterministic models arising in nuclear physics and engineering. Randomness is also important in the mathematics of **cryptography**, which is particularly important today and will continue to be in the future as sensitive information is transmitted across the Internet. Seemingly random numbers are used as the keys to encryption systems in use in digital communications.

In more complicated examples, randomness is closely tied to probability. Even seemingly irregular random phenomena exhibit some long-term regularity. **Probability theory** mathematically explains randomness. Mathematicians sometimes divide processes they study into **deterministic** or probabilistic (or **stochastic**) models. If a phenomenon can be modeled de-

algorithms rules or procedures used to solve mathematical problems

pseudo-random numbers numbers generated by a process that does not guarantee randomness; numbers produced by a computer using some highly complex function that simulates true randomness

random walk a mathematical process in a plane of moving a random distance in a random direction then turning through a random angle and repeating the process indefinitely

cryptography the science of encrypting information for secure transmission

probability theory the branch of mathematics that deals with quantities having random distributions

The seemingly random behavior of the Big Six Wheel actually follows the laws of probability.

deterministic having a single, well-defined solution

stochastic random, or relating to a variable at each moment

chaos theory the qualitative study of a periodic behavior in deterministic nonlinear dynamical systems

terministically, the process can be predicted with certainty using mathematical formulas and relationships. Stochastic models involve uncertainty, but with probability theory, the uncertain behavior of the phenomenon is better understood despite the haphazardness. One cannot predict the specific outcome of the coin-tossing experiment, but you can achieve an expectation and understanding of the process using probability theory. Through the use of probability theory, one understands much about topics such as nuclear physics.

Not all processes can be classified as deterministic or stochastic in an obvious manner. **Chaos theory** is a relatively recent area of mathematical study that helps explain the randomness that appears in some processes that are otherwise considered to be deterministic. The behavior of chaotic systems is dramatically influenced by their sensitivity to small changes in initial conditions. Mathematicians are currently developing methods to understand the underlying order of chaotic systems. Mathematicians apply chaos theory to clarify the apparent randomness of some processes. SEE ALSO Chaos; Cryptology.

Dick Jardine

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Rate of Change, Instantaneous

limit a mathematical concept in which numerical values get closer and closer to a given value or approach that value

velocity distance traveled per unit of time in a specific direction

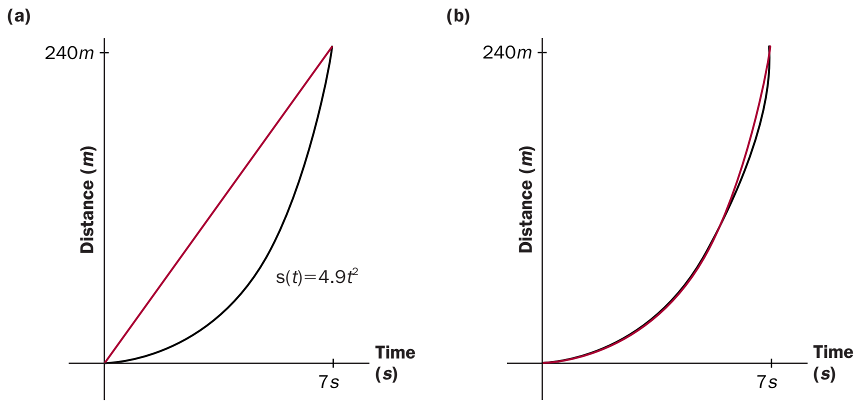
The instantaneous rate of change is the **limit** of the function that describes the average rate of change. It has many practical applications, and can be used to describe how an object travels through the air, in space, or across the ground. The changes in the speed of an airplane, a space shuttle, and a car all may be described using the instantaneous rate of change concept. When describing motion, this concept is also referred to as **velocity**.

There are many other uses of the instantaneous rate of change concept. Chemists use it to examine chemical reactions. Electrical engineers use this concept to describe the changes that occur in the current of an electric circuit. Economists use this concept to describe the profits and losses of a given business.

In order to understand the instantaneous rate of change concept, it is first necessary to understand the meaning of the terms *average rate of change* and *limit*.

Average Rate of Change

The average rate of change describes how one variable changes as the other variable increases by a single unit. For example, Galileo discovered that if the impact of air resistance is ignored, the distance that an object travels



with respect to time when dropped from a given height is described by the following function:

$$s(t) = 4.9t^2 \text{ where } t \geq 0.$$

In this function, t indicates the amount of time that has passed since the object was dropped. The evaluation of this function, $s(t)$, describes the total distance that the object has traveled by a given point in time. For example, after 2 seconds, the object will have traveled a total distance of $4.9(2^2)$ or approximately 19.6 meters. This is, of course, assuming that the object has not yet hit the ground.

In order to calculate the average rate of change described by the function $s(t)$, it is useful to examine a graph of this function. For this discussion, assume that the object is dropped from a height of 240 meters. The graph of the function, $s(t)$, is shown in (a) of the figure. Notice that the graph begins at 0 seconds and ends at 7 seconds. The curve in this graph suggests that the speed of the object is changing over time. Since this object is dropped from a height of 240 meters (m), after approximately 7 seconds the object will hit the ground ($4.9\,(7^2) = 240.1$ m).

slope the angle of a line relative to the x-axis

The average rate of change is the average distance that is traveled in a single second. To determine the average rate of change, a ratio is formed between the difference between the distance traveled and the amount of time that has passed. This can be expressed as $\dfrac{240\text{m} - 0\text{m}}{7\text{sec} - 0\text{sec}} = 34\,\dfrac{2}{7}$ m/sec. This same algebraic expression may also used to determine the **slope** of the line segment between the points (0, 0) and (7, 240). The colored line segment in (a) of the figure has a slope of $34\,\frac{2}{7}$. The slope of this line represents that average rate of change of the function $s(t)$. An interpretation of this slope is that on average the object covers a distance of $34\,\frac{2}{7}$ meters every second. If the original function had been a straight line, then the slope of that line would be equal to the average rate of change of that function.

It is important to recognize in the current example that $34\,\frac{2}{7}$ is the average rate of change for a 7-second time interval. The average rate of change can also be calculated for smaller time intervals. For example, the average rate of change may be calculated for each 1-second time interval. These calculations are shown in the first table. A graphical depiction of the average

AVERAGE RATE OF CHANGE FOR 1-SECOND TIME INTERVALS

Time Interval	Calculation	Average Rate of Change
0 to 1	$\frac{s(1) - s(0)}{1 - 0}$	4.9 m/sec
1 to 2	$\frac{s(2) - s(1)}{2 - 1}$	14.7 m/sec
2 to 3	$\frac{s(3) - s(2)}{3 - 2}$	24.5 m/sec
3 to 4	$\frac{s(4) - s(3)}{4 - 3}$	34.3 m/sec
4 to 5	$\frac{s(5) - s(4)}{5 - 4}$	44.1 m/sec
5 to 6	$\frac{s(6) - s(5)}{6 - 5}$	53.9 m/sec
6 to 7	$\frac{s(7) - s(6)}{7 - 6}$	63.7 m/sec

AVERAGE RATE OF CHANGE FOR SMALL TIME INTERVALS

For Intervals Ending at 1 Second		For Intervals Beginning at 1 Second	
Time Interval	Average Rate of Change	Time Interval	Average Rate of Change
0.9 to 1	9.31 (m/sec)	1 to 1.1	10.29 (m/sec)
0.99 to 1	9.751 (m/sec)	1 to 1.01	9.849 (m/sec)
0.999 to 1	9.7951 (m/sec)	1 to 1.001	9.8049 (m/sec)
0.9999 to 1	9.79951 (m/sec)	1 to 1.0001	9.80049 (m/sec)
0.99999 to 1	9.799951 (m/sec)	1 to 1.00001	9.800049 (m/sec)

rate of change for each of these 1-second time intervals is shown in (b) of the figure above. The slope of the colored line segments is the average rate of change for each time interval.

Notice that the average of the third column shown in the table is about $34\frac{1}{3}$ m/sec or approximately the average rate of change for the 7-second time interval. In other words, the average rate of change across these seven time intervals is equal to the average rate of change across the entire time interval. This will always be true. The average rate of change may be calculated for $\frac{1}{2}$-second time intervals, $\frac{1}{4}$-second time intervals, or even smaller time intervals.

The Limit Concept. Limit refers to a mathematical concept in which numerical values get closer and closer to a given value or approach that value. The limit of the function that describes the average rate of change is referred to as the instantaneous rate of change.

Returning to the previous example, if smaller and smaller time intervals are considered around the same value, the slope of the function $s(t)$ will approach or get closer and closer to a specific value. For example, smaller and smaller time intervals that all end at 1 second can be identified. As the second table suggests, the average rate of change is getting closer and closer to 9.8 m/sec. Smaller and smaller times intervals that all begin at 1 second can also be identified. As the table suggests, the average rate of change is again getting closer and closer to 9.8 m/sec. The value 9.8 m/sec is therefore the limit of the function that describes the average rate of change.

As the above example illustrates, determining the instantaneous rate of change is cumbersome using the methods of **algebra** and **geometry**. In **calculus**, the **derivative** concept is examined. Evaluating the derivative of a function at a given point is another way to determine the instantaneous rate of change. SEE ALSO Calculus; Limit.

Barbara M. Moskal

algebra the branch of mathematics that deals with variables or unknowns representing the arithmetic numbers

geometry the branch of mathematics that deals with the properties and relationships of points, lines, angles, surfaces, planes, and solids

calculus a method of dealing mathematically with variables that may be changing continuously with respect to each other

derivative the derivative of a function is the limit of the ratio of the change in the function; the change is produced by a small variation in the variable as the change in the variable is allowed to approach zero; an inverse operation to calculating an integral

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Ratio, Rate, and Proportion

If there are seven boys and twelve girls in a class, then the ratio of boys to girls can be expressed as 7 to 12, $\frac{7}{12}$, or 7:12. A ratio compares the size, or magnitude, of two numbers. Two other related concepts, rate and proportion, together with ratio, are used for solving many real-world problems that involve comparing different quantities.

SUMMARIZING THE CONCEPTS

A ratio compares the magnitude of two quantities. When the quantities have different units, then a ratio is called a rate. A proportion is a statement of equality between two ratios.

Calculating Ratios

Suppose a parking garage contains six blue cars and two green cars. The ratio of blue cars to green cars can be expressed as a fraction $\frac{6}{2}$. If the two green cars leave the garage, then there are zero green cars and the ratio becomes $\frac{6}{0}$. Division by zero, however, is not defined, so this form of the ratio is meaningless. Expressing a ratio as a fraction, $\frac{a}{b}$, is valid as long as b is not equal to zero. However, the ratio of blue to green cars can still be written as 6 to 0 or 6:0.

Ratios can be used to compare quantities of the same type of objects and of different types. There are two types of ratios that compare quantities of the same type. When the comparison is to part of the whole to the whole, then the ratio is a part-whole ratio. When the comparison is to part of the whole to another part of the whole, then the ratio is a part-part ratio.

For example, suppose there is a wall made up of twelve blocks, five white blocks and seven red blocks. The ratio of white blocks to the total number of blocks is $\frac{5}{12}$, which is a part-whole ratio. The ratio of white blocks to red blocks is $\frac{5}{7}$, which is a part-part ratio.

Figuring Rates

A ratio that compares quantities of different types is called a rate. A phone company charges \$0.84 for 7 minutes of long distance, and a student reads 10 pages in 8 minutes. The first rate is $\frac{\$0.84}{7}$ minutes, which is equal to $\frac{\$0.12}{1}$ minute (obtained by dividing both terms by 7). The second rate is $\frac{10\text{ pages}}{8}$ minutes, which is equal to $\frac{5\text{ pages}}{4}$ minutes.

The rate in the first example is called a unit rate. In a unit rate, the denominator quantity is 1. A unit rate is often used for comparing the cost of two similar items. If a 12-ounce box of cereal sells for \$2.40, and a 16-ounce box sells for \$2.88, which is the better buy? The unit rate of the first box is \$0.20/ounce ($\frac{\$2.40}{12}$ ounces), and the unit rate of the second box is \$0.18/ounce ($\frac{\$2.88}{16}$ ounces). Therefore, the second box is a better buy.

Understanding Proportions

When two ratios are equal, the mathematical statement of that equality is called a proportion. The statement that $\frac{2}{3} = \frac{4}{6}$ is a proportion. If $\frac{a}{b}$ is equal to $\frac{c}{d}$, then $\frac{a}{b} = \frac{c}{d}$ is called a proportion. To find out if two ratios form a proportion, one can evaluate the cross product. If $\frac{a}{b}$ and $\frac{c}{d}$ are ratios, then the two ratios form a proportion if $ad = bc$.

Proportions are used when three quantities are given, and the fourth quantity is an unknown. Suppose a person drives 126 miles in 3 hours. At

the same speed, how many miles would the driver travel in 4 hours? Because the rate of travel remains the same, a proportion can be written.

The unknown quantity, the distance traveled by the car in 4 hours, can be indicated by x. Therefore, the two ratios $\frac{126}{3}$ and $\frac{x}{4}$ form a proportion.

$$\frac{126 \text{ miles}}{3 \text{ hours}} = \frac{x}{4} \text{ hours}$$

Multiplying both sides by 4, or using cross multiplication, yields $x = 168$ miles. SEE ALSO Numbers, Rational.

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Reciprocal *See Inverses.*

Reflection *See Transformations.*

Restaurant Manager

Restaurant managers use mathematics to manage reservation schedules and to ensure that enough food is stocked to meet peak demand.

Night after night people are able to go to restaurants and, no matter what the time, order various meals. There is actually a science (and it should also be considered an art) to keeping an appropriate amount of food so that the restaurant does not run out and the food does not spoil. Restaurant managers are responsible for this task, and although it is not complicated to explain, it definitely takes skill to master.

Most restaurants have a par sheet to keep track of the amount of food in storage. Par—in food as in golf—means average or suggested amount. Every restaurant has stores of food, and when the quantity for a particular item falls below par, it needs to be reordered.

Knowing yield on products and how long food can be kept allows managers to order food in the correct quantities. This minimizes spoilage, as does a common method known as FIFO (first in, first out). FIFO is a basic rule in which storage shelves are stocked with the newest food being placed in the back. In this way, food is used in the order in which it arrives at the restaurant.

There is really no specified education necessary to become a restaurant manager; however, knowledge of the food business is certainly beneficial. Culinary education, job experience, and business management classes are some of the best ways to prepare for success in the field of managing restaurants. SEE ALSO Cooking, Measurements of.

Elizabeth Sweeney

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Robinson, Julia Bowman

American Logician and Number Theorist
1919–1985

Julia Bowman Robinson, the second daughter of Helen Hall and Ralph Bowers Bowman, was born in St. Louis, Missouri, on December 8, 1919. When Robinson was 9 years old, she contracted scarlet fever and a year later developed rheumatic fever. Julia's father hired a private teacher to work with her three mornings a week and, in one year, she had completed the state-required materials for the fifth through eighth grades. It was during this time that she first became fascinated with mathematics.

Early Mathematics Honors

By the time Robinson entered her junior year of high school, she was the only girl taking mathematics courses and advanced science courses such as physics. In 1936, when she graduated* from high school, she was awarded honors in mathematics and science, including the Bausch-Lomb medal for all-around excellence in science.

***When Julia Robinson graduated from high school, her parents gave her a slide rule that she treasured and named "Slippy."**

At the age of 16, Robinson enrolled at San Diego State College, now known as San Diego State University, where her older sister, Constance Bowman, attended classes. Julia majored in mathematics with the intention of earning teaching credentials.

Prior to her senior year, Julia transferred to the University of California at Berkeley to study to become a mathematician. It was there that she met Raphael M. Robinson, an assistant professor of mathematics at Berkeley. After receiving a Bachelor of Arts (B.A.) in 1940, she immediately began graduate studies and earned a graduate teaching fellowship in the mathematics department.

In 1941, during her second year of graduate school at Berkeley, Julia married Robinson. Because of a university rule forbidding spouses to teach concurrently in the same department, Julia gave up her teaching fellowship and attempted to start a family.

But Robinson later discovered the risk of bearing a child was too dangerous for her health due to the rheumatic fever she suffered as a child. In 1946, at the encouragement of her husband, Robinson turned her attention back to mathematics and in 1947 began to work at Berkeley toward a Doctor of Philosophy (Ph.D.) under the advisement of Alfred Tarski, a Polish mathematician.

Work on the Tenth Problem

In 1948 Robinson received her Ph.D. and began work on trying to solve the tenth on a list of twenty-three major problems posed by David Hilbert, a prominent mathematician at the beginning of the century. The Tenth Problem was:

> Given a **Diophantine equation**, with only **integer** solutions, with any number of unknown quantities and with rational integral numerical coefficients (and whole-number exponents) devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers. For

Diophantine equation a polynomial equation of several variables, with integer coefficients, whose solution is to be integers

integer a positive whole number, its negative counterpart, or zero

example, $x^2 + y^2 - 2 = 0$ has four Diophantine solutions—(1, 1), (−1, 1), (−1, −1), (1, −1)—but $x^2 + y^2 - 3 = 0$ has no solutions.

Striving to solve this problem occupied the largest portion of Robinson's professional career. The solution, which would be her major contribution to mathematics, was not completed until 1970. Many honors were bestowed upon Robinson as a result of her foundation for the solution to the Tenth Problem. In 1975 she became the first woman mathematician to be elected to the National Academy of Science. She later became the first woman officer of the American Mathematical Society (1978) and the first woman to serve as its president (1982). In 1983 she was awarded a MacArthur Fellowship, consisting of $60,000 a year for 5 years, in recognition of her contributions to mathematics.

In August 1984, Robinson was diagnosed with leukemia, and she died on July 30, 1985, at the age of 65.

Gay A. Ragan

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Roebling, Emily Warren

American Field Engineer and Lawyer
1843–1903

In the last quarter of the nineteenth century, the Equal Rights Party was organized; Susan B. Anthony and sixteen other women tried to vote but were arrested; and Congress passed a law that gave female federal employees equal pay for equal work. During this period in American history, Emily Warren Roebling oversaw the design and construction of the Brooklyn Bridge, which was heralded in those days as the eighth wonder of the world.

suspension bridge a bridge held up by a system of cables or cables and rods in tension; usually having two or more tall towers with heavy cables anchored at the ends and strung between the towers and lighter vertical cables extending downward to support the roadway

Emily Roebling's father-in-law, John August Roebling, was a pioneer in the construction of steel **suspension bridges**. He had designed the Brooklyn Bridge in 1865 to cross the East River in one uninterrupted span, thereby connecting the New York City boroughs of Brooklyn and Manhattan. The Brooklyn Bridge would be the first bridge to use steel for its 16-inch diameter cable wires. Its main span would be the longest in the world at that time.

From Assistant to Leader

Before he could complete the Brooklyn Bridge project, John Roebling died of tetanus in 1869 after a riverboat nudged a Brooklyn dock slip, crushing one of his feet between the pilings. His eldest son, who was also Emily's husband, Washington Roebling, was then named chief engineer. In order to help her husband, Emily Roebling began to study many topics related to civil engineering, including higher mathematics, bridge specifications, material strength, **catenary curves**, and cable construction.

But in 1872, Washington Roebling suffered a severe attack of what was called **caisson** disease, or "the bends," after working twelve hours in a watertight caisson deep beneath a pier. Unable to work at the construction site because of his injuries, Washington guided bridge operations from home, which overlooked the work site.

Emily Roebling played a critical role in the construction of the Brooklyn Bridge. Although some of her biographers appear to restrict her role to shuttling orders between her husband and the construction site, Roebling was very active in dealing with engineering issues that arose during the bridge's construction.

Respected "Chief" Engineer

At first, Emily Roebling carried messages, gave orders, and made inspections at the construction site. As time progressed, the number and difficulty of jobs that she handled increased. She soon began answering the questions of bridge officials and contractors. She answered their questions so well that many businessmen believed that she was the chief engineer.

Before the Brooklyn Bridge could be finished, many design adjustments had to be made by Washington Roebling. Because of unexpected load increases, expenses overran the original cost estimates. Many people began to doubt Washington's ability to continue as the chief engineer, and the Brooklyn mayor proposed his removal in 1882. At a meeting of the American Society of Civil Engineers, Emily Roebling represented her husband and defended him after questions arose regarding his ability to continue directing the bridge project. She successfully defended him, and public confidence was restored. She continued her project leadership role until the bridge was finished in 1883.

catenary curve the curve approximated by a free-hanging chain supported at each end; the curve generated by a point on a parabola rolling along a line

caisson a large cylinder or box that allows workers to perform construction tasks below the water surface; may be open at the top or sealed and pressurized

Emily Roebling's Legacy

Although Emily Roebling never planned on being an engineer, she contributed significantly to what was a huge engineering feat. She acquired a strong knowledge of bridge engineering, had a sharp mind and a natural mathematical ability, and was a good student of her husband. She was considered a peacemaker among the sometimes quarrelsome engineers, manufacturers, contractors, workmen, and board of trustees.

Throughout her work on the Brooklyn Bridge, Emily Roebling made many advances for women engineers, being perhaps the first woman field engineer. While her accomplishments went unrecognized by professional organizations, she is listed as one of the builders of the bridge on the dedication plaque.

In the years following the Brooklyn Bridge project, Roebling earned a law degree and became one of the first female lawyers in New York State. She published "The Journal of the Reverend Silas Constant" (1903) and was active in the women's group Daughters of the American Revolution.

William Arthur Atkins (with Philip Edward Koth)

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Roller Coaster Designer

The mathematics and physics of roller coaster design help ensure a safe ride, despite the human errors and mechanical malfunctions that sometimes cause problems.

A roller coaster designer is part mechanical engineer and part magician. His or her job is to keep the riders of the coaster perfectly safe while at the same time scaring the daylights out of them. It is really not magic, of course. The designer's secret tools are physics and mathematics.

Staying on Top

It may seem like magic when a roller coaster hangs upside down at the top of a loop, with nothing between it and the ground, but it is really nothing more than the principle of **inertia**. The physicist Galileo Galilei developed the concept of inertia in the sixteenth century. According to Galileo, objects that are in motion continue to move in the same direction and at the same speed unless an external force acts on them. If the forces acting on a roller coaster car could somehow switch off, at exactly the same moment when the car reached the top of a loop, the car would fly away, upside down, moving along a horizontal line. Now that would be magic!

In reality, both gravity and the pressure of the rails on the wheels push the car downward when it reaches the top of the loop. These forces, however, do not make the car crash straight down into the ground because of all the inertia they have to overcome. Instead, they push down just hard enough to deflect the car from its horizontal line and keep it on the tracks.

inertia tendency of a body that is at rest to remain at rest, or the tendency of a body that is in motion to remain in motion

acceleration the rate of change of an object's velocity

Testing the Limits

Any deviation of a roller-coaster car from a straight-line, constant-speed path is called **acceleration**. When a car plunges downhill, riders experience acceleration downward and forward. When it screeches around a curve, the riders experience acceleration to the left or right. And when it finally (or too soon, depending on one's point of view) starts to brake, the riders experience acceleration in reverse.

Ever since Galileo's experiment of dropping two balls off the Leaning Tower of Pisa, physicists have realized that all falling bodies near Earth's surface accelerate at the same rate, regardless of their mass. This rate is about 32 feet per second—a quantity that often is denoted by the letter g. A force three times stronger than Earth's gravity will produce an acceleration of 3 g.

Because the human body has adapted to Earth's gravity, it cannot tolerate dramatically greater accelerations in any direction. For example, a normal person can tolerate 6 g of positive acceleration (the kind that mashes the rider down into the floor of the coaster) for only a few seconds before

blacking out. A smart designer will not exceed those limits, and most roller coasters do not even come close.

The law of conservation of energy—taught in any high-school physics course—enables the designer to work out how a change in height translates into a change in velocity. If friction and air resistance were ignored, the car's speed would be exactly proportional to the square root of the vertical drop. (Thus, a coaster needs to be built four times taller to double the speed.)

In reality, the designer cannot ignore friction and air resistance. A full car will be slowed down more by friction than a half-empty car; a car on a hot day will go faster because its wheels are better lubricated. For this reason, designers do not rely on calculations alone. They use an **accelerometer** to test out scale models, and even the full-scale ride, under every conceivable condition.

accelerometer a device that measures acceleration

In a few high-tech rides, sensors monitor the speed of the car at every point. If the car is going too fast or too slow, the sensors can adjust the electric current running through the linear magnetic motor that drives it.

Part of the thrill of a roller coaster is the feeling of being out of control. There is no steering wheel to turn and no brake to push. But roller coaster riders hanging upside down in a car in the middle of a loop should relax: The engineers who designed the ride have used science and math to make sure that nothing bad will happen to the thrill seekers. SEE ALSO Rate of Change, Instantaneous.

Dana Mackenzie

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Rotation *See Transformations.*

Rounding

When an approximate value for a measurement, or other number, is needed, rounding can be used. For example, a person might say that he or she owns 100 books when the actual number of books that person owns might be 98 or 104. Sometimes the process of giving an approximate number is also called rounding off.

Suppose a measurement of 15.42 inches needs to be rounded to the nearest inch. The first step is to ask this question: "Is 15.42 inches closer to 15 inches or to 16 inches?" Because 15.42 is closer to 15 inches, 15.42 rounds to 15. Similarly, a measurement of 15.62 inches would be rounded to 16.

But what about a measurement of 15.5 inches? This measurement is exactly halfway between 15 and 16 inches. There are two ways to solve this dilemma. The first way, which is the most often used, is to round up when the digit is 5. Thus 15.5 rounds to 16. But what about 15.55? This is not midway between 15 and 16 because 15.55 is closer to 16 than to 15. So, when rounding, one should consider all the digits to the right of the place to which the number is being rounded (in this case, the units place).

The second way to round when the number is midway is often used in scientific calculation. To see how this works, consider these 4 measurements: 7.5, 4.5, 6.5, and 8.5. If one rounds all four measurements to the nearest inch, the result is 8, 5, 7, and 9, but this way of rounding may build in error. If the original measurements are added, the total is 27 inches. Compare this to the total for the rounded numbers, 29. Rounding created an error equal to 2 inches.

If, however, one rounds by the second, or scientific, method, the following rule is applied: Round when the number is midway so that the result is always an even number. What values will this give for the 4 numbers in the previous example? The result is 8, 4, 6, and 8. The total of these is 26, which is only 1 inch from the total of the original measurement. Rounding in this way tends, in the long run, to balance out the rounding errors because in a long list of numbers to be rounded, of those that are midway, about half will be even and half will be odd.

Sometimes the context of a problem will indicate which way to round a number properly. For example, in a survey taken to find how many pets live in a neighborhood, it is found that 7 neighbors have a total of 24 pets. Does this mean that the neighbors have $\frac{24}{7}$, or $3\frac{3}{7}$, pets each? It may be better to report that each neighbor has an average of about 3 pets each, although the actual calculation results in a fraction of a pet. SEE ALSO Estimation; Significant Figures or Digits.

Lucia McKay

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Glossary

abscissa: the x-coordinate of a point in a Cartesian coordinate plane

absolute: standing alone, without reference to arbitrary standards of measurement

absolute dating: determining the date of an artifact by measuring some physical parameter independent of context

absolute value: the non-negative value of a number regardless of sign

absolute zero: the coldest possible temperature on any temperature scale; $-273°$ Celsius

abstract: having only intrinsic form

abstract algebra: the branch of algebra dealing with groups, rings, fields, Galois sets, and number theory

acceleration: the rate of change of an object's velocity

accelerometer: a device that measures acceleration

acute: sharp, pointed; in geometry, an angle whose measure is less than 90 degrees

additive inverse: any two numbers that add to equal 1

advection: a local change in a property of a system

aerial photography: photographs of the ground taken from an airplane or balloon; used in mapping and surveying

aerodynamics: the study of what makes things fly; the engineering discipline specializing in aircraft design

aesthetic: having to do with beauty or artistry

aesthetic value: the value associated with beauty or attractiveness; distinct from monetary value

algebra: the branch of mathematics that deals with variables or unknowns representing the arithmetic numbers

algorithm: a rule or procedure used to solve a mathematical problem

algorithmic: pertaining to an algorithm

ambiguity: the quality of doubtfulness or uncertainty

analog encoding: encoding information using continuous values of some physical quantity

analogy: comparing two things similar in some respects and inferring they are also similar in other respects

analytical geometry: describes the study of geometric properties by using algebraic operations

anergy: spent energy transferred to the environment

angle of elevation: the angle formed by a line of sight above the horizontal

angle of rotation: the angle measured from an initial position a rotating object has moved through

anti-aliasing: introducing shades of gray or other intermediate shades around an image to make the edge appear to be smoother

applications: collections of general-purpose software such as word processors and database programs used on modern personal computers

arc: a continuous portion of a circle; the portion of a circle between two line segments originating at the center of the circle

areagraph: a fine-scale rectangular grid used for determining the area of irregular plots

artifact: something made by a human and left in an archaeological context

artificial intelligence: the field of research attempting the duplication of the human thought process with digital computers or similar devices; also includes expert systems research

ASCII: an acronym that stands for American Standard Code for Information Interchange; assigns a unique 8-bit binary number to every letter of the alphabet, the digits, and most keyboard symbols

assets: real, tangible property held by a business corporation including collectible debts to the corporation

asteroid: a small object or "minor planet" orbiting the Sun, usually in the space between Mars and Jupiter

astigmatism: a defect of a lens, such as within an eye, that prevents focusing on sharply defined objects

astrolabe: a device used to measure the angle between an astronomical object and the horizon

astronomical unit (AU): the average distance of Earth from the Sun; the semi-major axis of Earth's orbit

asymptote: the line that a curve approaches but never reaches

asymptotic: pertaining to an asymptote

atmosphere (unit): a unit of pressure equal to 14.7 lbs/in^2, which is the air pressure at mean sea level

atomic weight: the relative mass of an atom based on a scale in which a specific carbon atom (carbon-12) is assigned a mass value of 12

autogiro: a rotating wing aircraft with a powered propellor to provide thrust and an unpowered rotor for lift; also spelled "autogyro"

avatar: representation of user in virtual space (after the Hindu idea of an incarnation of a deity in human form)

average rate of change: how one variable changes as the other variable increases by a single unit

axiom: a statement regarded as self-evident; accepted without proof

axiomatic system: a system of logic based on certain axioms and definitions that are accepted as true without proof

axis: an imaginary line about which an object rotates

axon: fiber of a nerve cell that carries action potentials (electrochemical impulses)

azimuth: the angle, measured along the horizon, between north and the position of an object or direction of movement

azimuthal projections: a projection of a curved surface onto a flat plane

bandwidth: a range within a band of wavelengths or frequencies

base-10: a number system in which each place represents a power of 10 larger than the place to its right

base-2: a binary number system in which each place represents a power of 2 larger than the place to its right

base-20: a number system in which each place represents a power of 20 larger than the place to the right

base-60: a number system used by ancient Mesopotamian cultures for some calculations in which each place represents a power of 60 larger than the place to its right

baseline: the distance between two points used in parallax measurements or other triangulation techniques

Bernoulli's Equation: a first order, nonlinear differential equation with many applications in fluid dynamics

biased sampling: obtaining a nonrandom sample; choosing a sample to represent a particular viewpoint instead of the whole population

bidirectional frame: in compressed video, a frame between two other frames; the information is based on what changed from the previous frame as well as what will change in the next frame

bifurcation value: the numerical value near which small changes in the initial value of a variable can cause a function to take on widely different values or even completely different behaviors after several iterations

Big Bang: the singular event thought by most cosmologists to represent the beginning of our universe; at the moment of the big bang, all matter, energy, space, and time were concentrated into a single point

binary: existing in only two states, such as "off" or "on," "one" or "zero"

binary arithmetic: the arithmetic of binary numbers; base two arithmetic; internal arithmetic of electronic digital logic

binary number: a base-2 number; a number that uses only the binary digits 1 and 0

binary signal: a form of signal with only two states, such as two different values of voltage, or "on" and "off" states

binary system: a system of two stars that orbit their common center of mass; any system of two things

binomial: an expression with two terms

binomial coefficients: coefficients in the expansion of (x + y^n, where n is a positive integer

binomial distribution: the distribution of a binomial random variable

binomial theorem: a theorem giving the procedure by which a binomial expression may be raised to any power without using successive multiplications

bioengineering: the study of biological systems such as the human body using principles of engineering

biomechanics: the study of biological systems using engineering principles

bioturbation: disturbance of the strata in an archaeological site by biological factors such as rodent burrows, root action, or human activity

bit: a single binary digit, 1 or 0

bitmap: representing a graphic image in the memory of a computer by storing information about the color and shade of each individual picture element (or pixel)

Boolean algebra: a logic system developed by George Boole that deals with the theorems of undefined symbols and axioms concerning those symbols

Boolean operators: the set of operators used to perform operations on sets; includes the logical operators AND, OR, NOT

byte: a group of eight binary digits; represents a single character of text

cadaver: a corpse intended for medical research or training

caisson: a large cylinder or box that allows workers to perform construction tasks below the water surface, may be open at the top or sealed and pressurized

calculus: a method of dealing mathematically with variables that may be changing continuously with respect to each other

calibrate: act of systematically adjusting, checking, or standardizing the graduation of a measuring instrument

carrying capacity: in an ecosystem, the number of individuals of a species that can remain in a stable, sustainable relationship with the available resources

Cartesian coordinate system: a way of measuring the positions of points in a plane using two perpendicular lines as axes

Cartesian plane: a mathematical plane defined by the x and y axes or the ordinate and abscissa in a Cartesian coordinate system

cartographers: persons who make maps

catenary curve: the curve approximated by a free-hanging chain supported at each end; the curve generated by a point on a parabola rolling along a line

causal relations: responses to input that do not depend on values of the input at later times

celestial: relating to the stars, planets, and other heavenly bodies

celestial body: any natural object in space, defined as above Earth's atmosphere; the Moon, the Sun, the planets, asteroids, stars, galaxies, nebulae

central processor: the part of a computer that performs computations and controls and coordinates other parts of the computer

centrifugal: the outwardly directed force a spinning object exerts on its restraint; also the perceived force felt by persons in a rotating frame of reference

cesium: a chemical element, symbol Cs, atomic number 55

Chandrasekhar limit: the 1.4 solar mass limit imposed on a white dwarf by quantum mechanics; a white dwarf with greater than 1.4 solar masses will collapse to a neutron star

chaos theory: the qualitative study of unstable aperiodic behavior in deterministic nonlinear dynamical systems

chaotic attractor: a set of points such that all nearby trajectories converge to it

chert: material consisting of amorphous or cryptocrystalline silicon dioxide; fine-grained chert is indistinguishable from flint

chi-square test: a generalization of a test for significant differences between a binomial population and a multinomial population

chlorofluorocarbons: compounds similar to hydrocarbons in which one or more of the hydrogen atoms has been replaced by a chlorine or fluorine atom

chord: a straight line connecting the end points of an arc of a circle

chromakey: photographing an object shot against a known color, which can be replaced with an arbitrary background (like the weather maps on television newscasts)

chromosphere: the transparent layer of gas that resides above the photosphere in the atmosphere of the Sun

chronometer: an extremely precise timepiece

ciphered: coded; encrypyted

circumference: the distance around a circle

circumnavigation: the act of sailing completely around the globe

circumscribed: bounded, as by a circle

circumspheres: spheres that touch all the "outside" faces of a regular polyhedron

client: an individual, business, or agency for whom services are provided by another individual, business, or industry; a patron or customer

clones: computers assembled of generic components designed to use a standard operation system

codomain: for a given function *f*, the set of all possible values of the function; the range is a subset of the codomain

cold dark matter: hypothetical form of matter proposed to explain the 90 percent of mass in most galaxies that cannot be detected because it does not emit or reflect radiation

coma: the cloud of gas that first surrounds the nucleus of a comet as it begins to warm up

combinations: a group of elements from a set in which order is not important

combustion: chemical reaction combining fuel with oxygen accompanied by the release of light and heat

comet: a lump of frozen gas and dust that approaches the Sun in a highly elliptical orbit forming a coma and one or two tails

command: a particular instruction given to a computer, usually as part of a list of instructions comprising a program

commodities: anything having economic value, such as agricultural products or valuable metals

compendium: a summary of a larger work or collection of works

compiler: a computer program that translates symbolic instructions into machine code

complex plane: the mathematical abstraction on which complex numbers can be graphed; the *x*-axis is the real component and the *y*-axis is the imaginary component

composite number: an integer that is not prime

compression: reducing the size of a computer file by replacing long strings of identical bits with short instructions about the number of bits; the information is restored before the file is used

compression algorithm: the procedure used, such as comparing one frame in a movie to the next, to compress and reduce the size of electronic files

concave: hollowed out or curved inward

concentric: sets of circles or other geometric objects sharing the same center

conductive: having the ability to conduct or transmit

confidence interval: a range of values having a predetermined probability that the value of some measurement of a population lies within it

congruent: exactly the same everywhere; having exactly the same size and shape

conic: of or relating to a cone, that surface generated by a straight line, passing through a fixed point, and moving along the intersection with a fixed curve

conic sections: the curves generated by an imaginary plane slicing through an imaginary cone

continuous quantities: amounts composed of continuous and undistinguishable parts

converge: come together; to approach the same numerical value

convex: curved outward, bulging

coordinate geometry: the concept and use of a coordinate system with respect to the study of geometry

coordinate plane: an imaginary two-dimensional plane defined as the plane containing the x- and y-axes; all points on the plane have coordinates that can be expressed as x, y

coordinates: the set of n numbers that uniquely identifies the location of a point in n-dimensional space

corona: the upper, very rarefied atmosphere of the Sun that becomes visible around the darkened Sun during a total solar eclipse

corpus: Latin for "body"; used to describe a collection of artifacts

correlate: to establish a mutual or reciprocal relation between two things or sets of things

correlation: the process of establishing a mutual or reciprocal relation between two things or sets of things

cosine: if a unit circle is drawn with its center at the origin and a line segment is drawn from the origin at angle theta so that the line segment intersects the circle at (x, y), then x is the cosine of theta

cosmological distance: the distance a galaxy would have to have in order for its red shift to be due to Hubble expansion of the universe

cosmology: the study of the origin and evolution of the universe

cosmonaut: the term used by the Soviet Union and now used by the Russian Federation to refer to persons trained to go into space; synonomous with astronaut

cotton gin: a machine that separates the seeds, hulls, and other undesired material from cotton

cowcatcher: a plow-shaped device attached to the front of a train to quickly remove obstacles on railroad tracks

cryptography: the science of encrypting information for secure transmission

cubit: an ancient unit of length equal to the distance from the elbow to the tip of the middle finger; usually about 18 inches

culling: removing inferior plants or animals while keeping the best; also known as "thinning"

curved space: the notion suggested by Albert Einstein to explain the properties of space near a massive object, space acts as if it were curved in four dimensions

deduction: a conclusion arrived at through reasoning, especially a conclusion about some particular instance derived from general principles

deductive reasoning: a type of reasoning in which a conclusion necessarily follows from a set of axioms; reasoning from the general to the particular

degree: 1/360 of a circle or complete rotation

degree of significance: a determination, usually in advance, of the importance of measured differences in statistical variables

demographics: statistical data about people—including age, income, and gender—that are often used in marketing

dendrite: branched and short fiber of a neuron that carries information to the neuron

dependent variable: in the equation $y = f(x)$, if the function f assigns a single value of y to each value of x, then y is the output variable (or the dependent variable)

depreciate: to lessen in value

deregulation: the process of removing legal restrictions on the behavior of individuals or corporations

derivative: the derivative of a function is the limit of the ratio of the change in the function; the change is produced by a small variation in the variable as the change in the variable is allowed to approach zero; an inverse operation to calculating an integral

determinant: a square matrix with a single numerical value determined by a unique set of mathematical operations performed on the entries

determinate algebra: the study and analysis of equations that have one or a few well-defined solutions

deterministic: mathematical or other problems that have a single, well-defined solution

diameter: the chord formed by an arc of one-half of a circle

differential: a mathematical quantity representing a small change in one variable as used in a differential equation

differential calculus: the branch of mathematics primarily dealing with the solution of differential equations to find lengths, areas, and volumes of functions

differential equation: an equation that expresses the relationship between two variables that change in respect to each other, expressed in terms of the rate of change

digit: one of the symbols used in a number system to represent the multiplier of each place

digital: describes information technology that uses discrete values of a physical quantity to transmit information

digital encoding: encoding information by using discrete values of some physical quantity

digital logic: rules of logic as applied to systems that can exist in only discrete states (usually two)

dihedral: a geometric figure formed by two half-planes that are bounded by the same straight line

Diophantine equation: polynomial equations of several variables, with integer coefficients, whose solutions are to be integers

diopter: a measure of the power of a lens or a prism, equal to the reciprocal of its focal length in meters

directed distance: the distance from the pole to a point in the polar coordinate plane

discrete: composed of distinct elements

discrete quantities: amounts composed of separate and distinct parts

distributive property: property such that the result of an operation on the various parts collected into a whole is the same as the operation performed separately on the parts before collection into the whole

diverge: to go in different directions from the same starting point

dividend: the number to be divided; the numerator in a fraction

divisor: the number by which a dividend is divided; the denominator of a fraction

DNA fingerprinting: the process of isolating and amplifying segments of DNA in order to uniquely identify the source of the DNA

domain: the set of all values of a variable used in a function

double star: a binary star; two stars orbiting a common center of gravity

duodecimal: a numbering system based on 12

dynamometer: a device that measures mechanical or electrical power

eccentric: having a center of motion different from the geometric center of a circle

eclipse: occurrence when an object passes in front of another and blocks the view of the second object; most often used to refer to the phenomenon

that occurs when the Moon passes in front of the Sun or when the Moon passes through Earth's shadow

ecliptic: the plane of the Earth's orbit around the Sun

eigenvalue: if there exists a vector space such that a linear transformation onto itself produces a new vector equal to a scalar times the original vector, then that scalar is called an eigenfunction

eigenvector: if there exists a vector space such that a linear transformation onto itself produces a new vector equal to a scalar times the original vector, then that vector is called an eigenvector

Einstein's General Theory of Relativity: Albert Einstein's generalization of relativity to include systems accelerated with respect to one another; a theory of gravity

electromagnetic radiation: the form of energy, including light, that transfers information through space

elements: the members of a set

ellipse: one of the conic sections, it is defined as the locus of all points such that the sum of the distances from two points called the foci is constant

elliptical: a closed geometric curve where the sum of the distances of a point on the curve to two fixed points (foci) is constant

elliptical orbit: a planet, comet, or satellite follows a curved path known as an ellipse when it is in the gravitational field of the Sun or another object; the Sun or other object is at one focus of the ellipse

empirical law: a mathematical summary of experimental results

empiricism: the view that the experience of the senses is the single source of knowledge

encoding tree: a collection of dots with edges connecting them that have no looping paths

endangered species: a species with a population too small to be viable

epicenter: the point on Earth's surface directly above the site of an earthquake

epicycle: the curved path followed by planets in Ptolemey's model of the solar system; planets moved along a circle called the epicycle, whose center moved along a circular orbit around the sun

epicylic: having the property of moving along an epicycle

equatorial bulge: the increase in diameter or circumference of an object when measured around its equator usually due to rotation, all planets and the sun have equatorial bulges

equidistant: at the same distance

equilateral: having the property that all sides are equal; a square is an equilateral rectangle

equilateral triangle: a triangle whose sides and angles are equal

equilibrium: a state of balance between opposing forces

equinox points: two points on the celestial sphere at which the ecliptic intersects the celestial equator

escape speed: the minimum speed an object must attain so that it will not fall back to the surface of a planet

Euclidean geometry: the geometry of points, lines, angles, polygons, and curves confined to a plane

exergy: the measure of the ability of a system to produce work; maximum potential work output of a system

exosphere: the outermost layer of the atmosphere extending from the ionosphere upward

exponent: the symbol written above and to the right of an expression indicating the power to which the expression is to be raised

exponential: an expression in which the variable appears as an exponent

exponential power series: the series by which e to the x power may be approximated; $e^x = 1 + x + x^{2/2!} + x^{3/3!} + \ldots$

exponents: symbols written above and to the right of expressions indicating the power to which an expression is to be raised or the number of times the expression is to be multiplied by itself

externality: a factor that is not part of a system but still affects it

extrapolate: to extend beyond the observations; to infer values of a variable outside the range of the observations

farsightedness: describes the inability to see close objects clearly

fiber-optic: a long, thin strand of glass fiber; internal reflections in the fiber assure that light entering one end is transmitted to the other end with only small losses in intensity; used widely in transmitting digital information

fibrillation: a potentially fatal malfunction of heart muscle where the muscle rapidly and ineffectually twitches instead of pulsing regularly

fidelity: in information theory a measure of how close the information received is to the information sent

finite: having definite and definable limits; countable

fire: the reaction of a neuron when excited by the reception of a neurotransmitter

fission: the splitting of the nucleus of a heavy atom, which releases kinetic energy that is carried away by the fission fragments and two or three neutrons

fixed term: for a definite length of time determined in advance

fixed-wing aircraft: an aircraft that obtains lift from the flow of air over a nonmovable wing

floating-point operations: arithmetic operations on a number with a decimal point

fluctuate: to vary irregularly

flue: a pipe designed to remove exhaust gases from a fireplace, stove, or burner

fluid dynamics: the science of fluids in motion

focal length: the distance from the focal point (the principle point of focus) to the surface of a lens or concave mirror

focus: one of the two points that define an ellipse; in a planetary orbit, the Sun is at one focus and nothing is at the other focus

formula analysis: a method of analysis of the Boolean formulas used in computer programming

Fourier series: an infinite series consisting of cosine and sine functions of integral multiples of the variable each multiplied by a constant; if the series is finite, the expression is known as a Fourier polynomial

fractal: a type of geometric figure possessing the properties of self-similarity (any part resembles a larger or smaller part at any scale) and a measure that increases without bound as the unit of measure approaches zero

fractal forgery: creating a natural landscape by using fractals to simulate trees, mountains, clouds, or other features

fractal geometry: the study of the geometric figures produced by infinite iterations

futures exchange: a type of exchange where contracts are negotiated to deliver commodites at some fixed price at some time in the future

g: a common measure of acceleration; for example 1 *g* is the acceleration due to gravity at the Earth's surface, roughly 32 feet per second per second

game theory: a discipline that combines elements of mathematics, logic, social and behavioral sciences, and philosophy

gametes: mature male or female sexual reproductive cells

gaming: playing games or relating to the theory of game playing

gamma ray: a high-energy photon

general relativity: generalization of Albert Einstein's theory of relativity to include accelerated frames of reference; presents gravity as a curvature of four-dimensional space-time

generalized inverse: an extension of the concept of the inverse of a matrix to include matrices that are not square

generalizing: making a broad statement that includes many different special cases

genus: the taxonomic classification one step more general than species; the first name in the binomial nomenclature of all species

geoboard: a square board with pegs and holes for pegs used to create geometric figures

geocentric: Earth-centered

geodetic: of or relating to geodesy, which is the branch of applied mathematics dealing with the size and shape of the earth, including the precise location of points on its surface

geometer: a person who uses the principles of geometry to aid in making measurements

geometric: relating to the principles of geometry, a branch of mathematics related to the properties and relationships of points, lines, angles, surfaces, planes, and solids

geometric sequence: a sequence of numbers in which each number in the sequence is larger than the previous by some constant ratio

geometric series: a series in which each number is larger than the previous by some constant ratio; the sum of a geometric sequence

geometric solid: one of the solids whose faces are regular polygons

geometry: the branch of mathematics that deals with the properties and relationships of points, lines, angles, surfaces, planes, and solids

geostationary orbit: an Earth orbit made by an artificial satellite that has a period equal to the Earth's period of rotation on its axis (about 24 hours)

geysers: springs that occasionally spew streams of steam and hot water

glide reflection: a rigid motion of the plane that consists of a reflection followed by a translation parallel to the mirror axis

grade: the amount of increase in elevation per horizontal distance, usually expressed as a percent; the slope of a road

gradient: a unit used for measuring angles, in which the circle is divided into 400 equal units, called gradients

graphical user interface: a device designed to display information graphically on a screen; a modern computer interface system

Greenwich Mean Time: the time at Greenwich, England; used as the basis for universal time throughout the world

Gross Domestic Product: a measure in the change in the market value of goods, services, and structures produced in the economy

group theory: study of the properties of groups, the mathematical systems consisting of elements of a set and operations that can be performed on that set such that the results of the operations are always members of the same set

gyroscope: a device typically consisting of a spinning wheel or disk, whose spin-axis turns between two low-friction supports; it maintains its angular orientation with respect to inertial conditions when not subjected to external forces

Hagia Sophia: Instanbul's most famous landmark, built by the emperor Justinian I in 537 C.E. and converted to a mosque in 1453 C.E.

Hamming codes: a method of error correction in digital information

headwind: a wind blowing in the opposite direction as that of the course of a vehicle

Heisenberg Uncertainty Principle: the principle in physics that asserts it is impossible to know simultaneously and with complete accuracy the values of certain pairs of physical quantities such as position and momentum

heliocentric: Sun-centered

hemoglobin: the oxygen-bearing, iron-containing conjugated protein in vertebrate red blood cells

heuristics: a procedure that serves to guide investigation but that has not been proven

hominid: a member of family Hominidae; *Homo sapiens* are the only surviving species

Huffman encoding: a method of efficiently encoding digital information

hydrocarbon: a compound of carbon and hydrogen

hydrodynamics: the study of the behavior of moving fluids

hydrograph: a tabular or graphical display of stream flow or water runoff

hydroscope: a device designed to allow a person to see below the surface of water

hydrostatics: the study of the properties of fluids not in motion

hyperbola: a conic section; the locus of all points such that the absolute value of the difference in distance from two points called foci is a constant

hyperbolic: an open geometric curve where the difference of the distances of a point on the curve to two fixed points (foci) is constant

Hypertext Markup Language: the computer markup language used to create documents on the World Wide Web

hypertext: the text that contains hyperlinks, that is, links to other places in the same document or other documents or multimedia files

hypotenuse: the long side of a right triangle; the side opposite the right angle

hypothesis: a proposition that is assumed to be true for the purpose of proving other propositions

ice age: one of the broad spans of time when great sheets of ice covered the Northern parts of North America and Europe; the most recent ice age was about 16,000 years ago

identity: a mathematical statement much stronger than equality, which asserts that two expressions are the same for all values of the variables

implode: violently collapse; fall in

inclination: a slant or angle formed by a line or plane with the horizontal axis or plane

inclined: sloping, slanting, or leaning

incomplete interpretation: a statistical flaw

independent variable: in the equation $y = f(x)$, the input variable is x (or the independent variable)

indeterminate algebra: study and analysis of solution strategies for equations that do not have fixed or unique solutions

indeterminate equation: an equation in which more than one variable is unknown

index (number): a number that allows tracking of a quantity in economics by comparing it to a standard, the consumer price index is the best known example

inductive reasoning: drawing general conclusions based on specific instances or observations; for example, a theory might be based on the outcomes of several experiments

Industrial Revolution: beginning in Great Britain around 1730, a period in the eighteenth and nineteenth centuries when nations in Europe, Asia, and the Americas moved from agrarian-based to industry-based economies

inertia: tendency of a body that is at rest to remain at rest, or the tendency of a body that is in motion to remain in motion

inferences: the act or process of deriving a conclusion from given facts or premises

inferential statistics: analysis and interpretation of data in order to make predictions

infinite: having no limit; boundless, unlimited, endless; uncountable

infinitesimals: functions with values arbitrarily close to zero

infinity: the quality of unboundedness; a quantity beyond measure; an unbounded quantity

information database: an array of information related to a specific subject or group of subjects and arranged so that any individual bit of information can be easily found and recovered

information theory: the science that deals with how to separate information from noise in a signal or how to trace the flow of information through a complex system

infrastructure: the foundation or permanent installations necessary for a structure or system to operate

initial conditions: the values of variables at the beginning of an experiment or of a set at the beginning of a simulation; chaos theory reveals that small changes in initial conditions can produce widely divergent results

input: information provided to a computer or other computation system

inspheres: spheres that touch all the "inside" faces of a regular polyhedron; also called "enspheres"

integer: a positive whole number, its negative counterpart, or zero

integral: a mathematical operation similar to summation; the area between the curve of a function, the x-axis, and two bounds such as $x = a$ and $x = b$; an inverse operation to finding the derivative

integral calculus: the branch of mathematics dealing with the rate of change of functions with respect to their variables

integral number: integer; that is, a positive whole number, its negative counterpart, or zero

integral solutions: solutions to an equation or set of equations that are all integers

integrated circuit: a circuit with the transistors, resistors, and other circuit elements etched into the surface of a single chip of silicon

integration: solving a differential equation; determining the area under a curve between two boundaries

intensity: the brightness of radiation or energy contained in a wave

intergalactic: between galaxies; the space between the galaxies

interplanetary: between planets; the space between the planets

interpolation: filling in; estimating unknown values of a function between known values

intersection: a set containing all of the elements that are members of two other sets

interstellar: between stars; the space between stars

intraframe: the compression applied to still images, interframe compression compares one image to the next and only stores the elements that have changed

intrinsic: of itself; the essential nature of a thing; originating within the thing

inverse: opposite; the mathematical function that expresses the independent variable of another function in terms of the dependent variable

inverse operations: operations that undo each other, such as addition and subtraction

inverse square law: a given physical quality varies with the distance from the source inversely as the square of the distance

inverse tangent: the value of the argument of the tangent function that produces a given value of the function; the angle that produces a particular value of the tangent

invert: to turn upside down or to turn inside out; in mathematics, to rewrite as the inverse function

inverted: upside down; turned over

ionized: an atom that has lost one or more of its electrons and has become a charged particle

ionosphere: a layer in Earth's atmosphere above 80 kilometers characterized by the existence of ions and free electrons

irrational number: a real number that cannot be written as a fraction of the form *a*/*b*, where *a* and *b* are both integers and *b* is not zero; when expressed in decimal form, an irrational number is infinite and nonrepeating

isometry: equality of measure

isosceles triangle: a triangle with two sides and two angles equal

isotope: one of several species of an atom that has the same number of protons and the same chemical properties, but different numbers of neutrons

iteration: repetition; a repeated mathematical operation in which the output of one cycle becomes the input for the next cycle

iterative: relating to a computational procedure to produce a desired result by replication of a series of operations

iterator: the mathematical operation producing the result used in iteration

kinetic energy: the energy an object has as a consequence of its motion

kinetic theory of gases: the idea that all gases are composed of widely separated particles (atoms and molecules) that exert only small forces on each other and that are in constant motion

knot: nautical mile per hour

Lagrange points: two positions in which the motion of a body of negligible mass is stable under the gravitational influence of two much larger bodies (where one larger body is moving)

latitude: the number of degrees on Earth's surface north or south of the equator; the equator is latitude zero

law: a principle of science that is highly reliable, has great predictive power, and represents the mathematical summary of experimental results

law of cosines: for a triangle with angles A, B, C and sides a, b, c, $a^2 = b^2 + c^2 - 2bc \cos A$

law of sines: if a triangle has sides a, b, and c and opposite angles A, B, and C, then $\sin A/a = \sin B/b = \sin C/c$

laws of probability: set of principles that govern the use of probability in determining the truth or falsehood of a hypothesis

light-year: the distance light travels within a vaccuum in one year

limit: a mathematical concept in which numerical values get closer and closer to a given value

linear algebra: the study of vector spaces and linear transformations

linear equation: an equation in which all variables are raised to the first power

linear function: a function whose graph on the *x*-*y* plane is a straight line or line segment

litmus test: a test that uses a single indicator to prompt a decision

locus (pl: loci): in geometry, the set of all points, lines, or surfaces that satisfies a particular requirement

logarithm: the power to which a certain number called the base is to be raised to produce a particular number

logarithmic coordinates: the x and y coordinates of a point on a cartesian plane using logarithmic scales on the x- and y-axes.

logarithmic scale: a scale in which the distances that numbers are positioned, from a reference point, are proportional to their logarithms

logic circuits: circuits used to perform logical operations and containing one or more logic elements: devices that maintain a state based on previous input to determine current and future output

logistic difference equation: the equation $x_{(n+1)} = r \times x_{n(1-xn)}$ is used to study variability in animal populations

longitude: one of the imaginary great circles beginning at the poles and extending around Earth; the geographic position east or west of the prime meridian

machine code: the set of instructions used to direct the internal operation of a computer or other information-processing system

machine language: electronic code the computer can utilize

magnetic trap: a magnetic field configured in such a way that an ion or other charged particle can be held in place for an extended period of time

magnetosphere: an asymmetric region surrounding the Earth in which charged particles are trapped, their behavior being dominated by Earth's magnetic field

magnitude: size; the measure or extent of a mathematical or physical quantity

mainframes: large computers used by businesses and government agencies to process massive amounts of data; generally faster and more powerful than desktops but usually requiring specialized software

malfunctioning: not functioning correctly; performing badly

malleability: the ability or capability of being shaped or formed

margin of error: the difference between the estimated maximum and minimum values a given measurement could have

mathematical probability: the mathematical computation of probabilities of outcomes based on rules of logic

matrix: a rectangular array of data in rows and columns

mean: the arithmetic average of a set of data

median: the middle of a set of data when values are sorted from smallest to largest (or largest to smallest)

megabyte: term used to refer to one million bytes of memory storage, where each byte consists of eight bits; the actual value is 1,048,576 (2^{20})

memory: a device in a computer designed to temporarily or permanently store information in the form of binomial states of certain circuit elements

meridian: a great circle passing through Earth's poles and a particular location

metallurgy: the study of the properties of metals; the chemistry of metals and alloys

meteorologist: a person who studies the atmosphere in order to understand weather and climate

methanol: an alcohol consisting of a single carbon bonded to three hydrogen atoms and an O–H group

microcomputers: an older term used to designate small computers designed to sit on a desktop and to be used by one person; replaced by the term personal computer

microgravity: the apparent weightless condition of objects in free fall

microkelvin: one-millionth of a kelvin

minicomputers: a computer midway in size between a desktop computer and a main frame computer; most modern desktops are much more powerful than the older minicomputers and they have been phased out

minimum viable population: the smallest number of individuals of a species in a particular area that can survive and maintain genetic diversity

mission specialist: an individual trained by NASA to perform a specific task or set of tasks onboard a spacecraft, whose duties do not include piloting the spacecraft

mnemonic: a device or process that aids one's memory

mode: a kind of average or measure of central tendency equal to the number that occurs most often in a set of data

monomial: an expression with one term

Morse code: a binary code designed to allow text information to be transmitted by telegraph consisting of "dots" and "dashes"

mouse: a handheld pointing device used to manipulate an indicator on a screen

moving average: a method of averaging recent trends in relation to long term averages, it uses recent data (for example, the last 10 days) to calculate an average that changes but still smooths out daily variations

multimodal input/output (I/O): multimedia control and display that uses various senses and interaction styles

multiprocessing: a computer that has two or more central processers which have common access to main storage

nanometers: billionths of a meter

nearsightedness: describes the inability to see distant objects clearly

negative exponential: an exponential function of the form $y = e^{-x}$

net force: the final, or resultant, influence on a body that causes it to accelerate

neuron: a nerve cell

neurotransmitters: the substance released by a neuron that diffuses across the synapse

neutron: an elementary particle with approximately the same mass as a proton and neutral charge

Newtonian: a person who, like Isaac Newton, thinks the universe can be understood in terms of numbers and mathematical operations

nominal scales: a method for sorting objects into categories according to some distinguishing characteristic, then attaching a label to each category

non-Euclidean geometry: a branch of geometry defined by posing an alternate to Euclid's fifth postulate

nonlinear transformation: a transformation of a function that changes the shape of a curve or geometric figure

nonlinear transformations: transformations of functions that change the shape of a curve or geometric figure

nuclear fission: a reaction in which an atomic nucleus splits into fragments

nuclear fusion: mechanism of energy formation in a star; lighter nuclei are combined into heavier nuclei, releasing energy in the process

nucleotides: the basic chemical unit in a molecule of nucleic acid

nucleus: the dense, positive core of an atom that contains protons and neutrons

null hypothesis: the theory that there is no validity to the specific claim that two variations of the same thing can be distinguished by a specific procedure

number theory: the study of the properties of the natural numbers, including prime numbers, the number theorem, and Fermat's Last Theorem

numerical differentiation: approximating the mathematical process of differentiation using a digital computer

nutrient: a food substance or mineral required for the completion of the life cycle of an organism

oblate spheroid: a spheroid that bulges at the equator; the surface created by rotating an ellipse 360 degrees around its minor axis

omnidirectional: a device that transmits or receives energy in all directions

Öort cloud: a cloud of millions of comets and other material forming a spherical shell around the solar system far beyond the orbit of Neptune

orbital period: the period required for a planet or any other orbiting object to complete one complete orbit

orbital velocity: the speed and direction necessary for a body to circle a celestial body, such as Earth, in a stable manner

ordinate: the *y*-coordinate of a point on a Cartesian plane

organic: having to do with life, growing naturally, or dealing with the chemical compounds found in or produced by living organisms

oscillating: moving back and forth

outliers: extreme values in a data set

output: information received from a computer or other computation system based on the information it has received

overdubs: adding voice tracks to an existing film or tape

oxidant: a chemical reagent that combines with oxygen

oxidizer: the chemical that combines with oxygen or is made into an oxide

pace: an ancient measure of length equal to normal stride length

parabola: a conic section; the locus of all points such that the distance from a fixed point called the focus is equal to the perpendicular distance from a line

parabolic: an open geometric curve where the distance of a point on the curve to a fixed point (focus) and a fixed line (directrix) is the same

paradigm: an example, pattern, or way of thinking

parallax: the apparent motion of a nearby object when viewed against the background of more distant objects due to a change in the observer's position

parallel operations: separating the parts of a problem and working on different parts at the same time

parallel processing: using at least two different computers or working at least two different central processing units in the same computer at the same time or "in parallel" to solve problems or to perform calculation

parallelogram: a quadrilateral with opposite sides equal and opposite angles equal

parameter: an independent variable, such as time, that can be used to rewrite an expression as two separate functions

parity bits: extra bits inserted into digital signals that can be used to determine if the signal was accurately received

partial sum: with respect to infinite series, the sum of its first n terms for some n

pattern recognition: a process used by some artificial-intelligence systems to identify a variety of patterns, including visual patterns, information patterns buried in a noisy signal, and word patterns imbedded in text

payload specialist: an individual selected by NASA, another government agency, another government, or a private business, and trained by NASA to operate a specific piece of equipment onboard a spacecraft

payloads: the passengers, crew, instruments, or equipment carried by an aircraft, spacecraft, or rocket

perceptual noise shaping: a process of improving signal-to-noise ratio by looking for the patterns made by the signal, such as speech

perimeter: the distance around an area; in fractal geometry, some figures have a finite area but infinite perimeter

peripheral vision: outer area of the visual field

permutation: any arrangement, or ordering, of items in a set

perpendicular: forming a right angle with a line or plane

perspective: the point of view; a drawing constructed in such a way that an appearance of three dimensionality is achieved

perturbations: small displacements in an orbit

phonograph: a device used to recover the information recorded in analog form as waves or wiggles in a spiral grove on a flat disc of vinyl, rubber, or some other substance

photosphere: the very bright portion of the Sun visible to the unaided eye; the portion around the Sun that marks the boundary between the dense interior gases and the more diffuse

photosynthesis: the chemical process used by plants and some other organisms to harvest light energy by converting carbon dioxide and water to carbohydrates and oxygen

pixel: a single picture element on a video screen; one of the individual dots making up a picture on a video screen or digital image

place value: in a number system, the power of the base assigned to each place; in base-10, the ones place, the tens place, the hundreds place, and so on

plane: generally considered an undefinable term, a plane is a flat surface extending in all directions without end, and that has no thickness

plane geometry: the study of geometric figures, points, lines, and angles and their relationships when confined to a single plane

planetary: having to do with one of the planets

planisphere: a projection of the celestial sphere onto a plane with adjustable circles to demonstrate celestial phenomena

plates: the crustal segments on Earth's surface, which are constantly moving and rotating with respect to each other

plumb-bob: a heavy, conical-shaped weight, supported point-down on its axis by a strong cord, used to determine verticality in construction or surveying

pneumatic drill: a drill operated by compressed air

pneumatic tire: air-filled tire, usually rubber or synthetic

polar axis: the axis from which angles are measured in a polar coordinate system

pole: the origin of a polar coordinate system

poll: a survey designed to gather information about a subject

pollen analysis: microscopic examination of pollen grains to determine the genus and species of the plant producing the pollen; also known as palynology

polyconic projections: a type of map projection of a globe onto a plane that produces a distorted image but preserves correct distances along each meridian

polygon: a geometric figure bounded by line segments

polyhedron: a solid formed with all plane faces

polynomial: an expression with more than one term

polynomial function: a functional expression written in terms of a polynomial

position tracking: sensing the location and/or orientation of an object

power: the number of times a number is to be multiplied by itself in an expression

precalculus: the set of subjects and mathematical skills generally necessary to understand calculus

predicted frame: in compressed video, the next frame in a sequence of images; the information is based on what changed from the previous frame

prime: relating to, or being, a prime number (that is, a number that has no factors other than itself and 1)

Prime Meridian: the meridian that passes through Greenwich, England

prime number: a number that has no factors other than itself and 1

privatization: the process of converting a service traditionally offered by a government or public agency into a service provided by a private corporation or other private entity

proactive: taking action based on prediction of future situations

probability: the likelihood an event will occur when compared to other possible outcomes

probability density function: a function used to estimate the likelihood of spotting an organism while walking a transect

probability theory: the branch of mathematics that deals with quantities having random distributions

processor: an electronic device used to process a signal or to process a flow of information

profit margin: the difference between the total cost of a good or service and the actual selling cost of that good or service, usually expressed as a percentage

program: a set of instructions given to a computer that allows it to perform tasks; software

programming language processor: a program designed to recognize and process other programs

proliferation: growing rapidly

proportion: the mathematical relation between one part and another part, or between a part and the whole; the equality of two ratios

proportionately: divided or distributed according to a proportion; proportional

protractor: a device used for measuring angles, usually consisting of a half circle marked in degrees

pseudorandom numbers: numbers generated by a process that does not guarantee randomness; numbers produced by a computer using some highly complex function that simulates true randomness

Ptolemaic theory: the theory that asserted Earth was a spherical object at the center of the universe surrounded by other spheres carrying the various celestial objects

Pythagorean Theorem: a mathematical statement relating the sides of right triangles; the square of the hypotenuse is equal to the sums of the squares of the other two sides

Pythagorean triples: any set of three numbers obeying the Pythogorean relation such that the square of one is equal to the sum of the squares of the other two

quadrant: one-fourth of a circle; also a device used to measure angles above the horizon

quadratic: involving at least one term raised to the second power

quadratic equation: an equation in which the variable is raised to the second power in at least one term when the equation is written in its simplest form

quadratic form: the form of a function written so that the independent variable is raised to the second power

quantitative: of, relating to, or expressible in terms of quantity

quantum: a small packet of energy (matter and energy are equivalent)

quantum mechanics: the study of the interactions of matter with radiation on an atomic or smaller scale, whereby the granularity of energy and radiation becomes apparent

quantum theory: the study of the interactions of matter with radiation on an atomic or smaller scale, whereby the granularity of energy and radiation becomes apparent

quaternion: a form of complex number consisting of a real scalar and an imaginary vector component with three dimensions

quipus: knotted cords used by the Incas and other Andean cultures to encode numeric and other information

radian: an angle measure approximately equal to 57.3 degrees, it is the angle that subtends an arc of a circle equal to one radius

radicand: the quantity under the radical sign; the argument of the square root function

radius: the line segment originating at the center of a circle or sphere and terminating on the circle or sphere; also the measure of that line segment

radius vector: a line segment with both magnitude and direction that begins at the center of a circle or sphere and runs to a point on the circle or sphere

random: without order

random walks: a mathematical process in a plane of moving a random distance in a random direction then turning through a random angle and repeating the process indefinitely

range: the set of all values of a variable in a function mapped to the values in the domain of the independent variable; also called range set

rate (interest): the portion of the principal, usually expressed as a percentage, paid on a loan or investment during each time interval

ratio of similitude: the ratio of the corresponding sides of similar figures

rational number: a number that can be written in the form a/b, where a and b are intergers and b is not equal to zero

rations: the portion of feed that is given to a particular animal

ray: half line; line segment that originates at a point and extends without bound

real number: a number that has no imaginary part; a set composed of all the rational and irrational numbers

real number set: the combined set of all rational and irrational numbers, the set of numbers representing all points on the number line

realtime: occuring immediately, allowing interaction without significant delay

reapportionment: the process of redistributing the seats of the U. S. House of Representatives, based on each state's proportion of the national population

recalibration: process of resetting a measuring instrument so as to provide more accurate measurements

reciprocal: one of a pair of numbers that multiply to equal 1; a number's reciprocal is 1 divided by the number

red shift: motion-induced change in the frequency of light emitted by a source moving away from the observer

reflected: light or soundwaves returned from a surface

reflection: a rigid motion of the plane that fixes one line (the mirror axis) and moves every other point to its mirror image on the opposite side of the line

reflexive: directed back or turning back on itself

refraction: the change in direction of a wave as it passes from one medium to another

refrigerants: fluid circulating in a refrigerator that is successively compressed, cooled, allowed to expand, and warmed in the refrigeration cycle

regular hexagon: a hexagon whose sides are all equal and whose angles are all equal

relative: defined in terms of or in relation to other quantities

relative dating: determining the date of an archaeological artifact based on its position in the archaeological context relative to other artifacts

relativity: the assertion that measurements of certain physical quantities such as mass, length, and time depend on the relative motion of the object and observer

remediate: to provide a remedy; to heal or to correct a wrong or a deficiency

retrograde: apparent motion of a planet from east to west, the reverse of normal motion; for the outer planets, due to the more rapid motion of Earth as it overtakes an outer planet

revenue: the income produced by a source such as an investment or some other activity; the income produced by taxes and other sources and collected by a governmental unit

rhomboid: a parallelogram whose sides are equal

right angle: the angle formed by perpendicular lines; it measures 90 degrees

RNA: ribonucleic acid

robot arm: a sophisticated device that is standard equipment on space shuttles and on the International Space Station; used to deploy and retrieve satellites or perform other functions

Roche limit: an imaginary surface around a star in a binary system; outside the Roche limit, the gravitational attraction of the companion will pull matter away from a star

root: a number that when multiplied by itself a certain number of times forms a product equal to a specified number

rotary-wing design: an aircraft design that uses a rotating wing to produce lift; helicopter or autogiro (also spelled autogyro)

rotation: a rigid motion of the plane that fixes one point (the center of rotation) and moves every other point around a circle centered at that point

rotational: having to do with rotation

round: also to round off, the systematic process of reducing the number of decimal places for a given number

rounding: process of giving an approximate number

sample: a randomly selected subset of a larger population used to represent the larger population in statistical analysis

sampling: selecting a subset of a group or population in such a way that valid conclusions can be made about the whole set or population

scale (map): the numerical ratio between the dimensions of an object and the dimensions of the two or three dimensional representation of that object

scale drawing: a drawing in which all of the dimensions are reduced by some constant factor so that the proportions are preserved

scaling: the process of reducing or increasing a drawing or some physical process so that proper proportions are retained between the parts

schematic diagram: a diagram that uses symbols for elements and arranges these elements in a logical pattern rather than a practical physical arrangement

schematic diagrams: wiring diagrams that use symbols for circuit elements and arranges these elements in a logical pattern rather than a practical physical arrangement

search engine: software designed to search the Internet for occurences of a word, phrase, or picture, usually provided at no cost to the user as an advertising vehicle

secant: the ratio of the side adjacent to an acute angle in a right triangle to the side opposite; given a unit circle, the ratio of the x coordinate to the y coordinate of any point on the circle

seismic: subjected to, or caused by an earthquake or earth tremor

self-similarity: the term used to describe fractals where a part of the geometric figure resembles a larger or smaller part at any scale chosen

semantic: the study of how words acquire meaning and how those meanings change over time

semi-major axis: one-half of the long axis of an ellipse; also equal to the average distance of a planet or any satellite from the object it is orbiting

semiconductor: one of the elements with characteristics intermediate between the metals and nonmetals

set: a collection of objects defined by a rule such that it is possible to determine exactly which objects are members of the set

set dancing: a form of dance in which dancers are guided through a series of moves by a caller

set theory: the branch of mathematics that deals with the well-defined collections of objects known as sets

sextant: a device for measuring altitudes of celestial objects

signal processor: a device designed to convert information from one form to another so that it can be sent or received

significant difference: to distinguish greatly between two parameters

significant digits: the digits reported in a measure that accurately reflect the precision of the measurement

silicon: element number 14, it belongs in the category of elements known as metalloids or semiconductors

similar: in mathematics, having sides or parts in constant proportion; two items that resemble each other but are not identical

sine: if a unit circle is drawn with its center at the origin and a line segment is drawn from the origin at angle theta so that the line segment intersects the circle at (x, y), then y is the sine of theta

skepticism: a tendency towards doubt

skew: to cause lack of symmetry in the shape of a frequency distribution

slope: the ratio of the vertical change to the corresponding horizontal change

software: the set of instructions given to a computer that allows it to perform tasks

solar masses: dimensionless units in which mass, radius, luminosity, and other physical properties of stars can be expressed in terms of the Sun's characteristics

solar wind: a stream of particles and radiation constantly pouring out of the Sun at high velocities; partially responsible for the formation of the tails of comets

solid geometry: the geometry of solid figures, spheres, and polyhedrons; the geometry of points, lines, surfaces, and solids in three-dimensional space

spatial sound: audio channels endowed with directional and positional attributes (like azimuth, elevation, and range) and room effects (like echoes and reverberation)

spectra: the ranges of frequencies of light emitted or absorbed by objects

spectrum: the range of frequencies of light emitted or absorbed by an object

sphere: the locus of points in three-dimensional space that are all equidistant from a single point called the center

spin: to rotate on an axis or turn around

square: a quadrilateral with four equal sides and four right angles

square root: with respect to real or complex numbers s, the number t for which $t^2 = s$

stade: an ancient Greek measurement of length, one stade is approximately 559 feet (about 170 meters)

standard deviation: a measure of the average amount by which individual items of data might be expected to vary from the arithmetic mean of all data

static: without movement; stationary

statistical analysis: a set of methods for analyzing numerical data

statistics: the branch of mathematics that analyzes and interprets sets of numerical data

stellar: having to do with stars

sterographics: presenting slightly different views to left and right eyes, so that graphic scenes acquire depth

stochastic: random, or relating to a variable at each moment

Stonehenge: a large circle of standing stones on the Salisbury plain in England, thought by some to be an astronomical or calendrical marker

storm surge: the front of a hurricane, which bulges because of strong winds; can be the most damaging part of a hurricane

stratopause: the boundary in the atmosphere between the stratosphere and the mesosphere usually around 55 kilometers in altitude

stratosphere: the layer of Earth's atmosphere from 15 kilometers to about 50 kilometers, usually unaffected by weather and lacking clouds or moisture

sublimate: change of phase from a solid to a gas

sublunary: "below the moon"; term used by Aristotle and others to describe things that were nearer to Earth than the Moon and so not necessarily heavenly in origin or composition

subtend: to extend past and mark off a chord or arc

sunspot activity: one of the powerful magnetic storms on the surface of the Sun, which causes it to appear to have dark spots; sunspot activity varies on an 11-year cycle

superconduction: the flow of electric current without resistance in certain metals and alloys while at temperatures near absolute zero

superposition: the placing of one thing on top of another

suspension bridge: a bridge held up by a system of cables or cables and rods in tension; usually having two or more tall towers with heavy cables anchored at the ends and strung between the towers and lighter vertical cables extending downward to support the roadway

symmetric: to have balanced proportions; in bilateral symmetry, opposite sides are mirror images of each other

symmetry: a correspondence or equivalence between or among constituents of a system

synapse: the narrow gap between the terminal of one neuron and the dendrites of the next

tactile: relating to the sense of touch

tailwind: a wind blowing in the same direction of that of the course of a vehicle

tangent: a line that intersects a curve at one and only one point in a local region

tectonic plates: large segments of Earth's crust that move in relation to one another

telecommuting: working from home or another offsite location

tenable: defensible, reasonable

terrestrial refraction: the apparent raising or lowering of a distant object on Earth's surface due to variations in atmospheric temperature

tessellation: a mosaic of tiles or other objects composed of identical repeated elements with no gaps

tesseract: a four-dimensional cube, formed by connecting all of the vertices of two three-dimensional cubes separated by the length of one side in four-dimensional space

theodolite: a surveying instrument designed to measure both horizontal and vertical angles

theorem: a statement in mathematics that can be demonstrated to be true given that certain assumptions and definitions (called axioms) are accepted as true

threatened species: a species whose population is viable but diminishing or has limited habitat

time dilation: the principle of general relativity which predicts that to an outside observer, clocks would appear to run more slowly in a powerful gravitational field

topology: the study of those properties of geometric figures that do not change under such nonlinear transformations as stretching or bending

topspin: spin placed on a baseball, tennis ball, bowling ball, or other object so that the axis of rotation is horizontal and perpendicular to the line of flight and the top of the object is rotating in the same direction as the motion of the object

trajectory: the path followed by a projectile; in chaotic systems, the trajectory is ordered and unpredictable

transcendental: a real number that cannot be the root of a polynomial with rational coefficients

transect: to divide by cutting transversly

transfinite: surpassing the finite

transformation: changing one mathematical expression into another by translation, mapping, or rotation according to some mathematical rule

transistor: an electronic device consisting of two different kinds of semiconductor material, which can be used as a switch or amplifier

transit: a surveyor's instrument with a rotating telescope that is used to measure angles and elevations

transitive: having the mathematical property that if the first expression in a series is equal to the second and the second is equal to the third, then the first is equal to the third

translate: to move from one place to another without rotation

translation: a rigid motion of the plane that moves each point in the same direction and by the same distance

tree: a collection of dots with edges connecting them that have no looping paths

triangulation: the process of determining the distance to an object by measuring the length of the base and two angles of a triangle

trigonometric ratio: a ratio formed from the lengths of the sides of right triangles

trigonometry: the branch of mathematics that studies triangles and trigonometric functions

tropopause: the boundry in Earth's atmosphere between the troposphere and the stratosphere at an altitude of 14 to 15 kilometers

troposphere: the lowest layer of Earth's atmosphere extending from the surface up to about 15 kilometers; the layer where most weather phenomena occur

ultra-violet radiation: electromagnetic radiation with wavelength shorter than visible light, in the range of 1 nanometer to about 400 nanometer

unbiased sample: a random sample selected from a larger population in such a way that each member of the larger population has an equal chance of being in the sample

underspin: spin placed on a baseball, tennis ball, bowling ball, or other object so that the axis of rotation is horizontal and perpendicular to the line of flight and the top of the object is rotating in the opposite direction from the motion of the object

Unicode: a newer system than ASCII for assigning binary numbers to keyboard symbols that includes most other alphabets; uses 16-bit symbol sets

union: a set containing all of the members of two other sets

upper bound: the maximum value of a function

vaccuum: theoretically, a space in which there is no matter

variable: a symbol, such as letters, that may assume any one of a set of values known as the domain

variable star: a star whose brightness noticeably varies over time

vector: a quantity which has both magnitude and direction

velocity: distance traveled per unit of time in a specific direction

verify: confirm; establish the truth of a statement or proposition

vernal equinox: the moment when the Sun crosses the celestial equator marking the first day of spring; occurs around March 22 for the northern hemisphere and September 21 for the southern hemisphere

vertex: a point of a graph; a node; the point on a triangle or polygon where two sides come together; the point at which a conic section intersects its axis of symmetry

viable: capable of living, growing, and developing

wavelengths: the distance in a periodic wave between two points of corresponding phase in consecutive cycles

whole numbers: the positive integers and zero

World Wide Web: the part of the Internet allowing users to examine graphic "web" pages

yield (interest): the actual amount of interest earned, which may be different than the rate

zenith: the point on the celestial sphere vertically above a given position

zenith angle: from an observer's viewpoint, the angle between the line of sight to a celestial body (such as the Sun) and the line from the observer to the zenith point

zero pair: one positive integer and one negative integer

ziggurat: a tower built in ancient Babylonia with a pyramidal shape and stepped sides

Topic Outline

APPLICATIONS

CAREERS

DATA ANALYSIS

FUNCTIONS & OPERATIONS

GRAPHICAL REPRESENTATIONS

IDEAS AND CONCEPTS

MEASUREMENT

NUMBER ANALYSIS

Ratio, Rate, and Proportion
Rounding
Scientific Notation
Sequences and Series
Significant Figures or Digits
Step Functions
Symbols
Zero

NUMBER SETS

Bases
Field Properties
Fractions
Integers
Number Sets
Number System, Real
Numbers: Abundant, Deficient, Perfect, and Amicable
Numbers, Complex
Numbers, Forbidden and Superstitious
Numbers, Irrational
Numbers, Massive
Numbers, Rational
Numbers, Real
Numbers, Whole

SCIENCE APPLICATIONS

Absolute Zero
Alternative Fuel and Energy
Astronaut
Astronomer
Astronomy, Measurements in
Banneker, Benjamin
Brain, Human
Chaos
Comets, Predicting
Cosmos
Dating Techniques
Earthquakes, Measuring
Einstein, Albert
Endangered Species, Measuring
Galileo, Galilei
Genome, Human
Human Body
Leonardo da Vinci
Light
Light Speed
Mitchell, Maria
Nature
Ozone Hole
Poles, Magnetic and Geographic
Solar System Geometry, History of
Solar System Geometry, Modern Understandings of
Sound
Space Exploration
Space, Growing Old in
Spaceflight, History of
Spaceflight, Mathematics of
Sun
Superconductivity
Telescope
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Scale Drawings and Models

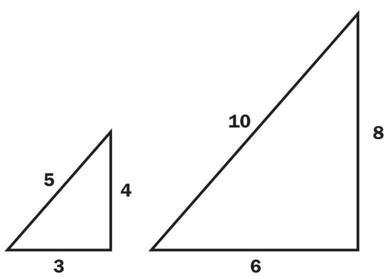
S

Scale drawings are based on the geometric principle of similarity. Two figures are similar if they have the same shape even though they may have different sizes. Any figure is similar to itself, so, in this specific case, the similar figures do actually have the same size as well as the same shape.

Scale Drawing and Models in Geometry

In traditional Euclidean geometry, two **polygons** are similar if their corresponding angles are equal and their corresponding sides are in proportion. For example, the two right triangles shown below are similar, because the length of each side in the large triangle is twice the length of the corresponding side in the small triangle.

polygon a geometric figure bounded by line segments



In transformational geometry, two figures are said to be similar if one of them can be "mapped" onto the other one by a transformation that expands or contracts all dimensions by the same factor. Such a transformation is usually called a dilation, a size change, or a size transformation. For example, in the preceding illustration, the small triangle can be mapped onto the large triangle by a dilation with scale factor 2. The ratio of the length of a side in the large triangle to the corresponding side in the small triangle is 2-to-1, often written as 2:1.

Another example is that a square with sides of length 2 centimeters (cm) is similar to a square with sides of length 10 cm, since the 2 cm square can be mapped onto the 10 cm square by a dilation with magnitude (scale factor) 5. This simply means that each dimension of the 2 cm square is multiplied by 5 to produce the 10 cm square. We call 5 the scale factor or magnitude of the dilation and we say that the ratio of the dimensions of the

larger square to those of the smaller square is 5:1 (or "5-to-1"). The smaller square has been "scaled up" by a factor of 5.

*This process of scaling up or down to produce similar figures with different sizes has numerous applications in many fields, such as in copying machines where one enlarges or reduces the original.

A figure can also be "scaled down".* Starting with a square with sides of length 10 cm, we can produce a square whose sides have length 2 cm by scaling the larger square by a factor of $\frac{1}{5}$. Note that all squares are similar because any given square can be mapped onto any other square by a dilation of magnitude b/a where b/a is the ratio of the sides of the second square to that of the first. This is not true for polygons in general. For example, any two triangles whose corresponding angles are unequal cannot be similar.

Maps

One familiar type of scale drawing is a map. When someone plans a trip, it is convenient to have a map of the general region in which that trip will take place. Obviously, the map will not be the actual size, but, if it is to be helpful, it should look similar to the region, but on a much smaller scale. Any map that is designed to help people plan out a journey will have a "legend" printed on it showing, for example, how many miles or kilometers are represented by each inch or centimeter. It would be terribly confusing to the traveler to have the scale vary on different parts of the same map, so the scale is uniform for the entire map, which returns to the mathematical concept of similarity. The ratio of each distance in the real world to its corresponding distance on the map will be the same for all corresponding distances between the two.

It is no coincidence that mathematicians have adopted the word "mapping" to refer to the correspondence between the points in one set and the points of another set. The dilation referred to in the earlier paragraph is often called a "similarity mapping."

Models

Almost everyone is familiar with toys that are scale models of cars, airplanes, boats, and space shuttles, as well as many other familiar objects. Here again, similarity is applied in the manufacturing of these scale models. A popular series of scale model cars has the notation "1:18" on the box. This is a ratio that tells how the model compares in size to the real car. In this case, all the dimensions of the model car are $\frac{1}{18}$ the size of the corresponding dimensions of the real car. So the model is $\frac{1}{18}$ as long, $\frac{1}{18}$ as wide, and $\frac{1}{18}$ as high as the real car. If the real car is 180 inches long, then the model car will be $\frac{1}{18}$ as long or 10 inches in length. Because all the dimensions of the model car are scaled from the real car by the same factor, the model looks realistic.

This same similarity principle is used in reverse by the people who design automobiles. They usually start with a sketch, drawn more or less to scale, of the car they want to design; translate this sketch into a true scale drawing, often with the help of computer-assisted design software; and build a scale model to the specifications in the drawing. Then, if everything is approved by the executives of the automobile company, the scale drawings and models are given to automotive engineers who design and build a full-size prototype of the vehicle. If the prototype also receives approval from senior executives, then the vehicle is put into full-scale production.

Other Uses

Scale drawings and models are also very important for architects, builders, carpenters, plumbers, and electricians. Architects work in a similar manner as automotive designers. They start with sketches of ideas showing what they want their buildings to look like, then translate the sketches into accurate scale drawings, and finally build scale models of the buildings for clients to evaluate before any construction on the actual structure is begun. If the client approves the design, then detailed scale drawings, called blueprints, are prepared for the whole building so that the builders will know where to put walls, doors, restrooms, stairwells, elevators, electrical wiring, and all other features of the building.

On each blueprint is a legend similar to that found on a map that tells how many feet are represented by each inch on the blueprint. In addition, a blueprint tells the scale of the drawing to the actual building. All of the people involved in the construction of the building have access to these blueprints so that every feature of the building ends up being the right size and in the right place.

Artists such as painters and sculptors often use scale drawings or scale models as preliminary guides for their work. An artist commissioned to paint a large mural often makes preliminary small sketches of the drawings she plans to put on the wall. These are essentially scale drawings of the final work.

Similarly, a sculptor usually makes a small scale model, perhaps of clay or plaster, before beginning the actual sculpture, which might be in marble, granite, or brass. Mount Rushmore, one of the great national monuments in the United States, is a gigantic sculpture featuring the heads of Presidents George Washington, Thomas Jefferson, Abraham Lincoln, and Theodore Roosevelt carved into the natural granite of the Black Hills of

Architect Frank Lloyd Wright (1867–1959) views a model of the Guggenheim Museum in 1945. The actual facility was completed in New York City in 1959. Such models help designers view a project in three dimensions, whereas blueprints show only two.

South Dakota. The sculptor of this imposing work, Gutzon Borglum, first created plaster scale models of the heads in his studio before ever going to the mountain. His models had a simple 1:12 scaling ratio; one inch on the model would represent one foot on the mountain. He made very careful measurements on the models, multiplied these measurements by 12, and used these latter values to guide his work in sculpting the mountain. When Borglum's work was finished, each of the faces of the four presidents was more than 60 feet high from the chin to the top of the head.

From arts and crafts to large-scale manufacturing processes, the use of similarity in scale drawings and models is fundamental to creating products that are both useful and pleasing to the eye. SEE ALSO Computer-Aided Design; Mount Rushmore, Measurement of; Photocopier; Ratio, Rate and Proportion; Transformations.

Stephen Robinson

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Scientific Method, Measurements and the

A common misconception about the scientific method is that it involves the use of exact measurements to prove **hypotheses**. What is the real nature of science and measurement, and how are these two processes related?

hypothesis a proposition that is assumed to be true for the purpose of proving other propositions

The Nature of Scientific Inquiry

According to one definition, the scientific method is a mode of research in which a hypothesis is tested using a controlled, carefully documented, and replicable (reproducible) experiment. In fact, scientific inquiry is conducted in many different ways. Even so, the preceding definition is a reasonable summary or approximation of much scientific work.

Science, at heart, is an effort to understand and make sense of the world outside and inside ourselves. Carl Sagan, in *The Demon-Haunted World: Science as a Candle in the Dark*, noted that the scientific method has been a powerful tool in this effort. He concluded that the knowledge generated by this method is the most important factor in extending and bettering our lives. The steps of the scientific method follow.

1. Recognize a problem.
2. Formulate a hypothesis using existing knowledge.
3. Test the hypothesis by gathering data.

4. Revise the hypothesis (if necessary).
5. Test the new hypothesis (if necessary).
6. Draw a conclusion.

Scientists often use controlled experiments. Theoretically, controlled experiments eliminate alternative explanations by varying only one factor at a time. Does this guarantee that all alternative explanations are eliminated? To answer this question, consider the role measurement plays in the scientific method.

The Nature of Measurement

Measurement addresses a fundamental question that arises in many scientific studies (and everyday activities): "How big or small is it?" Answering this question can involve either direct or indirect measurements. For example, determining the size of a television screen can be done directly by using a tape measure. Many other measurements, including most of those done by scientists, must be done indirectly.

For example, if a teacher wants to know how much algebra her students learned during the semester, she cannot examine their brains directly for this knowledge. So she gives them a test, which indicates what they have stored in their brains. Likewise, astronomers cannot directly measure the distance to a faraway star, its temperature or chemical composition, or whether it has orbiting planets. They must devise ways of measuring such attributes indirectly.

Even when measurement is done directly, it is essentially an estimation rather than an exact process. One reason for this is the difference between **discrete quantities** and **continuous quantities**. To determine how much there is of a discrete quantity, one simply has to count the number of items in a collection. To determine how much there is of a continuous quantity, however, one must measure it.

discrete quantities amounts composed of separate and distinct parts

continuous quantities amounts composed of continuous and undistinguishable parts

Although measurement is a more complicated process than counting, it involves the following fundamentally simple idea: Divide the continuous quantity into equal-size parts (units) and count the number of units. In effect, the measurement process transforms a continuous quantity into one that is countable (a discrete quantity). For example, measuring the diagonal of a television involves, in effect, subdividing it into segments 1-inch in length and counting the number of these units.

Unlike counting a discrete quantity, measurement can involve a part of a unit. A unit of measurement can be subdivided into smaller units (such as tenths of an inch), which can be further subdivided (such as hundredths of an inch), and so on, forever. In other words, if one measures the diagonal of a 34-inch television, it is extremely unlikely to be exactly 34 inches in length. It could be, for instance, 33.9, 34.1, 33.999, or 34.0001 inches in length.

Absolutely precise measurement is impossible because measurement devices cannot exactly replicate a standard unit. For example, the inch demarcations on a yardstick are not exactly 1 inch each, even when the stick was first manufactured. Nonexact units further result from repeated expansion and contraction of the stick because of repeated heating and cooling.

Accurately measuring small amounts of chemical solutions requires the proper measuring devices and attention to detail. Replicate samples, as shown here, are a mainstay of scientific measurement.

Even if measuring devices were perfectly accurate, it would not be humanly possible to read them with perfect precision. In brief, even direct measurement involves some degree or margin of error.

The Relation Between Measurement and the Scientific Method

Testing a hypothesis requires collecting data. There are basically two types of data: categorical (name) data, such as gender, and numerical (number) data. The latter can involve a discrete quantity (such as the number of men who had a particular voting preference) or a continuous one (such as the amount of math anxiety). Continuous data must be measured on a scale.

Because measurements are always imprecise to some degree, scientists' conclusions may be incorrect. So scientists work hard to develop increasingly accurate measurement devices. The possibility of measurement error is also a reason why scientific experiments need to be replicated.

Indeed, scientific conclusions based on any type of data may be imperfect for a number of reasons. One is that scientists' instruments or tests do not always measure what the scientists intend. Another problem is that scientists frequently cannot collect data on every possible case, and so only a sample of cases are measured, and a sample is not always representative of all the cases. For example, a biologist may find a result that is true for her sample of men, but the conclusions may not apply to women or even all men.

Measurement is an integral aspect of the scientific method. However, using the scientific method does not guarantee exact results and definitive proof of a hypothesis. Rather, it is a sound approach to constructing an increasingly accurate and thorough understanding of our world. SEE ALSO Accuracy and Precision; Data Collection and Interpretation; Problem Solving, Multiple Approaches to.

Arthur J. Baroody

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Scientific Notation

Scientific notation is a method of writing very large and very small numbers. Ordinary numbers are useful for everyday measurement, such as daily temperatures and automobile speeds, but for large measurements like astronomical distances, scientific notation provides a way to express these numbers in a short and concise way.

The basis of scientific notation is the power of ten. Since many large and small numbers have a few **integers** with many zeros, the power of ten can be used to shorten the length of the written number.

integer a positive whole number, its negative counterpart, or zero

A number written in scientific notation has two parts. The first part is a number between 1 and 10, and the second part is a power of ten. Mathematically, writing a number in scientific notation is the expression of the number in the form $n \times 10^x$ where n is a number greater than 1 but less than 10 and x is an exponent of 10. An example is 15,653 written as 1.5653×10^4. This type of notation is also used for very small numbers such as 0.0000000072, which, in scientific notation, is rewritten as 7.2×10^{-9}.

Some examples of measurements where scientific notation becomes useful follow.

- The wavelength for violet light is 40-millionths of a centimeter = 4×10^{-5}cm.
- Some black holes are measured by the amount of solar masses they could contain. One black hole was measured as 10,000,000 or 1.0×10^7 **solar masses**.
- The weight of an alpha particle, which is emitted in the radioactive decay of Plutonium-239, is 0.000,000,000,000,000,000,000,000,006,645 kilograms (6.645×10^{-27} kilograms).
- A computer hard disk could hold 4 gigabytes (about 4,000,000,000 bytes) of information. That is $4.0 \text{ x } 10^9$ bytes.
- Computer calculation speeds are often measured in nanoseconds. A nanosecond is 0.000000001 seconds, or 1.0×10^{-9} seconds.

solar masses dimensionless units in which mass, radius, luminosity, and other physical properties of stars can be expressed in terms of the Sun's characteristics

POWERS, EXPONENTS, AND SCIENTIFIC NOTATION

The process of calculating scientific notation comes from the rules of exponents.

- Any number raised to the power of zero is one.
- Any number raised to the power of one is equal to itself.
- Any number raised to the nth power is equal to the product of the number used as a factor n times.

Converting Numbers into Scientific Notation

Complete the following steps to convert large and small numbers into scientific notation. First, identify the significant digits and move the decimal place to the right or left so that only one integer is on the left side of the decimal. Rewrite the number with the new decimal place and include only the identified significant digits. Then, following the number, write a mul-

tiplication sign and the number 10. Raise the 10 to the exponent that represents the number of places you moved the decimal point.

If the number is large and you moved the decimal point to the left, the exponent is positive. Conversely, if the number is small and you moved the decimal point to the right, the exponent is negative.

If a chemist is trying to discuss the number of electrons expected in a sample of atoms, the number may be 1,100,000,000,000,000,000,000,000,000. Using three significant figures the number is written 1.10×10^{27}. Along the same lines the weight of a particular chemical may be 0.00000000000000000000721 grams. In scientific notation the example would be written 7.21×10^{-20}. SEE ALSO Measurement, Metric System of; Numbers, Massive; Powers and Exponents.

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Sequences and Series

A sequence is an ordered listing of numbers such as {1, 3, 5, 7, 9}. In mathematical terms, a sequence is a function whose domain is the natural number set {1, 2, 3, 4, 5, . . .}, or some subset of the natural numbers, and whose range is the set of real numbers or some subset thereof. Let f denote the sequence {1, 3, 5, 7, 9}. Then $f(1) = 1$, $f(2) = 3$, $f(3) = 5$, $f(4) = 7$, and $f(5) = 9$. Here the domain of f is {1, 2, 3, 4, 5} and the range is {1, 3, 5, 7, 9}.

The terms of a sequence are often designated by subscripted variables. So for the sequence {1, 3, 5, 7, 9}, one could write $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, $a_4 = 7$, and $a_5 = 9$. A shorthand designation for a sequence written in this way is $\{a_n\}$. It is sometimes possible to get an explicit formula for the general term of a sequence. For example, the general nth term of the sequence {1, 3, 5, 7, 9} may be written $a_n = 2n - 1$, where n takes on values from the set {1, 2, 3, 4, 5}.

The sequence {1, 3, 5, 7, 9} is finite, since it contains only five terms, but in mathematics, much work is devoted to the study of infinite sequences.* For instance, the sequence of "all" odd natural numbers is written {1, 3, 5, 7, 9, . . .}, where the three dots indicate that there is an infinite number of terms following 9. Note that the general term of this sequence is also $a_n = 2n - 1$, but now n can take on any natural number value, however large. Other examples of infinite sequences include the even natural numbers, all multiples of 3, the digits of pi (π), and the set of all prime numbers.

***A famous infinite sequence is the so-called Fibonacci sequence {1, 1, 2, 3, 5, 8, 13, 21, . . .} in which the first two terms are each 1 and every term after the second is the sum of the two terms immediately preceding it.**

The amount of money in a bank account paying compound interest at regular intervals is a sequence. If $100 is deposited at an annual interest rate of 5 percent compounded annually, then the amounts of money in the account at the end of each year form a sequence whose general term can be computed by the formula $a_n = 100(1.05)^n$, where n is the number of years since the money was deposited.

Series

A series is just the sum of the terms of a sequence. Thus {1, 3, 5, 7, 9} is a sequence, but $1 + 3 + 5 + 7 + 9$ is a series. So long as the series is finite the sum may be found by adding all the terms, so $1 + 3 + 5 + 7 + 9 = 25$. If the series is infinite, then it is not possible to add all the terms by the ordinary addition **algorithm**, since one could never complete the task.

algorithm a rule or procedure used to solve a mathematical problem

Nevertheless, there are infinite series that have finite sums. An example of one such series is: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$, where, again, the three dots indicate that this series continues according to this pattern without ever ending. Now, obviously someone cannot sit down and complete the term by term addition of this series, even in a lifetime, but mathematicians reason in the following way. The sum of the first term is 1; the sum of the first two terms is 1.5; the sum of the first three terms is 1.75; the sum of the first four terms is 1.875; the sum of the first five terms is 1.9375; and so on. These are called the first five **partial sums**. Mathematically, it is possible to argue that no matter how many partial sums one takes, the nth partial sum will always be slightly less than 2 (in the above example), for any value of n. On the other hand, each new partial sum will be closer to 2 than the one before it. Therefore, one would say that the limiting value of the sequence of partial sums is 2 and the sum is defined as the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$. This infinite series is said to converge to 2. A case where the partial sums grow without bound is described as a series that diverges or that has no finite sum. SEE ALSO Fibonacci, Leonardo Pisano.

partial sum with respect to infinite series, the sum of its first n terms

Stephen Robinson

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Shapes, Efficient

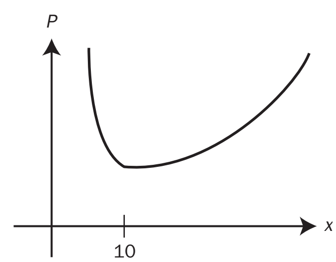
Eighteen ounces of *Salubrious Cereal* is packaged in a box $2\frac{1}{8}''$ (inches) $\times$ $7\frac{11}{16}'' \times 11''$. The box's volume is about 180 cubic inches; its total surface area, A, is about 249 square inches. Could the manufacturer keep the same volume but reduce A by changing the dimensions of the package? If so, the company could save money.

Consider $3'' \times 6'' \times 10''$ and $4'' \times 5'' \times 9''$ boxes. Their volume is 180 cubic inches, but $A = 216$ and 202 square inches, respectively. Clearly, for a fixed volume, surface area can vary, suggesting that a certain shape might produce a minimum surface area.

The problem can be rephrased this way: Let x, y, and z be the sides of a box such that $xyz = V$ for a fixed V. Find the minimum value of the function $A(x, y, z) = 2xy + 2xz + 2yz$. This looks complicated. An appropriate

strategy would be to first solve a simpler two-dimensional problem in order to develop methods for attacking the three-dimensional problem.

Consider the following: Let the area of a rectangular region be fixed at 100 square inches. Find the minimum perimeter of the region. Letting x and y be the sides of the rectangle means that $xy = 100$, and you are asked to minimize the perimeter, $P(x, y) = 2x + 2y$. Begin by solving the first equation for y so that the second can be expressed as a function of one variable, namely $P(x) = 2x + \frac{2(100)}{x}$. The problem can now be solved in several ways. First, graph the function, obtaining the graph below.



Using the techniques available on the graphing calculator, you could find that the minimum function value occurs at $x = 10$, making $y = 10$, and $P = 40$.

Consider yet another simplification to our "minimum area" problem—let the base of the box be a square, thereby eliminating one variable. If the base is x by x and the height is y, then a fixed $V = x^2y$ and a variable $A(x, y) = 2x^2 + 4xy$. Since $y = \frac{V}{x^2}$, $A(x) = 2x^2 + \frac{4V}{x}$. Using calculus, $A'(x) = 4x - \frac{4V}{x^2} = 0$, giving $x = y = \sqrt[3]{V}$. The optimal solution is a cube.

Calculus makes it clear that the minimum surface area occurs when the cereal box is a cube, which means the area equals $6V^{2/3}$. In the original problem, a cube with edges equal to $\sqrt[3]{180} \approx 5.65$ gives a surface area of approximately 192 square inches, a savings of 23% from the original box. It should be noted that nonrectangular shapes may reduce the surface area further. For example, a cylinder with a radius of 2.7 inches and a volume of 180 cubic inches would have a total surface area of 179 square inches. A sphere, while not a feasible shape for a cereal box, would have a surface area of 154 square inches if the volume is 180 cubic inches. The larger question then becomes: Of all three dimensional shapes with a fixed volume, which has the least surface area?

History of the Efficient Shape Problem

Efficient shapes problems emerged when someone asked the question: Of all plane figures with the same perimeter, which has the greatest area? Aristotle (384 B.C.E.–322 B.C.E.) may have known this problem, having noted that a geometer knows why round wounds heal the slowest. Zenodorus (c. 180 B.C.E.) wrote a treatise on isometric figures and included the following propositions, not all of which were proved completely.

1. Of all regular polygons with equal perimeter, the one with the most sides has the greatest area.
2. A circle has a greater area than any regular polygon of the same perimeter.
3. A sphere has a greater volume than solid figures with the same surface area.

Since Zenodorus draws on the work of Archimedes (287 B.C.E.–212 B.C.E.), it is possible that Archimedes contributed to the solution of the problem. Zenodorus used both direct and indirect proofs in combination with inequalities involving sides and angles in order to establish inequalities among areas of triangles. Pappus (c. 320 C.E.) extended Zenodorus's work, adding the proposition that of all circular segments with the same circumference, the semicircle holds the greatest area. In a memorable passage, Pappus wrote that through instinct, living creatures may seek optimal solutions. In particular he was thinking of honey bees, noting that they chose a hexagon for the shape of honeycombs—a wise choice since hexagons will hold more honey for the same amount of material than squares or equilateral triangles.

After Pappus, problems such as these did not attract successful attention until the advent of calculus in the late 1600s. While calculus created tools for the solution of many optimization problems, calculus was not able to prove the theorem that states that of all plane figures with a fixed perimeter, the circle has the greatest area. Using methods drawn from synthetic geometry, Jacob Steiner (1796–1863) advanced the proof considerably. Schwarz, in 1884, finally proved the theorem. Research on efficient shapes is still flourishing.

One important aspect of these problems is that they come in pairs. Paired with the problem of minimizing the perimeter of a rectangle of fixed area is the dual problem of maximizing the area of a rectangle with fixed perimeter. For example, the makers of a cereal box may want to have a large area for advertising. Similarly, paired with the problem of minimizing the surface area of a shape of fixed volume is the problem of maximizing the volume of a shape of fixed surface area. Standard calculus problems involve finding the dimensions of a cylinder that minimizes surface area for a fixed volume or its pair. The cylinder whose height equals its diameter has the least surface area for a fixed volume. SEE ALSO Minimum Surface Area; Dimensional Relationships.

Don Barry

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Significant Figures or Digits

Imagine that a nurse takes a child's temperature. However, the mercury thermometer being used is marked off in intervals of one degree. There is a 98° F mark and a 99° F mark, but nothing in between. The nurse announces that the child's temperature is 99° F. How does someone interpret this information? The answer becomes clear when one has an understanding of significant digits.

The Importance of Precision

One possibility, of course, is that the mercury did lie exactly on the 99° F mark. What are the other possibilities? If the mercury were slightly above or below the 99° F mark (so that his actual temperature was, say, 99.1° F or 98.8° F), the nurse would probably still have recorded it as 99° F. However, if his actual temperature was 98.1° F, the nurse should have recorded it as 98° F. In fact, the temperature would have been recorded as 99° F only if the mercury lay within half a degree of 99° F—in other words, if the actual temperature was between 98.5° F and 99.5° F. (This interval includes its left endpoint, 98.5° F, but not its right endpoint, because 99.5° F would be rounded up to 100° F. However, any number just below 99.5° F, such as 99.49999° F, would be rounded down to 99° F.) One can conclude from this analysis that, in the measurement of 99° F, only the first 9 is guaranteed to be accurate—the temperature is definitely 90-something degrees. The second 9 represents a rounded value.

Now suppose that, once again, the child's temperature is being taken, but this time, a more standard thermometer is being used, one that is marked off in intervals of one tenth of a degree. Once again, the nurse announces that his temperature is 99° F. This announcement provides more information than did the previous measurement. This time, if his temperature were actually 98.6° F, it would have been recorded as such, not as 99° F. Following the reasoning of the previous paragraph, one can conclude that his actual temperature lies at most halfway toward the next marker on the thermometer—that is, within half of a tenth of a degree of 99° F. So the possible range of values this time is 98.95° F to 99.05° F—a much narrower range than in the previous case.

When marking down a patient's temperature in a medical chart, it may not be necessary to make it clear whether a measurement is precise to one half of a degree or one twentieth of a degree. But in the world of scientific experiments, such precision can be vital. It is clear that simply recording a measurement of 99° F does not, by itself, give any information as to the level of precision of this measurement. Thus, a method of recording data that will incorporate this information is needed. This method is the use of significant digits (also called significant figures).

In the second of the two scenarios described earlier, the nurse is able to round off the measurement to the nearest tenth of a degree, not merely to the nearest degree. This fact can be communicated by including the tenths digit in the measurement—that is, by recording the measurement as 99.0° F rather than 99° F. The notation 99.0° F indicates that the first two digits are accurate and the third is a rounded value, whereas the notation 99° F indicates that the first digit is accurate and the second is a rounded value.

The number 99.0° F is said to have three significant digits and the number 99° F to have two.

What Is Significant?

When one looks at a measurement, how can she tell how many significant digits it has? Any number between 1 and 9 is a significant digit. When it comes to zeros, the situation is a little more complicated. To see why, suppose that a certain length is recorded to be 90.30 meters long. This measurement has four significant digits—the zero on the end (after the three) indicates that the nine, first zero, and three are accurate and the last zero is an estimate. Now suppose that she needs to convert this measurement to kilometers. The measurement now reads 0.09030 kilometers. She now has five digits in the measurement, but she has not suddenly gained precision simply by changing units. Any zero that appears to the left of the first nonzero digit (the 9, in this case) cannot be considered significant.

Now suppose that she wants to convert that same measurement to millimeters, so that it reads 90,300 millimeters. Once again, she has not gained any precision, so the rightmost zero is not significant. However, the rule to be deduced from this situation is less clear. How should she consider zeros that lie to the right of the last nonzero digit? The zero after the three in 90.30 is significant because it gives information about precision—it is not a "necessary" zero, in the sense that mathematically 90.30 could be written 90.3. However, the two zeros after the three in 90300 are mathematically necessary as place-holders; 90300 is certainly not the same number as 9030 or 903. So you cannot conclude that they are there to convey information about precision. They might be—in the example, the first one is and the second one is not—but it is not guaranteed. The following rules summarize the previous discussion.

1. Any nonzero digit is significant.
2. Any zero that lies between non-zero digits is significant.
3. Any zero that lies to the left of the first non-zero digit is not significant.
4. If the number contains a decimal point, then any zero to the right of the last non-zero digit is significant.
5. If the number does not contain a decimal point, then any zero to the right of the last non-zero digit is not significant.

Here are some examples:

0.030700 — five significant digits (all but the first two zeros);

400.00 — five significant digits;

400 — one significant digit (the four); and

5030 — three significant digits.

Consider the last example. One has seen earlier that the zero in the ones column cannot be considered to be significant, because it may be that the measurement was rounded to the nearest multiple of ten. If, however, someone is measuring with an instrument that is marked in intervals of one unit, then recording the measurement as 5030 does not convey the full level of

precision of the measurement. One cannot record the measurement as 5030.0, because that indicates a jump from three to five significant digits, which is too many. In such a circumstance, avoid the difficulty by recording the measurement using scientific notation, i.e. writing 5030 as 5.030×10^3. Now the final zero lies to the right of the decimal point, so by rule four, it is a significant digit.

Calculations Involving Significant Figures

Often, when doing a scientific experiment, it is necessary not only to record data but to do computations with the information. The main principle involved is that one can never gain significant digits when computing with data; a result can only be as precise as the information used to get it.

In the first example, two pieces of data, 10.30 and 705, are to be added. Mathematically, the sum is 715.30. However, the number 705 does not have any significant digits to the right of the decimal point. Thus the three and the zero in the sum cannot be assumed to have any degree of accuracy, and so the sum is simply written 715. If the first piece of data were 10.65 instead of 10.30, then the actual sum would be 715.65. In this case, the sum would be rounded up and recorded as 716. The rule for numbers to be added (or subtracted) with significant digits is that the sum/difference cannot contain a digit that lies in or to the right of any column containing an insignificant digit.

In a second example, two pieces of data, 10.30 and 705, are to be multiplied. Mathematically, the product is 7261.5. Which of these digits can be considered significant? Here the rule is that when multiplying or dividing, the product/quotient can have only as many significant digits as the piece of data containing the fewest significant digits. So in this case, the product may only have three significant digits and would be recorded as 7260.

In the final example, the diameter of a circle is measured to be 10.30 centimeters. To compute the circumference of this circle, this quantity must be multiplied by π. Since π is a known quantity, not a measurement, it is considered to have infinitely many significant digits. Thus the result can have as many significant digits as 10.30—that is, four. (To do this computation, take an approximation of π to as many decimal places as necessary to guarantee that the first four digits of the product will be accurate.) A similar principle applies if someone wishes to do any strictly mathematical computation with data: for instance, if one wants to compute the radius of the circle whose diameter she has measured, multiply by 0.5. The answer, however, may still have four significant digits, because 0.5 is not a measurement and hence is considered to have infinitely many significant digits. SEE ALSO Accuracy and Precision; Estimation; Rounding.

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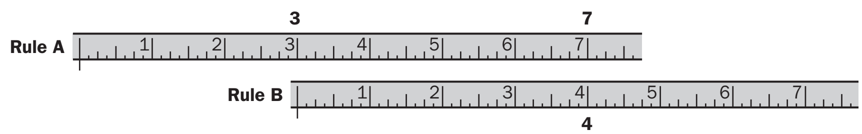
Sine *See Trigonometry.*

Slide Rule

Pocket calculators only came into common use in the 1970s. Digital computers first appeared in the 1940s, but were not in widespread use by the general public until the 1980s. Before pocket calculators, there were mechanical desktop calculators, but these could only add and subtract and at best do the most basic of multiplications.

A tool commonly used by engineers and scientists who dealt with math in their work was the slide rule. A slide rule is actually a simple form of what is called an analog computer, a device that does computation by measuring and operating on a continuously varying quantity, as opposed to the discreet units used by digital computers. While many analog computers work by measuring and adding voltages, the slide rule is based on adding distances.



A simple slide rule looks like two rulers that are attached to each other so that they can slide along one edge. In fact any two number lines that can be moved past each other make up a slide rule. To add two numbers on a slide rule, look at the above illustration, which shows how to add 3 and 4. Start by finding 3 on rule A. Then slide rule B so that the zero point of rule B lines up with 3 on rule A. Find 4 on rule B and it will be lined up with the sum on rule A, which is 7. Reverse these steps to subtract two numbers.

The slide rule was invented by William Oughtred (1574–1660), an English mathematician. Oughtred designed the common slide rule, which has a movable center rule between fixed upper and lower rules. Many different scales are printed on it, and a sliding cursor helps the user better view the alignment between scales. He also designed less common circular slide rules that worked on the same principles.

The previous example has already illustrated how to add and subtract using a slide rule. One can also multiply and divide with a slide rule using **logarithms**. In fact, slide rules were invented specifically for multiplying and dividing large numbers using logarithms shortly after John Napier invented logarithms.

logarithm the power to which a certain number called the base is to be raised to produce a particular number

To multiply 3,750 and 225 using a slide rule, first find $\log_{10}$ 3,750 and $\log_{10}$ 225. Use the slide rule to add these logarithms. Then find the antilog of the sum, which is the product of 3,750 and 225.

Division using a slide rule is the same as multiplication, except one subtracts the logarithms. Of course, one must have logarithm tables handy when multiplying or dividing with a slide rule, to look up the logarithms.

One can compute powers on a slide rule by taking the log of a log. Since A^x is the same as x times $\log A$, one can do this multiplication by adding $\log x$ and $\log (\log A)$. Special log-log scales on a slide rule make it possible to calculate powers. Roots, such as $\sqrt{2}$ can be considered fractional powers, so the slide rule with a log-log scale could be used to compute roots as

well. Additional scales were provided to look up trigonometric functions such as **sine**, **cosine**, and **tangent**.

sine if a unit circle is drawn with its center at the origin and a line segment is drawn from the origin at angle theta so that the line segment intersects the circle at (x, y), then y is the sine of theta

cosine if a unit circle is drawn with its center at the origin and a line segment is drawn from the origin at angle theta so that the line segment intersects the circle at (x, y), then x is the cosine of theta

tangent if a unit circle is drawn with its center at the origin at angle theta so that the line segment intersects the circle at (x, y), then the ratio $\frac{y}{x}$ is the tangent of theta

The accuracy of a slide rule is limited by its size, the quality to which its scales are marked, and the ability of the user to read the scales. Typical engineering slide rules were accurate to within 0.1 percent. This degree of accuracy was considered sufficient for many engineering and scientific applications. However, the use of slide rules declined rapidly after the electronic calculator became inexpensive and widely used. This was due, in large part, to the fact that it is much easier and convenient to perform applications, such as multiplication, on a calculator than it is to look up logarithms. SEE ALSO Analog and Digital; Bases; Logarithms; Mathematical Devices, Early.

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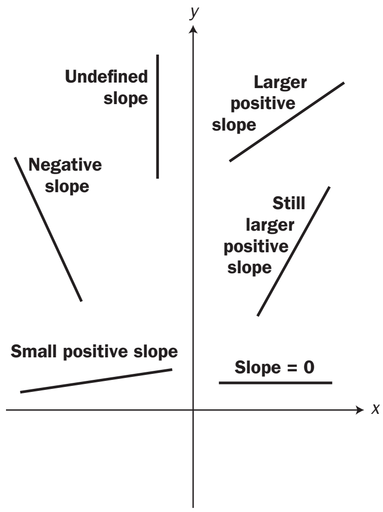
Slope

Slope is a quantity that measures the steepness of a line. To figure out how to define slope, think about what it means for one line to be steeper than another. Intuitively, one would say the steeper line "climbs faster." To make this mathematically precise, consider the following. If two points on the steeper line that are horizontally one unit apart are compared with two points on the less-steep line that are horizontally one unit apart, the pair on the steeper line will be farther apart vertically. Thus the slope of a line is defined to be the ratio of the vertical distance to the horizontal distance between any two points on that line. Sometimes this ratio is called "rise to run," meaning the vertical change relative to the horizontal change.

Several interesting things can be noticed about the slope of a line. First of all, the definition does not tell us which two points on the line to choose. Fortunately, this does not matter because any pair of points on the same line will yield the same slope.

In fact, this is really the defining characteristic of a line: When a line is "straight," this means that its steepness never varies, unlike a parabola or a circle, which "bend" and therefore climb more steeply in some places than in others.

The slope of a line indicates whether the line slants upwards from left to right, slants downwards from left to right, or is flat. If a line slants upwards, movement from one point to a second point is such that one unit to the right of the first will yield an increase (that is, a positive difference) in the y-coordinates of the points. If the line slants downward, the points' y-coordinates will decrease (a negative difference) as one unit is moved to the right. If the line is horizontal, the y-coordinate will not change. Therefore, an upward-slanting line has positive slope, a downward-slanting line has negative slope, and a horizontal line has a slope of zero.



A slope also involves division, which raises the possibility of dividing by zero. This would occur whenever the slope of a vertical line is computed (and only then). Therefore, vertical lines are considered to have an undefined slope. Alternatively, some people consider the slope of a vertical line to be infinite, which makes sense intuitively because a vertical line climbs as steeply as possible. SEE ALSO Graphs, Effects of Parameter Changes; Infinity; Lines, Parallel and Perpendicular; Lines, Skew.

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Soap Bubbles *See Minimum Surface Area.*

Solar System Geometry, History of

Humanity's understanding of the geometry of the solar system has developed over thousands of years. This journey towards a more accurate model of our planetary system has been marked throughout by controversy and misconception. The science of direct observation would ultimately play the most important role in bringing order to our understanding of the universe.

Ancient Conceptions and Misconceptions

Archaeologists have found artifacts demonstrating that ancient Babylonians and Egyptians made numerous observations of **celestial** events such as full moons, new moons, and **eclipses**, as well as the paths of the Sun and Moon relative to the stars. The ancient Greeks drew upon the work of the ancient Babylonians and Egyptians, and they used that knowledge to learn even more about the heavens.

As far back as the sixth century B.C.E., Greek astronomers and mathematicians attempted to determine the structure of the universe. Anaximander (c. 611 B.C.E.–546 B.C.E.) proposed that the distance of the Moon from Earth is 19 times the **radius** of Earth. Pythagoras (c. 572 B.C.E.–497 B.C.E.),

celestial relating to the stars, planets, and other heavenly bodies

eclipse occurrence when an object passes in front of another and blocks the view of the second object; most often used to refer to the phenomenon that occurs when the Moon passes in front of the Sun or when the Moon passes through Earth's shadow

radius the line segment originating at the center of a circle or sphere and terminating on the circle or sphere; also, the measure of that line segment

HOW FAR IS THE MOON FROM EARTH?

In contrast to Anaximander's smaller estimate, Claudius Ptolemy would later prove that the distance of the Moon from Earth is actually approximately 64 times the radius of Earth.

whose name is given to the **Pythagorean Theorem**, was perhaps the first Greek to believe that the Earth is spherically shaped.

The ancient Greeks largely believed that Earth was stationary, and that all of the planets and stars rotated around it. Aristarchus (c. 310 B.C.E.–230 B.C.E.) was one of the few who dared question this hypothesis, and is known as the "Copernicus of antiquity." His bold statement was recorded by the famous mathematician Archimedes (c. 287 B.C.E.–212 B.C.E.) in the book *The Sand Reckoner.* Aristarchus is quoted as saying that the Sun and stars are motionless, Earth orbits around the Sun, and Earth rotates on its axis. Neither Archimedes nor any of the other mathematicians of the time believed Aristarchus. Consequently, Aristarchus's remarkable insight about the movements of the solar system would go unexplored for the next 1,800 years.

Pythagorean Theorem a mathematical statement relating the sides of right triangles; the square of the hypotenuse is equal to the sum of the squares of the other two sides

circumference the distance around a circle

stade an ancient Greek measurement of length, one stade is approximately 559 feet (about 170 meters)

congruent exactly the same everywhere

longitude one of the imaginary great circles beginning at the poles and extending around Earth; the geographic position east or west of the prime meridian

circumnavigation the act of sailing completely around a globe

Eratosthenes (c. 276 B.C.E.–194 B.C.E.), a Greek mathematician who lived during the same time as Archimedes, was another key contributor to the early understanding of the universe. Eratosthenes was the librarian at the ancient learning center in Alexandria in Egypt. Among his many mathematical and scientific contributions, Eratosthenes used an ingenious system of measurement to produce the most accurate estimate of Earth's **circumference** that had been computed up to that time.

At Syene, which is located approximately 5,000 **stades** from Alexandria, Eratosthenes observed that a vertical stick had no shadow at noon on the summer solstice (the longest day of the year). In contrast, a vertical stick at Alexandria produced a shadow a little more than 7° from vertical (approximately $\frac{1}{50}$th of a circle) on the same day. Using the property that two parallel lines (rays of the Sun) cut by a transversal (the line passing through the center of Earth and Alexandria) form **congruent** alternate interior angles, Eratosthenes deduced that the distance between Syene and Alexandria must be $\frac{1}{50}$th of the circumference of Earth. As a result, he computed the circumference to be approximately 250,000 stades. Thus, Eratosthenes's estimate equates to about 25,000 miles, which is remarkably close to Earth's actual circumference of 24,888 miles. Apparently, Eratosthenes's interest in this calculation stemmed from the desire to develop a convenient measure of **longitude** to use in the **circumnavigation** of the world.

The most far-reaching contribution during the Greek period came from Claudius Ptolemy (c. 85 C.E.–165 C.E.). His most important astronomical work, *Syntaxis*, (better known as *Almagest*) became the premier textbook on astronomy until the work of Kepler and Galileo over 1,500 years later. Mathematically, *Almagest* explains both plane and spherical **trigonometry**, and includes tables of trigonometric values and **square roots**.

trigonometry the branch of mathematics that studies triangles and trigonometric functions

square root with respect to real or complex numbers s, the number t for which $t^2 = s$

Ptolemy's text firmly placed Earth at the center of the universe without any possibility of its movement. His work also reflected the Greek belief that celestial bodies generally possess divine qualities that have spiritual significance for humans. Ptolemy synthesized previous explanations for planetary motion given by the Greek mathematicians Apollonius, Eudoxus, and Hipparchus into a grand system capable of accounting for almost any orbital phenomenon. This scheme included the use of circular orbits (cycles), orbits whose center lies on the path of a larger orbit ("epicycles"), and eccentric orbits, where Earth is displaced slightly from the center of the orbit. Despite its explanatory power, however, the sheer complexity of Ptolemy's model of

the universe ultimately drove Nicolas Copernicus (1473–1543) to a completely different conclusion about Earth's place in the solar system.

The Dawn of Direct Observation

By the first half of the sixteenth century, Ptolemy's **geocentric** model of the universe had taken on an almost mystical quality. Not only were scientists unquestioning in their devotion to the model and its implications, but the leadership of the Catholic Church in Europe regarded Ptolemy's view as doctrine nearly equal in status to the Bible. It is ironic, therefore, that the catalyst for the scientific revolution would come from Copernicus, a Catholic trained as a mathematician and astronomer.

geocentric Earth-centered

Copernicus's primary goal in his astronomical research was to simplify and improve the techniques used to calculate the positions of celestial objects. To accomplish this, he replaced Ptolemy's geocentric model and its complicated system of compounded circles with a radically different **heliocentric** view that placed the Sun at the center of the solar system. He retained Ptolemy's concept of uniform motion of the planets in circular orbits, but now the orbits were concentric rather than overlapping circles.

heliocentric Sun-centered

Copernicus's ideas were very controversial given the faith in Ptolemy's model. Consequently, despite the fact that his model was completed in 1530, his theory was not published until after his death in 1543. The book containing Copernicus's theory, *De Revolutionibus Orbium Caelestium* ("On the Revolutions of the Heavenly Spheres"), was so mathematically technical that only the most competent astronomer could understand it. Yet its effects would be felt throughout the scientific community. Copernicus's new theory was published at a critical juncture, at the intersection of the ancient and modern worlds. It would set the stage for the new science of direct observation known as **empiricism**, which would follow less than a century later.

empiricism the view that the experience of the senses is the single source of knowledge

Johannes Kepler (1571–1630) was a product of the struggle between ancient and modern beliefs. A teacher of mathematics, poetry and writing, he made remarkable breakthroughs in **cosmology** as a result of his work with the astronomer Tycho Brahe (1546–1601). Brahe had hired Kepler as an assistant after reading his book *Mysterium Cosmographicum.* In it, Kepler posits a model of the solar system in which the orbits of the six known planets around the Sun are shown as **inspheres** and **circumspheres** of the five regular polyhedra: the tetrahedron, hexahedron (cube), octahedron, dodecahedron and icosahedron.

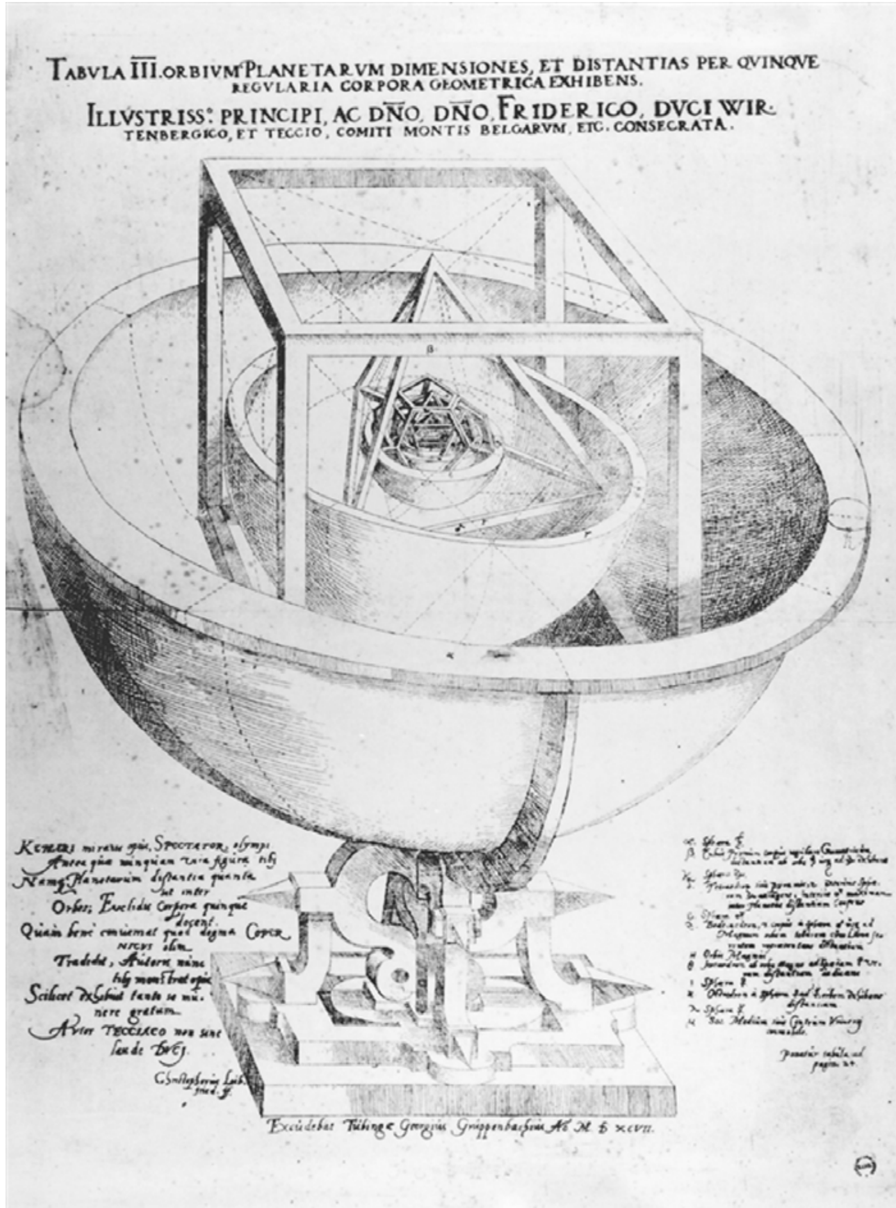
cosmology the study of the origin and evolution of the universe

inspheres spheres that touch all the "inside" faces of a regular polyhedron; also called "enspheres"

circumspheres spheres that touch all the "outside" faces of a regular polyhedron

Kepler's fascination with these geometric solids reflected his devotion to ancient Greek tradition. Plato had given a prominent role to the regular polyhedra, also known as Platonic solids, when he mystically associated them with fire, air, water, earth and the universe, respectively. When Brahe died suddenly and Kepler inherited his fastidious observations of the planets, Kepler set about to verify his own model of the solar system using the data. This began an eight-year odyssey that, far from confirming his model, completely changed the way Kepler looked at the universe. To his credit, unlike the vast majority of mathematicians and scientists of his time, Kepler recognized that if the model and the data did not agree, then the data should be assumed to be correct. Little by little, Kepler's romantic notions about

This illustration of a model shows the orbits of the planets as suggested by Johannes Kepler.



focus one of the two points that define an ellipse; in a planetary orbit, the Sun is at one focus and nothing is at the other focus

radius vector a line segment with both magnitude and direction that begins at the center of a circle or sphere and runs to a point on the circle or sphere

semi-major axis one-half of the long axis of an ellipse; also equal to the average distance of a planet or any satellite from the object it is orbiting

the structure of the solar system gave way to new, more scientific ideas, resulting in Kepler's Three Laws of Planetary Motion:

First Law The planets move about the Sun in elliptical orbits with the Sun at one **focus**.

Second Law The **radius vector** joining a planet to the Sun sweeps over equal areas in equal intervals of time.

Third Law The square of the time of one complete revolution of a planet about its orbit is proportional to the cube of the orbit's **semi-major axis**.

The first two laws were published in *Astronomia Nova* in 1609, with the third law following ten years later in *Harmonice Mundi*. Kepler's calculations were so new and different that he had to invent his mathematical tools as he went along. For example, the primary method used to confirm the sec-

ond law involved summing up triangle-shaped slices to compute the **elliptical** area bounded by the orbit of Mars. Kepler gives one account in *Astronomia Nova* of computing and summing 180 triangular slices, and repeating this process forty times to ensure accuracy! This work with **infinitesimals**, which was adapted from Archimedes's method of computing circular areas, directly influenced Isaac Newton's development of the **calculus** about a half-century later.

Interestingly, Kepler's work with the second law led him to the first law, which finally convinced him that planetary orbits were not circles at all (as Copernicus had assumed). However, as Kepler's account in the *Astronomia Nova* makes clear, once he had deduced the equation governing Mars's orbit, due to a computation error he did not quite realize that the equation was that of an **ellipse**. This is not as strange as it might first appear, considering that Kepler discovered his laws prior to Descartes's development of **analytical geometry**, which would occur about a decade later. Thus, with the elliptical equation in hand, he suddenly decided that he had come to a dead end and chose to pursue the hypothesis that he felt all along must be true—that the orbit of Mars was elliptical! In short order Kepler realized that he had already confirmed his own hypothesis. In the publication of his second law, Kepler coined the term "focus" in referring to an elliptical orbit, though the importance of the foci for defining an ellipse was well understood in Greek times.

elliptical a closed geometric curve where the sum of the distances of a point on the curve to two fixed points (foci) is constant

infinitesimals functions with values arbitrarily close to zero

calculus a method of dealing mathematically with variables that may be changing continuously with respect to each other

ellipse one of the conic sections, it is defined as the locus of all points such that the sum of the distances from two points called the foci is constant

analytical geometry describes the study of geometric properties by using algebraic operations

No discussion of either the geometry of the solar system or the science of direct observation is complete without noting the key contributions of Galileo Galilei (1564–1642). Like Kepler, Galileo's science was closely tied to mathematics. In his investigations of the **velocity** of falling objects, the **parabolic** flight paths of projectiles, and the structures of the Moon and the planets Galileo introduced new ways of mathematically describing the world around us. As he wrote in his book *Saggiatore*, "The Book [of Nature] is . . . written in the language of mathematics, and its characters are triangles, circles, and geometric figures without which it is . . . impossible to understand a single word of it; without these one wanders about in a dark labyrinth."

velocity distance traveled per unit of time in a specific direction

parabolic an open geometric curve where the distance of a point on the curve to a fixed point (focus) and a fixed line (directrix) is the same

Using the latest technology (which included telescopes), Galileo's observations of Jupiter and its moons as well as the phases of Venus left no doubt about the validity of Copernicus's heliocentric theory. His close inspection of the surface of the Moon also convinced him that the ancient Greek assumption about the Moon and the planets being perfect spheres was simply not **tenable**. Galileo paid dearly for his findings, incurring the wrath of the Catholic Church and a skeptical scientific community. But as a result of his struggle, mathematics was no longer viewed as either the servant of philosophy or theology as it had been for millennia. Instead, mathematics came to be regarded as a means of validating theoretical models, ushering in a revolutionary era of scientific discovery.

tenable defensible, reasonable

Through the work of Copernicus, Kepler and Galileo, the mistaken beliefs, pseudoscience, and romanticism of the ancients were gradually replaced by modern observational science, whose goal was and continues to be the congruence of theory with available data. Though the scientific view of the solar system continues to evolve to this day, it does so resting firmly on the foundation of the breakthroughs of the sixteenth and seventeenth centuries.

SEE ALSO Galilei, Galileo; Solar System Geometry, Modern Understandings of.

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Solar System Geometry, Modern Understandings of

Aristarchus (c. 310 B.C.E.–230 B.C.E.), an ancient Greek mathematician and astronomer, made the first claim that the planets of the solar system orbit the Sun rather than Earth. However, it was Nicolas Copernicus (1473–1543) who would spur modern investigations that would ultimately overthrow the ancient view of a **geocentric** universe. Johannes Kepler (1571–1630) and Galileo Galilei (1564–1642) initially carried out these investigations. Through the work of Copernicus, Kepler, and Galileo, the geocentric view of circular orbits with constant **velocities** was gradually replaced by a **heliocentric** perspective in which planets travel in **elliptical** orbits of changing velocities. Kepler and Galileo worked during the beginning of what has come to be known as the "Century of Genius," a remarkable time of mathematical and scientific discovery lasting from the early 1600s through the early 1700s. Isaac Newton (1642–1727), perhaps the greatest mathematician of the modern period, lived his entire life during the Century of Genius, building on the foundation laid by Kepler and Galileo.

geocentric Earth-centered

heliocentric Sun-centered

velocity distance traveled per unit of time in a specific direction

elliptical a closed geometric curve where the sum of the distances of a point on the curve to two fixed points (foci) is constant

Advances in Celestial Mechanics

Isaac Newton was born the year that Galileo died, so there is a sense that the investigations begun by Kepler and Galileo continued in an unbroken chain through the life of Newton. Newton was a mathematician and scientist who was profoundly influenced by the work of his predecessors. Mathematically, he was able to synthesize the work of Kepler and others into perhaps the most useful mathematical tool ever devised, the **calculus**. Scientifically, among Newton's many accomplishments, his work in celestial mechanics can be considered a generalization and explanation of Kepler's three laws of planetary motion. As early as 1596, Kepler had made refer-

calculus a method of dealing mathematically with variables that may be changing continuously with respect to each other

ence to a force (he used the word "souls") that seemed to emanate from the Sun, resulting in the planets' orbits. Despite many years of work, Kepler was unable to specifically identify this force. Newton not only named the force, but developed a way of mathematically describing its magnitude in a two-body system:

$$F = \frac{Gm_1m_2}{r^2}$$

where m_1 and m_2 are the masses of the two bodies (for instance, the Sun and Earth), r is the distance between the bodies, and G is the gravitational constant. This formula is known now as the "Universal Law of Gravity."

As a result of his work on the two-body problem, Newton concluded that such a system may be bound or unbound or in "steady-state," referring to the situation in which the potential energy of the system is greater than, less than, or equal to (respectively) the **kinetic energy** of the system. If bound, then one body will orbit the central mass on an elliptical path, as Kepler had stated in his second law of motion; however, if unbound, the relative orbit will be **hyperbolic**; and if steady-state, the relative orbit will be **parabolic**. The two-body problem was applicable not only to planets of the solar system, but to the paths of asteroids and comets as well. Of course, the reality of the solar system is that forces beyond just the two bodies in question influence gravitation and orbital paths. Newton noted in his influential book *Principia* that he believed the solution to a three-body problem would exceed the capacity of the human mind.

kinetic energy the energy an object has as a consequence of its motion

hyperbolic an open geometric curve where the difference of the distances of a point on the curve to two fixed points (foci) is constant

parabolic an open geometric curve where the distance of a point on the curve to a fixed point (focus) and a fixed line (directrix) is the same

From the 1700s onward, mathematical contributions to our understanding of the solar system came from individuals who, unlike before, were not necessarily astronomers. Lagrange, Euler and Laplace—all considered eminent mathematicians—provided new tools for more accurately describing the complexities of the universe. Joseph-Louis Lagrange (1736–1813) studied **perturbations** of planetary orbits. He lent his name to what has come to be called "Lagrangian points," which are equally spaced locations in front and behind a planet in its orbit, containing bodies (such as asteroids) that obey the principles of the three-body problem. Lagrange and Leonhard Euler (1707–1783) shared the 1772 prize from the Académie des Sciences of Paris for their work on the three-body problem, though it appears that Euler was the first to work on the general problem for bodies outside the solar system. In his masterwork *Mécanique Céleste*, Pierre-Simon Laplace (1749–1827) showed that the eccentricities and inclinations of planetary orbits to each other always remain small, constant, and self-correcting. He also addressed the question of the stability of the solar system and sowed the seeds for the discovery of Neptune.

perturbations small displacements in an orbit

The Discoveries of Neptune and Pluto

Despite the seismic nature of the mathematical and scientific discoveries through the seventeenth and eighteenth centuries, mathematicians and astronomers of the time could scarcely have imagined the discoveries that would be made in the following two centuries. Perhaps the greatest surprise of the period following the time of Newton, Lagrange, Euler, and Laplace was the extent to which mathematics would be used not only to confirm astronomical theories but also to predict as yet undetected phenomena.

Until 1781, there were only six known planets in the solar system. That year, an amateur astronomer discovered the planet that would later be named Uranus. By about 1840 enough irregularities had been noted in the orbit of Uranus, including those contained in Laplace's book, that astronomers were actively seeking the cause. One popular theory was that Newton's law of gravitation began to break down over large distances. The other primary theory assumed the opposite—it used Newton's law to predict the existence of an eighth, previously undetected planet that was affecting the orbit of Uranus. In fact, observations of the planet that would come to be known as Neptune had been made by various astronomers as far back as Galileo, but it was assumed to be another star.

It was John Adams (1819–1892) of England and Urbain Le Verrier (1811–1877) of France, working simultaneously using Newton's formulas and the calculus, who accomplished what seemed impossible at the time: making accurate predictions of the position of an unknown planet based solely on the gravitational effects on a known planet. Remarkably, the astronomers were even able to determine the new planet's mass without direct observation. In 1846, an astronomer in Berlin very close to Le Verrier's predicted position observed Neptune. This use of mathematics as a prediction, rather than just a confirmation tool, established it as an essential method for astronomical investigation.

Interestingly, even Neptune's orbit did not behave as expected, and the existence of Neptune did not completely account for the irregularities in the orbits of either Uranus or Jupiter. As a result, in 1905 Percival Lowell (1855–1916) hypothesized the existence of a ninth planet. Lowell would die before finding the planet, but in 1930 the discovery of Pluto was made using the telescope at Lowell's observatory in Flagstaff, Arizona. The planet's orbit was found to be highly eccentric and inclined at a much greater angle to the **ecliptic** than the other planets. In addition, its orbit actually intersects the interior of the orbit of Neptune, allowing it to approach Uranus more closely than Neptune. Though Pluto's orbit obeys Kepler's and Newton's laws, astronomers consider the orbit to be unpredictable over a long period of time (2×10^7 years).

ecliptic the plane of the Earth's orbit around the Sun

This unpredictability, or sensitivity to initial conditions, is what mathematicians and astronomers refer to as chaos. In a chaotic region, nearby orbits diverge exponentially with time. Though a recent innovation, chaos theory confirms the work of the mathematician Henri Poincaré (1854–1912) on perturbation series, which he did at the beginning of the last century. Though Poincaré's work called into question Laplace's conclusion about the stability of the solar system, it is now believed that our planetary scheme is an example of bounded chaos, where motion is chaotic but the orbits are relatively stable within a billion year timeframe. The realization that, as a result of chaos, even few-body systems can in fact be very complex in their behavior is arguably the greatest discovery about the geometry of the solar system in the final two decades of the twentieth century.

Relativity and the New Geometries

A major discovery of the first two decades of the twentieth century was Albert Einstein's (1879–1955) theory of relativity. In 1859, Le Verrier had found a small discrepancy between the predicted orbit of Mercury and its

actual path. Le Verrier believed the cause of the difference was some undetected planet or ring of material that was too close to the Sun to be observed. However, if this was not the case, Newton's theory of gravitation appeared to be inadequate to explain the discrepancy.

In 1905, Poincaré posited an initial form of the special theory of relativity, and argued that Newton's law was not valid and proposed gravitational waves that moved with the velocity of light. Only weeks later, Einstein published his first paper on special relativity, where he suggested that the velocity of light is the same whether the light is emitted by a body at rest or by a body in uniform motion. Then, in 1907, Einstein began to wonder how Newtonian gravitation would have to be modified to fit in with special relativity. This ultimately resulted in his general theory of relativity, whose **litmus test** was the discrepancy in the orbit of Mercury. Einstein applied his theory of gravitation and discovered that the difference was not due to unseen planets or a ring of material, but could be accounted for by his theory of relativity. Einstein found **Euclidean geometry**, the geometry typically taught in high school, to be too limited for describing all gravitational systems that might be encountered in space. As a result, he ascribed a prominent role to the non-Euclidean geometries that had recently been developed during the nineteenth century.

litmus test a test that uses a single indicator to prompt a decision

Euclidean geometry the geometry of points, lines, angles, polygons, and curves confined to a plane

non-Euclidean geometry a branch of geometry defined by posing an alternate to Euclid's fifth postulate

equidistant at the same distance

In the early 1800s, Janos Bolyai (1802–1860) and Nicolai Lobachevsky (1793–1856) had boldly published their revolutionary views of one kind of **non-Euclidean geometry**. Essentially, they independently confirmed that a geometry was possible where there existed not just one but an infinite number of parallel lines to a given line through a point not on the line. In the classical mechanics of Newton, a key assumption was that rays of light travel in parallel lines that are everywhere **equidistant** from each other. In contrast, Bolyai and Lobachevsky claimed that parallel lines converged and diverged, though still not intersecting. This kind of geometry is known as Lobachevskian or Hyperbolic, whereas Euclidean or plane geometry is referred to as Parabolic. There is also another kind of non-Euclidean geometry, known as Elliptic, which resulted from the later work of Poincaré.

The great mathematician Karl Freidrich Gauss (1777–1855) had early on advanced the idea that space was curved, even suggesting a method for measuring the curvature. However, the connection between Gauss's work with the curvature of space and the new geometries was not immediately apparent. In fact, the immediate effect of the non-Euclidean geometries was to cause a crisis in the mathematical community: Was there any longer a relationship between mathematics and reality? Euclidean geometry had been the basis not only for all of mathematics up to and including the calculus, but also for all scientific observation and measurement, and even the philosophies of Emmanuel Kant (1724–1804) and his followers.

Karl Freidrich Gauss made a number of contributions to the field astronomy, including his theories of perturbations between planets, which led to the discovery of Neptune.

Einstein immediately saw how the various geometries co-existed in space. The genius of his insight was the realization that, whereas light might travel in a straight line in some restricted situation (say, on Earth) when the vast expanse of the universe is considered, light rays are actually traveling in non-Euclidean paths due to gravity from masses such as the Sun. In fact, a modern theoretical model of the universe based on Einstein's general theory of relativity depends on a more complex non-Euclidean geometry than either Hyperbolic or Elliptic. The assumption is that the universe is not Eu-

clidean, and it is the concentration of matter in space (which is not uniform) that determines the extent of the deviation from the Euclidean model.

The modern period of observational science has revolutionized the way we view our universe. The geometry of our solar system is infinitely more complex than was considered possible during the time of Kepler, Galileo or even Newton. With new tools such as the Hubble telescope available to twenty-first century astronomers, it is difficult to imagine the discoveries that await us. SEE ALSO Astronomer; Chaos; Solar System Geometry, History of.

Paul G. Shotsberger

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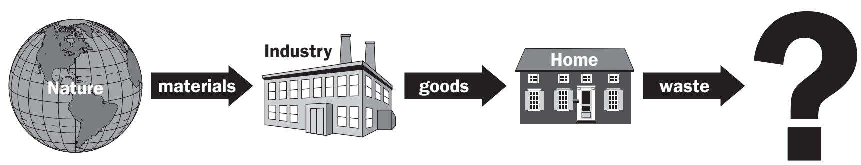
Solid Waste, Measuring

We usually know where things come from. The food we buy comes from farms, practically everything else we own comes from factories, and the raw materials to make them come from mines and forests.

But were does it all go once we finish using it or it breaks? We put it on the curb and—thanks to trash collectors—it seems to disappear. Yet in reality, nothing disappears: We just change useful things into trash, just as we change natural resources into things we can use. This article examines how mathematics lets us calculate the amounts of trash we must either divert to beneficial uses or dispose in landfills or incinerators.



Types and Fates of Solid Waste

In general, there are two kinds of trash, or solid waste. Industrial solid waste is generated when factories convert raw materials into products. Municipal solid waste is the trash we throw away after we buy and use those products. This article discusses only municipal solid waste, or MSW.

The U.S. Environmental Protection Agency estimates that about 223,230,000 tons of MSW were generated in the United States in 2000. Almost all the waste met one of two fates:

- Disposal at a landfill or incinerator; or
- Recycling or reuse (called diversion).

A small amount of the generated waste was dumped illegally, but these amounts are difficult to quantify, and are not addressed in this article.

Waste Disposal. The most common waste disposal method in the United States is landfilling. A landfill essentially is a big hole in the ground where we bury our garbage. Engineers design a landfill based on the amount and types of solid waste it needs to receive over its projected "lifetime." Once it is full and cannot hold any more trash, a new landfill must be dug. Knowing how much waste is generated by a community helps waste managers and community planners determine how big to make a new landfill, as well as how many trash trucks are needed to collect and transport the waste.

Calculations for a community start with individual households. In 1998 the average person in the United States produced 4.46 pounds of waste per day. From this, the amount of waste produced by a household of four people over a year is calculated as follows.

$$4.46 \text{ lbs / person / day} \times 365 \text{ days / year} \times 4 \text{ people / household} = 6{,}511.6 \text{ lbs / household / year}$$

To calculate how much waste must be managed by a community of 160,000 people (that is, 40,000 households of four people each), it is more convenient to convert pounds to tons, starting with the number derived from above.

First,

$$6{,}511.6 \text{ lbs / household / year} \times 1 \text{ ton / } 2{,}000 \text{ lbs} = 3.3 \text{ tons / household / year}$$

So,

$$3.3 \text{ tons / household / year} \times 40{,}000 \text{ households} = 132{,}000 \text{ tons / year}$$

Assuming that engineers designed a landfill to be 60 feet from bottom to top, and the community has equipment to compact the waste into the landfill at a rate of 2 cubic yards per ton, how many acres of land will be needed each year to accommodate the community's waste? The next equation shows the calculation.

$$132{,}000 \text{ tons / year} \times 2 \text{ yd}^3 \text{ / ton} \times 9 \text{ ft}^3 \text{ / yd}^3 \times 1 \text{ / } 60 \text{ ft} \times 1 \text{ acre / } 43{,}560 \text{ ft}^2 = 0.91 \text{ acres / year}$$

This is equivalent to filling a football field with waste each year. And, if the landfill charges the community a disposal fee of \$30 for every ton, that generates \$3,960,000 every year (132,000 tons / year × \$30 / ton). This number illustrates how expensive landfill disposal can be, not only from disposal fees, but also in terms of land use constraints and the costs of protecting the environment.

Reuse and Recycling. As the U.S. population continues to increase, Americans keep throwing away more trash each year. To decrease the amount of municipal solid waste (MSW) that is disposed in landfills, consumers, businesses, and industries can divert certain materials from disposal via reuse and recycling. Even better, they can reduce the amount of trash generated in the first place. The more materials that are reduced, reused, and recycled, the less quickly landfills will fill up.

The equations below show how to calculate the percentage of MSW produced (generated) that is diverted from disposal via reuse and recycling. Reduction is difficult to quantify, and is not considered in the following equations.

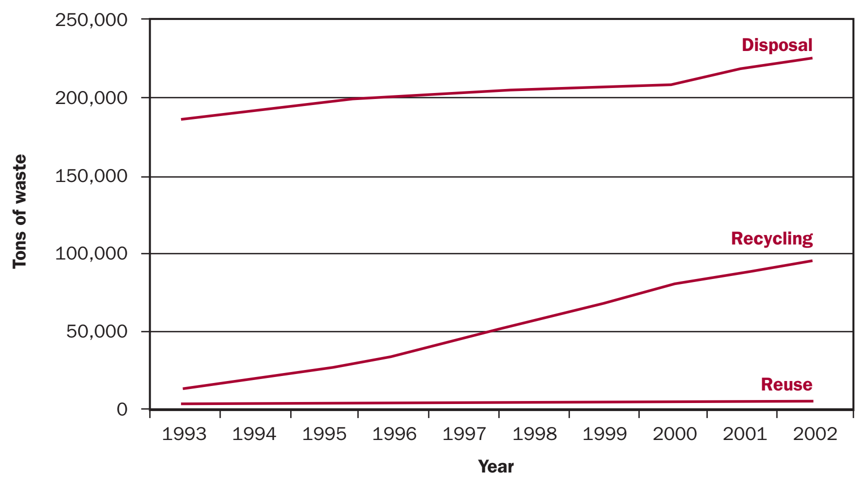
$$\text{Generation} = \text{Disposal} + (\text{Recycling} + \text{Reuse})$$

$$= \text{Disposal} + \text{Diversion}$$

So,

$$\%\ \text{Diversion} = (\text{Diversion} / \text{Generation}) \times 100\%$$

Suppose the community planner contacted the disposal and recycling facilities in her area and found how many tons of MSW were disposed, recycled, and reused in recent years. (Most states require landfills to weigh all waste they receive, and to account for its origin.) Substituting these numbers into the series of equations above, she can derive the percentage of waste generated in 2002 that was being kept from landfills by recycling and reuse. Her results are shown in the table, and the graph is shown below.



As the table shows, by 2002 our sample community was able to keep nearly one-third of its MSW out of its landfill. This number is fairly representative of most communities in the United States that have strong MSW recycling and reuse programs.

"Pay-As-You-Throw." Most U.S. communities have added recycling to their waste management programs to help divert waste from disposal. But community leaders know that not everything can be reused or recycled. A better alternative is to keep things from becoming waste in the first place. And yet there is a catch: consumers like to buy things! Although most people are genuinely concerned about the environment, it will take more than concern to prompt citizens into reducing or rethinking what they buy, and therefore what they will eventually throw away.

SOLID WASTE DIVERTED FROM LANDFILL DISPOSAL

	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002
Disposal	185,545	191,065	196,652	199,945	200,564	203,301	205,770	207,936	215,840	224,000
Recycle	14,118	19,107	24,581	32,932	44,570	54,735	65,846	77,976	85,120	92,800
Reuse	2,017	2,123	2,235	2,352	2,476	2,606	2,744	2,888	3,040	3,200
Total Generation	201,680	212,295	223,468	235,229	247,610	260,642	274,360	288,800	304,000	320,000
% Diversion	8%	10%	12%	15%	19%	22%	25%	28%	29%	30%

One motivation for reducing waste is money: namely, making consumers pay for what they throw away. Some communities have implemented such programs, called "Pay-As-You-Throw," or PAYT. Without PAYT, a household either pays a company every month or year to take their trash, or the household pays for its trash collection in its taxes. In other words, the household pays the same amount no matter whether the family throws away one bag or twenty bags.

But what if the household must pay for every bag or can it sets out for the trash collector? Suppose \$300 of its taxes has gone toward trash pickup. Now, what if instead the household pays \$1.00 for every bag set out? If the family sets out four bags a week, that would be 4 bags × \$1.00 / bag × 52 wks / year = \$208 per year.

Although this yields a savings of \$92 (\$300 − \$208), it seems like the cost is greater because the household sees the \$4.00 disappear every week and will not see the tax savings until annual tax season.

But what if the household could further reduce its garbage from four bags to three bags a week by being more careful about what is thrown away and by recycling more? In this case, the household would pay only \$3.00 per week (3 bags × \$1.00 per bag) instead of \$4.00. Compared to the \$4.00 per week fee, householders would see their dollar savings every time they put out the trash.

This way of looking at monetary benefits is called a market incentive. If appealing to our concern for the environment is not enough to motivate some of us to reduce waste generation, then most of us will do it if we can save money (and hence if it costs us money to *not* do it).

Applying PAYT in a Sample Community. Essentially, PAYT is as much psychology as it is mathematics, but still there is math involved. The amount charged must be affordable yet enough to cover the cost of disposing the waste generated. How does a community decide a fair charge?

First the community must determine how much waste there will be and how much it will cost to dispose. To find out, start with the amount of waste generated under the old system; that is, before PAYT measures were implemented. Using 2002 as a base year, and using numbers from the previous table, we know this is 320,000 tons. Divide this amount by the number of people in the community to find how much waste is generated by each resident.

For this calculation, the 2002 population must first be estimated. Suppose that census records showed a 1990 population of 154,000 and a 2000 population of 159,000. This is an increase of 5,000 residents over 10 years, or 500 people per year. Because the most recent census year was 2000, and the base year is 2002, there are 2 years' worth of people to add. Hence, 159,000 + (500 × 2) = 160,000.

The amount of waste generated per resident can now be determined.

$$320{,}000 \text{ tons} \div 160{,}000 = 2 \text{ tons per resident}$$

It is important to note that this amount includes the trash that business like offices and restaurants generate. This is why the rate of trash generation is so much greater than the previously stated rate of 4.46 pounds, which represents only what people throw away from their homes.

Recycling household materials such as glass containers can save residents money, especially if a community has a "Pay-As-You-Throw" program. Disposing of less trash also benefits the environment, reduces the consumption of natural resources, and creates recycling-related jobs.

It usually takes a year or two to start a PAYT program, and often the community will grow during that time. How many people will be in the community when the program starts, say in 2004? Use the previous per-year increase in population to make the adjustment for 2 more years into the future. That is, 160,000 + (500 × 2 years) = 161,000. Now, how much waste would those people generate in 2004, the first year of PAYT, if there were no PAYT prior to that year? The answer is 2 tons / person × 161,000 persons = 322,000 tons.

But once the community has a PAYT program in place, the amount of waste generated should start to decrease. Usually a community starting a PAYT program will look to other communities of about the same size that already have such programs, and then assume they will have a similar decrease. Suppose our sample community discovered that waste generation in nearby Payville decreased from 320,000 tons to 176,000 tons after they implemented their PAYT program. The decrease for Payville is 176,000 tons ÷ 320,000 tons × 100 percent = 55 percent.

Now our sample community can apply this rate to the waste it estimates will be generated annually with PAYT, and then determine the monthly total. The annual waste expected with PAYT is 322,000 tons × 0.55 = 177,100 tons. Hence, 177,100 tons / year × 1 year / 12 months = 14,758 tons / month.

Next, estimate how much the PAYT program will cost. Some costs, like building rent, probably will not change much. Others generally will increase over time. For instance, the salary of city employees will probably be greater in 2004 than in 2002. These increases can be estimated the same way population increases were projected: namely, by taking an average of historical increases over a period of time and projecting the annual increase to future years. Conversely, certain costs will decrease as waste decreases. The city may not need as many trucks or as many people to pick up trash as before. Assume that these scenarios have already been considered when estimating costs in the list below.

Building mortgage or rent per month	$100,000
Salaries of secretary and file clerk per month	$5,000
Salaries of truck drivers, waste collectors per month	$250,000
Truck gasoline & maintenance per month	$55,000
Cost of garbage bags to your city per month	$20,000
Landfill fees per ton per month ($30 / ton × 14,758 tons / month)	$442,740
Total Costs	872,740

Now the dollar charge per bag can be determined. First you need to know how much an average bag weighs. A bag full of cat litter will weigh more than a bag of the same size filled with tissue. A simple way to find an average weight per bag is to go to a landfill, select several bags at random, weigh them, add the weight of each, and then divide by the number of bags. Suppose five bags weighed 10, 20, 5, 25, and 40 lbs. Hence, (10 + 20 + 5 + 25 + 40) / 5 = 20 lbs per bag. Converting to tons, 20 lbs / bag ÷ 2,000 lbs / ton = 0.01 ton / bag.

Next, the number of bags that will result from the 14,758 tons that are expected is calculated by 14,758 tons / month $\times$ 1 bag / 0.01 ton = 1,475,800 bags / month. Finally, knowing that total costs per month are \$872,740, how much is that per bag?

$$\$872,740 \div 1,475,800 \text{ bags} = \$0.59 \text{ per bag}$$

This price will cover the estimated costs. Now the community leaders must use psychology again and ask whether it is too high, in which case citizens may refuse to accept PAYT. But if the price is too low, the idea of market incentives applies, and citizens won't be enticed to save money by reducing the trash they throw out (that is, because the money potentially saved would be minimal). So the community leaders must adjust the price up or down to yield the best result.

This example gives insight into the mathematics needed to create a waste disposal system that can pay for itself, is good for the environment, and is fair. People who throw away less trash should be paying less than people who throw away more trash.

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Somerville, Mary Fairfax

Scottish-born English Mathematics Writer
1780–1872

Mary Fairfax Somerville was born December 26, 1780, the fifth of seven children, to Margaret Charters and Lieutenant William George Fairfax. The family lived in Scotland during an era in which it was customary to provide daughters with an education emphasizing domestic skills and social graces. When Somerville was nine years old, her father expressed disappointment with her reading, writing, and arithmetic skills, and the following year, Somerville was sent to an expensive and exclusive boarding school. She was very unhappy and left there, her only full-time formal schooling, at the end of the year.

When she was 13, Somerville encountered **algebra** by chance when looking at a ladies' fashion magazine. At this time, such things as riddles, puzzles, and simple mathematical problems were common features in such publications. As a result of her curiosity, Somerville persuaded her younger brother's tutor to buy her copies of Bonny Castle's *Algebra* and Euclid's *Elements.*

algebra the branch of mathematics that deals with variables or unknowns representing the arithmetic numbers

Mary Fairfax Somerville was called the "Queen of Nineteenth-Century Science" by the London *Morning Post.*

Somerville's father was unhappy with the books she was reading, fearing negative effects on her domestic skills and social graces, and forbade her to read such materials. Despite her father's restrictions, Somerville secretly continued to read about mathematics by candlelight during the night while others in her family slept.

In 1804, at the age of 24, she married her cousin, Captain Samuel Greig of the Russian Navy, and they eventually had two sons, Woronzow (1805–1865) and William George (1806–1814). After the death of Greig in 1807, Somerville began, for the first time, to openly study mathematics and investigate physical astronomy. One of her most helpful mentors during this time was William Wallace (1768–1843), a Scottish mathematics master at a military college. It was upon his advice that Somerville obtained a small library of French books to provide her with a sound background in mathematics. These works were important in forming her mathematical style and account, in part, for her mastery of French mathematics.

Another supporter of her mathematical studies was her cousin, Dr. William Somerville (1777–1860), who was a surgeon in the British Navy. They were married in 1812 and had four children. With the encouragement of her husband, Somerville explored her interest in science and mathematics that steadily deepened and expanded into four books that would become her most important works.

In 1831, she published *The Mechanism of Heavens*, which was a "translation" of the first four books of the French mathematician, Laplace's *Mécanique Céleste*. Her work was noted as being far superior to previous attempts (mere translations of only Laplace's first book) at bringing such work to English readers.

In 1834 Somerville's second book, *On the Connexion of the Physical Sciences*, an account of the connections among the physical sciences, was met with even greater success and distinctions. She and Caroline Herschel were elected to the Royal Astronomical Society in 1835, becoming the first women so honored.

In 1838, William Somerville's failing health caused the family to move to the warmer climate of Italy, where William died in 1860. Mary spent the remaining years of her life in Italy and in 1848, at age 68, published her third and most successful book, *Physical Geography*, which was widely used in schools and universities for many years. Her last scientific book, *On Molecular and Microscopic Science*, a summary of the most recent discoveries in chemistry and physics, was published in 1869 when she was 89.

quaternion a form of complex number consisting of a real scalar and an imaginary vector component with three dimensions

During the last years of her life, Mary Somerville was involved in many projects, including writing a book on **quaternions** and reviewing a volume, *On the Theory of Differences*. Her reading, studying, and writing about mathematics continued until the very day of her death. Somerville died on November 29, 1872, at the age of 92.

Gay A. Ragan

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Sound

Sound is produced by the vibration of some sort of material. In a piano, a hammer strikes a steel string, causing the string to vibrate. A guitar string is plucked and vibrates. A violin string vibrates when it is bowed. In a saxophone, the reed vibrates. In an organ pipe, the column of air vibrates. When a person speaks or sings, two strips of muscular tissue in the throat vibrate. All of the vibrating objects produce sound waves.

Characteristics of Waves

All waves, including sound waves, share certain characteristics: they travel through material at a certain speed, they have frequency, and they have wavelength. The frequency of a wave is the number of waves that pass a point in one second. The wavelength of a wave is the distance between any two corresponding points on the wave. For all waves, there is a simple mathematical relationship between these three quantities called the wave equation. If the frequency is denoted by the symbol f, and the wavelength is denoted by the symbol λ, and the symbol v denotes the velocity, then the wave equation is $v = f\lambda$. In a given medium, a sound wave with a shorter wavelength will have a higher frequency.

Waves also have amplitude. Amplitude is the "height" of the wave, or how "big" the wave is. The amplitude of a sound wave determines how loud is the sound.

Longitudinal Waves. Sound waves are longitudinal waves. That means that the part of the medium vibrating moves back and forth instead of up and down or from side to side. Regions where the particles of the medium are pushed together are called compressions. Places where the particles are pulled apart are called rarefactions. A sound wave consists of a series of compressions and rarefactions moving through the medium.

As with all types of waves, the medium is left in the same place after the wave passes. A person listening to a radio across the room receives sound waves that are moving through the air, but the air is not moving across the room.

Speed of Sound Waves. Sound travels through solids, liquids, and gases at different speeds. The speed depends on the springiness of the medium. Steel, for example, is much springier than air, so sound travels through steel about fifteen times faster than it travels through air.

At 0° C, sound travels through dry air at about 331 meters per second. The speed of sound increases with temperature and humidity. The speed of sound in air is related to many important thermodynamic properties of air. Since the speed of sound in air measures how fast a wave of pressure will move through air, anything that depends on air pressure will be expected to behave differently near the speed of sound. This characteristic caused designers of high-speed aircraft many problems. Before the design problems were overcome, several test pilots lost their lives while trying to "break the sound barrier."

The speed of sound changes with temperature. The speed increases by 0.6 meters per second (m/s) for each Celsius degree rise in temperature (T). This information can be used to construct a formula for the speed (v) of sound at any temperature:

$$v = (331 + 0.60T)\text{m/s}$$

At average room temperature of 20° C, the speed of sound is close to 343 meters per second.

Frequency and Pitch. Sounds can have different frequencies. The frequency of the sound is the number of times the object vibrates per second. Frequency is measured in vibrations per second, or hertz (Hz). One Hz is one vibration per second. We perceive different frequencies of sound as different pitches; as a consequence, the higher the frequency, the higher the pitch. The normal human ear can detect sounds with frequencies between 20 Hz and 20,000 Hz. As humans age, the upper limit usually drops. Dogs can hear much higher frequencies, up to 50,000 Hz. Bats can detect frequencies up to 100,000 Hz.

logarithmic scale a scale in which the distances that numbers are positioned, from a reference point, are proportional to their logarithms

Intensity and Perception of Sound

Sound waves, like all waves, transport energy from one place to another. The rate at which energy is delivered is called power and is measured in watts. Sound intensity is measured in watts per square meter (W/m^2).

The human ear can detect sound intensity levels as low as 10^{-12} W/m^2 and as high as 1.0 W/m^2. This is an incredible range. Because of the wide range of sound intensity that humans can hear, humans perceive loudness instead of intensity. A sound with ten times the intensity in watts per square meter is perceived as only about twice as loud.

Since humans do not directly perceive sound intensity, a **logarithmic scale** for loudness was developed. The unit of loudness is the decibel, named after Alexander Graham Bell. The threshold of hearing—0 dB—represents a sound intensity of 10^{-12} W/m^2. Each tenfold increase in intensity corresponds to 10 dB on the loudness scale. Thus 10 dB is ten times the sound intensity of 0 dB. A sound of 20 dB is ten times the intensity of a 10 dB sound and one hundred times the intensity of a 0 dB sound. The list below shows the loudness of some common sounds.

The sound at a heavy metal concert is about a billion times the intensity of a whisper, but is perceived as only about 800 times as loud because humans perceive sound on a logarithmic scale.

Source	*Loudness (dB)*
Jet engine	140
Threshold of pain	120
Rock concert	115

Source	*Loudness (dB)*
Subway train	100
Factory	90
Busy street	70
Normal speech	60
Library	40
Whisper	20
Quiet breathing	10
Threshold of hearing	0

Even short exposure to sounds above 120 dB will cause permanent damage to hearing. Longer exposure to sounds just below 120 dB will also cause permanent damage. SEE ALSO Logarithms; Powers and Exponents.

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Space, Commercialization of

Since the space program began in the early 1960s, there has been talk of the possible commercial exploitation of space. One of the earliest satellites launched into space was the giant metallic Mylar balloon called "Echo." Radio signals were reflected from the satellite in an early test of satellite communications. By the early part of the twenty-first century, the space around Earth was filled with commercial communications satellites. Television, radio, telephone, data transfer, and many other forms of information pass through these satellite systems. Excluding the use of two cans and a piece of string, it would be hard to find any form of information transfer that does not involve satellites.

In the early days of space exploration, many people thought that the apparent weightlessness in orbiting satellites would allow for types of manufacturing that would not be possible in a normal gravity environment. However, improved processing techniques in Earth-based facilities have proven to be more economical than space-based facilities. Manufacturing remains an underdeveloped form of space commercialization.

Nonetheless, commercial exploitation of space remains a real possibility. In recent years, several small companies have been formed. Former astronauts or former National Aeronautics and Space Administration (NASA) scientists or engineers lead many of these small companies. These people believe that the commercial exploitation of space has economic potential. Whatever their goals as a company, they are all convinced that a profit can be made in space.

Space Stations

In 1975, participants in an Ames Research Center (a part of NASA) summer workshop created the design for what they considered to be a commercially viable space station. The space station was to be a toroidal structure (like a wheel) 2 kilometers (km) in diameter, and it was supposed to be built in orbit at the same distance as the Moon.

The group selected either the L4 or L5 lunar **Lagrange points** as the planned station's orbital position. These are points in space 60° ahead of or behind the Moon in orbit. These points were chosen because they are gravitationally stable and objects placed at one of these two points tend to stay in position. The station would be large enough to hold 10,000 colonists, who would live and work in a habitat that formed the rim of the wheel. At one revolution per minute, **centrifugal** acceleration at the rim would equal 9.8 m/s^2, which would simulate Earth's gravity. Engineers thought that the colonists could breathe oxygen extracted from moon rocks. The habitat tube would include 63 hectares of farmland, enough to grow food to feed the colonists. The team calculated that the colony would cost $200 billion, which would be equivalent to $500 billion in 2001. The engineers predicted this enormous cost could be recovered within 30 years in savings achieved by having the colonists assemble solar-powered satellites in orbit.

Lagrange points two positions in which the motion of a body of negligible mass is stable under the gravitational influence of two much larger bodies (where one larger body is moving about the other)

centrifugal the outwardly directed force a spinning object exerts on its restraint; also the perceived force felt by persons in a rotating frame of reference

microgravity the apparent weightless conditions of objects in free fall

payloads the passengers, crew, instruments, or equipment carried by an aircraft, spacecraft, or rocket

This was an ambitious, even grandiose plan. Current space station plans are much more modest by comparison. The International Space Station holds seven crew members and generates negligible income. It will never generate enough income to cover its $60-billion construction cost. NASA is still trying to form alliances with companies interested in using the space station's **microgravity** environment for commercial purposes such as renting research modules on the space station to pharmaceutical, biotechnology, or electronics companies. Thus far, private companies have shown little interest.

In contrast to other forms of space commercialization, the satellite communications business is booming and shows no signs of slowing down. The strong demand for cellular telephone service and satellite television broadcasts keeps that part of the space industry commercially successful. In all, space-related businesses accounted for over $121 billion per year in 1998.

The International Space Station is expected to grow to a mass of over 500,000 kilograms by 2004. As of 2001, all items are shipped from Earth, with space-based operations limited to assembly. This is obviously a major constraint in mission planning. Demand for space-based construction is projected to cost $20 billion per year in 2005, rising to $500 billion per year by 2020 as human space exploration progresses. This cost is primarily due to the high cost of launching **payloads** from Earth's surface.

Successful commercial exploitation of space may still be speculative, with the one exception of satellite communications.

Low-Cost Launch Vehicles

One major obstacle to commercially successful space manufacturing is the enormous cost of launching a vehicle into Earth orbit. Launch costs using either expendable rockets or the Space Shuttle are currently between $10,000 and $20,000 per kilogram. This high cost led NASA to invest in a prototype launch vehicle called the "X-33." It was hoped that this prototype would lead to a lightweight, fully reusable space plane. Lockheed Mar-

tin was building the prototype and originally planned to follow it with a commercial vehicle called the "VentureStar." However, NASA withdrew funding for the project, leaving it 75 percent complete. Instead NASA created a new program, the Space Launch Initiative, to continue research and development of reusable launch vehicles.

Space-Based Manufacturing

While the communications and weather satellite business is profitable, the space industry may need to diversify. For many years, NASA has tried to interest private companies in space-based manufacturing, with the idea that medicines involving large protein molecules, certain types of ultra-pure semiconductor materials, and other products could be manufactured with better quality in an orbital station with near-zero gravity. However, many experts now believe that space-based manufacturing will never be profitable, even if cheaper launch vehicles become available.

In-orbit assembly of solar-powered satellites—the main purpose of the giant space colony conceived in 1975—seems to offer more hope of profitability, especially with the anticipated growth in global energy consumption. A solar collector in **geostationary orbit** would be in full sunlight all the time, except when Earth is near the equinox points in its orbit. It would also be above the atmosphere, so it would receive about 8 times as much light as a solar collector on the ground. Engineers have calculated that a dish 10 kilometers in diameter could generate about 5 billion watts of power, 5 times the output of a conventional power plant. The power could be transmitted by microwave beams from the satellites to antenna arrays on Earth that would convert the microwave energy directly into electrical energy with high efficiency. By using a large antenna array, the microwaves can be kept at a safe intensity level. The array could be mounted several meters off of the ground. The space underneath would receive about 50 percent of normal sunlight, so it could be used to grow shade-tolerant crops.

geostationary orbit an Earth orbit made by an artificial satellite that has a period equal to the Earth's period of rotation on its axis (about 24 hours)

While promising, solar power is a long way from reality. In 1997, NASA completed a study that reexamined the costs of solar-powered satellites. NASA's conclusion was that solar power could be made profitable only if launch costs drop below $400 per kilogram, a fraction of the current $10,000 to $20,000 per kilogram launch costs. This seems like an impossible cost reduction, but new technologies are always expensive before their costs rapidly decrease with the innovations brought forth by healthy competition.

Space Tourism

Until solar power or some other form of space manufacturing becomes more practical, the most likely source of income might be tourism. Former Apollo astronaut Edwin "Buzz" Aldrin, Jr., the second man to walk on the Moon, has formed a private foundation, the ShareSpace Foundation, to promote space tourism. "People have come up to me and asked, 'When do we get a chance to go?'" Aldrin says. A 1997 survey of 1,500 Americans showed that 42 percent were interested in flying on a space cruise. Space tourism was encouraged by a 1998 NASA study that concluded it could grow into a $10-billion-a-year industry in a few decades.

The First Space Tourist. Space tourism became a reality in 2001, at least for one wealthy individual. Dennis Tito, an investment banker from Cali-

fornia, originally paid an estimated $20 million to the Russian Space Agency, RKK Energia, for the privilege of being transported to the Russian Space Station Mir. However, before that event could take place, the aging Mir was allowed to burn up in the atmosphere. Tito subsequently made arrangements with the Russian Space Agency to be transported to the International Space Station (ISS) as a passenger on a Russian Soyuz flight. The other agencies operating the International Space Station originally objected strongly, but when it became apparent that Russia was going to fly Tito to the Space Station in spite of their objections, they reluctantly agreed to allow him to board. On April 30, 2001, Tito officially became the world's first space tourist when he boarded the International Space Station *Alpha*.

Tito was required to sign liability releases and to agree to pay for anything he damaged while on board. NASA argued that the presence of Tito on the space station, which was not designed to receive paying guests, would hamper scientific work. The schedule of activities on board the ISS was curtailed during Tito's visit to allow for possible disruption. In spite of these difficulties, NASA and the other agencies operating the space station anticipate further requests from paying tourists.

Mining in Space

Many space entrepreneurs are also considering the possibility of mining minerals and valuable ores from Earth's Moon and the asteroid belt. There are strong hints from radar data that there may be ice caps at the lunar poles in deep craters that never receive sunlight. This discovery has raised the possibility of a partially self-sustaining Moon colony. However, the most promising commercial possibilities for mining in space may come from near-Earth asteroids (NEAs). These are asteroids that intersect Earth's orbit. Many of these asteroids are easier to reach than the Moon. Certain kinds of asteroids are rich in iron, nickel, cobalt and platinum-group metals. A two-kilometer wide asteroid of the correct kind holds more ore than has been mined since the beginning of civilization.

It would be difficult, impractical, expensive, and dangerous to transport this ore to Earth's surface. Thus, asteroid ores would be processed in space, and the metals used to build satellites, spaceships, hotels, and solar power satellites. Surprisingly, the most valuable resource to be mined from the asteroids might turn out to be water. This water could be supplied to the space hotels and other satellites, or solar energy could be used to break down the water into hydrogen and oxygen that could then be used as rocket fuel. Since all of the materials are already in orbit, rockets built in space and launched using fuel from water would cost much less than rockets launched from Earth.

SpaceDev is a publicly-held commercial space exploration and development company that already has plans in the works for asteroid prospecting. The company has plans to send a spacecraft known as the "Near Earth Asteroid Prospector" (NEAP) to the asteroid 4660 Nereus by 2002. NEAP will cost about $50 million, but this is inexpensive compared to other spacecraft. If this mission is successful, it will be the first commercial deep-space mission. NEAP will carry several scientific instruments, and SpaceDev hopes to make a profit by selling the data collected by these instruments to other entrepreneurs, governments, and university researchers. SpaceDev also

hopes to carry other scientific instruments. The company currently offers space for one ejectable payload for $9 million, and three custom non-ejectable payloads for $10 million each. NASA classified NEAP as a Mission of Opportunity, which means that research groups can compete for NASA funds to place scientific instruments onboard the spacecraft.

The Space Enterprise Zone

If any of the companies interested in the commercial exploitation of space are going to make a profit, they must avoid having to compete with NASA. Several companies and interest groups, such as the Space Frontier Foundation, have proposed that NASA turn over all low-orbit space endeavors to private companies. This would include the Space Shuttle and the International Space Station. NASA, they feel, should concentrate on deep space exploration. Rick Tumlinson, the president of the Space Frontier Foundation, says "NASA astronauts shouldn't be driving the space trucks, they should be going to Mars!"

NASA is also studying how to transfer technology to private industry. This has been a goal of NASA from its very beginning. In 1998 Congress passed the Commercial Space Act, which requires that NASA draft a plan for the privatization of the space shuttle. The law also established a regulatory framework for licensing the next generation of reusable launch vehicles.

To encourage even more privatization of space, several space entrepreneurs have proposed that Congress create a space enterprise zone similar to the enterprise zones in many inner cities. Companies beginning new space-related businesses, such as building reusable launch vehicles, would not be taxed for several years.

In spite of the obvious problems, most people in the space industry remain optimistic. As a group, they are convinced that the commercial exploitation of space will eventually become profitable, and that commercial operations will be established in Earth orbit, on the Moon, in the asteroid belt and on other planets. For years, science fiction writers and established scientists have looked far ahead and predicted such extraordinary ideas as transforming Mars into a habitable planet. There is even a word invented to describe this process, "terraforming." The number of space entrepreneurs may remain small, but they share a vision that commercial exploits like terraforming Mars are possible and even necessary if we are to survive as a species. SEE ALSO Space Exploration; Spaceflight, History of; Spaceflight, Mathematics of.

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Space Exploration

On July 20, 1969, the people of Earth looked up and saw the Moon in a way they had never seen it before. It was the same Moon that humans had been observing in the sky since the dawn of their existence, but on that July evening, for the first time in history, two members of their own species walked on its surface. At that time it seemed that Neil Armstrong's "giant leap for mankind" would mark the beginning of a bold new era in the exploration of other worlds by humans.

In 1969, some people believed that human scientific colonies would be established on the lunar surface by the 1980s, and that a manned mission to Mars would surely be completed before the turn of the twenty-first century. By the end of 1972, a dozen humans would explore the surface of the Moon. However, 28 years later, as revelers around the world greeted the new millenium, the number of lunar visitors remained at twelve, and a manned mission to Mars seemed farther away than it did in the summer of 1969. How is it that the United States could arguably produce the greatest engineering feat in human history in less than a decade and then fail to follow through with what seemed to be the next logical steps? The answers are complex, but have mostly to do with the tremendous costs of human missions and the American public's willingness to pay for them.

The Space Race Fuels Space Exploration

Gross Domestic Product a measure in the change in the market value of goods, services, and structures produced in the economy

The Apollo program, which sent humans to the Moon, was hugely expensive. At its peak in 1965, 0.8 percent of the **Gross Domestic Product** (GDP) of the United States was spent on the program. In 2000, the budget for all of the National Aeronautics and Space Administration (NASA) was about 0.25 percent of GDP, with little public support for increased spending on space programs. The huge budgets for the Apollo program were made palatable to Americans in the 1960s because of ongoing competition between the United States and the Soviet Union. The United States had been humiliated by the early successes of the Soviet Union's space program, including the launch of Sputnik.

The Soviet launch of Sputnik, the first man-made satellite, in 1957, and the early failures of American rockets put the United States well behind its greatest rival in what is now commonly known as the "space race." In 1961, when Soviet cosmonaut Yuri Gagarin became the first human to ride a spacecraft into orbit around the Earth, the leaders of the United States felt the need to respond in a very dramatic way. In less than a year, then-President John F. Kennedy set a national goal of sending a man to the Moon and returning him safely to Earth by the end of the decade.

The American public, feeling increasingly anxious about perceived Soviet superiority in science and engineering, gave their support to gigantic

spending increases for NASA to fund the race to the Moon. Neil Armstrong's first footprint in the lunar dust on July 20, 1969, achieved the first half of Kennedy's goal. Four days later, splashdown in the Pacific Ocean of the Apollo 11 capsule carrying Armstrong, fellow Moon-walker Edwin "Buzz" Aldrin, and command pilot Michael Collins completed the achievement.

Americans were jubilant over this success but growing increasingly weary of spending such huge sums of money on the space program when there were pressing needs at home. By 1972, with the United States mired in an unpopular war in Vietnam, the public's willingness to fund any more forays, whether to the Moon or southeast Asia, had been exhausted. On December 14, 1972, astronaut Eugene Cernan stepped onto the ladder of the lunar module Challenger, leaving the twentieth century's final human footprint on the lunar surface.

When Neil Armstrong became the first person to walk on the Moon, his first words on that historic occasion were: "That's one small step for a man, one giant leap for mankind."

Robotic Exploration of Space

Some scientists contend that the same amount of scientific data could have been obtained from the Moon by robotic missions costing a tenth as much as the Apollo missions. As the twenty-first century dawns, the enormous expense and complexity of human space flight to other worlds has scientists relying almost solely on robotic space missions to explore the solar system and beyond. Many scientists believe that the robotic exploration of space will lead to more discoveries than human missions since many robotic missions may be launched for the cost of a single human mission. NASA's current estimate of the cost of a single human mission to Mars is about $55 billion, more than the cost of the entire Apollo program. In the early-twenty-first century, human space travel is limited to flights of NASA's fleet of space shuttles, which carry satellites, space telescopes, and other scientific instruments to be placed in Earth's orbit. The shuttles also ferry astronaut-scientists to and from the Earth-orbiting International Space Station.

Numerous robotic missions are either in progress or are being planned for launch through the year 2010. In October of 1997, the Cassini mission to Saturn was launched. Cassini flew by Jupiter in December of 2000 and will reach Saturn in July of 2004. Cassini is the best instrumented spacecraft ever sent to another planet. It carries the European built Huygens probe, which will be launched from Cassini to land on Saturn's giant moon, Titan. The probe carries an entire scientific laboratory, which will make measurements of the atmospheric structure and composition, wind and weather, energy flux, and surface characteristics of Titan. It will transmit its findings to the Cassini orbiter, which will relay them back to Earth. The orbiter will continue its four-year tour of Saturn and its moons, measuring the system's **magnetosphere** and its interaction with the moons, rings, and the **solar wind**; the internal structure and atmosphere of Saturn; the rings; the atmosphere and surface of Titan; and the surface and internal structure of the other moons of Saturn.

magnetosphere an asymmetric region surrounding the Earth in which charged particles are trapped, their behavior being dominated by Earth's magnetic field

solar wind a stream of particles and radiation constantly pouring out of the Sun at high velocities; partially responsible for the formation of the tails of comets

In April, 2001, NASA launched the Mars 2001 Odyssey with a scheduled arrival in October of 2001. Odyssey carries instruments to study the mineralogy and structure of the Martian surface, the abundance of hydrogen, and the levels of radiation that may pose a danger to future human explorers of Mars. NASA's Genesis mission launched on August 8, 2001. This

spacecraft was sent to study the solar wind, collecting data for two years. This information will help scientists understand more about the composition of the Sun. In July of 2002, Contour is scheduled to launch with the purpose of flying by and taking high-resolution pictures of the nucleus and **coma** of at least two comets as they make their periodic visits to the inner solar system.

coma the cloud of gas that first surrounds the nucleus of a comet as it begins to warm up

In November of 2002, the Japanese Institute of Space and Aeronautical Science is scheduled to launch the MUSES-C spacecraft, which will bring back surface samples from an asteroid. The European Space Agency will launch Rosetta in January of 2003 for an eight-year journey to rendezvous with the comet 46 P/Wirtanen. Rosetta will spend two years studying the comet's nucleus and environment and will make observations from as close as one kilometer. Six international robotic missions to Mars, one to Mercury, one to Jupiter's moons Europa and Io, and one to the comet P/Tempel 1 are planned for the period between 2003–2004. Scheduled for an early 2007 launch will be the Solar Probe, which will fly through the Sun's corona sending back information about its structure and dynamics. The Solar Probe is the last launch currently scheduled for the decade 2001–2010, although this could change as scientists propose new missions and governments grant funding for these missions.

The Andromeda galaxy (M31) and its two dwarf elliptical satellite galaxies. Its trillions of stars gathered to form a large elliptical shape.

The Future of Space Exploration

Beyond the first decade of the twenty-first century, plans for the exploration of space remain in the "dream" stage. Scientists at the world's major space facilities, universities, and other scientific laboratories are working to find breakthroughs that will make space travel faster, less expensive, and more hospitable to humans. One technology that is being met with some enthusiasm in the space science community is the "solar sail," which works somewhat like the wind sail on a sailboat except that it gets its propulsion from continuous sunlight. A spacecraft propelled by a solar sail would need no rockets, engines, or fuel, making it significantly lighter than rocket-propelled crafts. It could be steered in space in the same manner as a sailboat is steered in water, by tilting the sail at the correct angle. The sail itself would be a thin aluminized sheet. Powered by continuous sunlight, the solar sailcraft could cruise to the outermost reaches of the solar system and even into interstellar space. Estimates are that this technology will not be fully developed until sometime near the end of 2010.

In January of 2001, scientists at Israel's Ben Gurion University reported that they had shown that the rare nuclear material americium-242m (Am-242m) could sustain a **nuclear fission** reaction as a thin metallic film less than a thousandth of a millimeter thick. If true, this could enable Am-242m to act as a nuclear fuel propellant for a relatively lightweight spacecraft that could reach Mars in as little as two weeks instead of the six to eight months required for current spacecraft. However, like the solar sail, this method of propulsion is also far from the implementation stage. Producing enough of it to power spacecraft will be difficult and expensive, and no designs for the type of reactor necessary for this type of nuclear fission have even been proposed. Other scientists are looking into the possibility that antimatter could

nuclear fission a reaction in which an atomic nucleus splits into fragments

be harnessed and used in jet propulsion, but the research is still in the very early stages. It may be that one or more of these technologies will propel spacecraft around the solar system in the decades ahead, but none of them hold the promise of reaching the holy grail of interstellar travel at near light speed.

Moving at the Speed of Light. The difficulty with interstellar space travel is the almost incomprehensible vastness of space. If, at some future time, scientists and engineers could build a spacecraft that could travel at light speed, it would still require more than four years to reach Proxima Centauri, the nearest star to our solar system. Consequently, to achieve the kind of travel among the stars made popular in science fiction works, such as Star Trek or Star Wars, an entirely different level of technology than any we have yet seen is required. The Pioneer 10 and Voyager 1 spacecraft launched in the 1970s have traveled more than 6.5 billion miles and are on their way out of our solar system, but at the speed they are traveling it would take them tens of thousands of years to reach Proxima Centauri.

Incremental increases in speed, while helpful within the solar system, will not enable interstellar travel. That will require a major breakthrough in science and technology. Einstein's **relativity** theory excludes bodies traveling at or beyond the speed of light. As a result, scientists are looking for ways to circumvent this barrier by distorting the fabric of space-time itself to create wormholes, which are shortcuts in space-time, or by using warp drives, which are moving segments of space-time. These would require the expenditure of enormous amounts of energy. To study the feasibility of such seemingly far-fetched proposals, NASA, in 1996, established the Breakthrough Propulsion Physics Program. It may seem to the people of Earth, at the beginning of the twenty-first century, that these dreams of creating reality from science fiction are next to impossible—but we should remember that sending humans to the Moon was nothing more than science fiction at the turn of the twentieth century. SEE ALSO Light Speed; Space, Growing Old in; Spaceflight, History of; Spaceflight, Mathematics of.

Stephen Robinson

relativity the assertion that measurements of certain physical quantities such as mass, length, and time depend on the relative motion of the object and observer

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Space, Growing Old in

Radiation and vacuum, large variations in temperature, and lack of oxygen make it impossible for humans to survive unprotected in space. Therefore, mini-ecosystems are needed to sustain life. But would there be any benefit to living and growing old in mini-ecosystems in space? Would the elderly live longer or suffer less from ailments such as arthritis?

On Earth, the direction of gravity (G) is perpendicular to the surface, and the intensity is 1 **g**. The intensity of gravity on the Earth's Moon is 0.17 g and on Mars it is 0.3 g. Even 200 miles away from Earth, the forward motion of a spacecraft counterbalances Earth's gravity, resulting in continuous free-fall around the planet with a resultant acceleration of about 10^{-6} g or 1 micro *g*. This is not weightlessness but **microgravity**.

g a common measure of acceleration; for example 1 g is the acceleration due to gravity at the Earth's surface, roughly 32 feet per second per second

microgravity the apparent weightless condition of objects in free fall

Astronauts orbit the Earth in about 90-minute cycles and go through 16 day/night cycles every 24 hours. With the clock seemingly ticking faster and the reduced influence of *g*, what are the possibilities of growing old in space? Data collected so far indicate that the clock of living organisms ignores the 90-minute cycle and "freeruns" at slightly more than 24 hours, relying on the internal genetic clock. But microgravity also has more obvious consequences.

All life on Earth has evolved in 1 *g* and, therefore, has developed systems to sense and use *g*. Only by going into space can one fully understand how *g* has affected life on Earth. Biological organisms respond to changes in *g* direction and intensity. Plants grow and align with the direction of *g*. Humans change the direction *g* enters their body by changing position with respect to the surface of the Earth. For example, when standing up, *g* intensity (Gz) is greatest; when lying down, *g* pulling across the chest (Gx) is least intense.

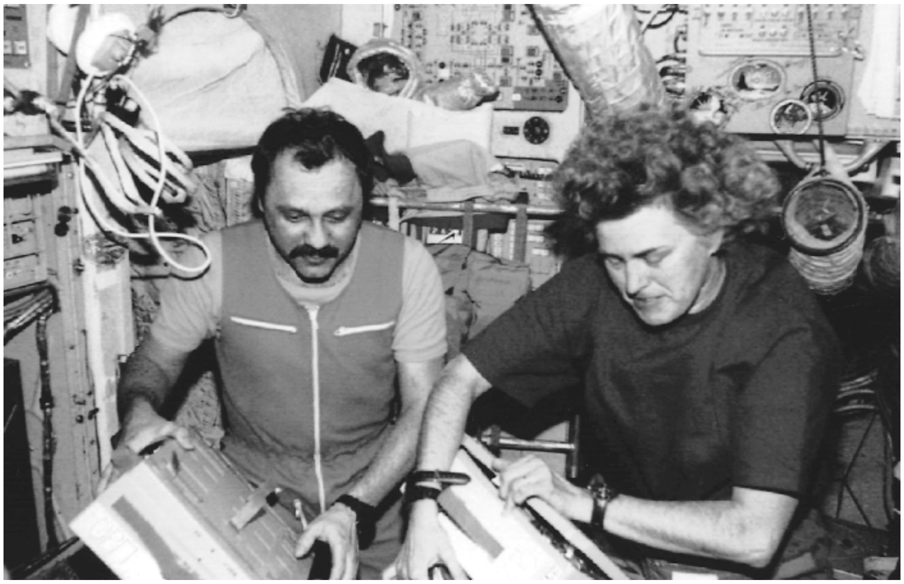
The Effects of Microgravity on the Human Body

Microgravity is not a threat to life. Astronauts endure physical and psychological stresses for short periods of time and survive to complete a mission. The longer they live in space, however, the more difficult they find it to recover and re-adapt to Earth.

After less than five months on the Mir space station, David Wolf lost 23 lbs, 40 percent of his muscle mass, and 12 percent of his bone mass. It took one year to recover this bone mass. In space, adult humans lose 2 percent bone mass per month, compared to about 1 percent per year on Earth.

When astronauts return to Earth, long rehabilitation is needed: the weight of the body cannot be supported because of loss of calcium from bone; there is degradation of ligaments and cartilage in joints; and astronauts experience decreased lower limb muscle mass and strength. It is even hard to sit up without fainting because in microgravity blood volume is reduced, the heart grows smaller, and unused blood vessels in the legs are now no longer able to resist gravitational pull by pumping blood up towards the head.

Astronauts find that even standing and walking can be a chore. They may need to first walk with their feet apart for balance. Because the vestibular system in the inner ear that senses *g* and acceleration received no such input in space, leg movement coordination and proper balance are disturbed.



Shannon Lucid, shown here with Yuri Usachev, managed to walk off the space shuttle Atlantis, after 188 days in space. Lucid logged almost 400 hours on the stationary bicycle and treadmill aboard the space station Mir.

Exercise alone may not prevent adaptation to microgravity nor help maintain Earth-health-status to reduce this long rehabilitation.

One way to help astronauts prepare for the increased gravitational pull back on Earth may be to provide "artificial gravity" while on the spacecraft, using an on-board centrifuge for short exposures to *g*. Because the effects of microgravity are similar to the effects of prolonged exposure to Gx (the gravitational pull across the chest while lying in bed), tests on Earth have used healthy sleeping volunteers to mimic the effects of spaceflight microgravity, and the results have been used to research the best treatments.

On Earth, the symptoms astronauts suffer during spaceflight are associated with growing old. In astronauts the symptoms are fully reversible after their return to Earth. Their age has not changed, nor does it affect their life span back on Earth. In typical, earth-bound aging, however, these same symptoms are believed to be inevitable and irreversible. Because the symptoms are also seen after prolonged bed rest, it is believed that even healthy inactive people develop aging symptoms much earlier than had they been more active and used *g* to greater advantage.

If a human were to live in space forever, the body would adapt completely to microgravity. Some functions needed on Earth but not necessary in space would degrade or even disappear. Changes would progress until the body reached a new steady state, what would then be "normal" for space. At that point, the body may never be able to re-adapt to Earth.

On the other hand, consider someone on Earth with the same symptoms, pinned to a wheelchair by *g*—perhaps because of paralysis after a spinal cord injury. Such a person may experience tremendous freedom in being able to move about in microgravity using only the upper body as do the astronauts.

Would We Live Longer in Space?

These appropriate changes to the space environment tell us nothing about whether humans would live less or longer in space. To find out, scientists

are using specimens with much shorter life spans. Experiments with fruit flies (*Drosophila melanogaster*) exposed for short periods of their adult life to space and then studied back on Earth have shown no change in their life span.

Many genes related to aging in organisms like fruit flies and the roundworm (*Caenorhabditis elegans*) are similar to the aging genes of humans. Because fruit flies and roundworms live 4-8 weeks, it is easier to study them in space both from birth and over several generations.

Markers of biological aging like telomeres studied extensively in aging research on Earth will also be studied in space. Telomeres are fragments of non-coding deoxyribonucleic acid (DNA; that is, DNA that does not give rise to proteins) found on the ends of each chromosome. When a cell divides, telomeres shorten until they eventually become so short that the cell can no longer divide and dies. The role of g and microgravity on such mechanisms may help unravel the mysteries of growing old both in space and on Earth.

So what is growing old? On Earth we use it to describe the debilitating changes that occur as one's age increases, eventually leading to death. Clearly, this does not apply to what happens in space, suggesting that perhaps growing old on Earth should be redefined. Space has taught scientists a great deal about growing old on Earth. The degenerative changes seen in astronauts and in humans aging on Earth indicate how important it is to be active and to use *g* to its fullest on Earth. In so doing, humans may live healthier, if not longer, lives. SEE ALSO Space Exploration; Spaceflight, History of.

Joan Vernikos

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Spaceflight, History of

In the early-thirteenth century, the Chinese invented the rocket by packing gunpowder into a tube and setting fire to the powder. Half a millenium later, the British Army colonel, William Congreve, developed a rocket that could carry a 20-pound warhead nearly three miles. In 1926, the American physicist Robert Goddard built and flew the first liquid-fueled rocket, becoming the father of rocket science in the United States. In Russia, a mathematics teacher named Konstantin Tsiolkovsky derived and published the mathematical theory and equations governing rocket propulsion.

Tsiolkovsky's work was largely ignored until the Germans began to employ it to build rocket-based weapons in the 1920s under the leadership of

mathematician and physicist Hermann Oberth. In October of 1942, the Germans launched the first rocket to penetrate the lower reaches of space. It reached a speed of 3,500 miles per hour, giving it a range of about 190 miles. The technology and design of the German rocket were essentially the same as those pioneered by Goddard. The successors of this first rocket would be the infamous V-2 ballistic missiles, which the Nazis launched against England in 1944 and 1945. This assault came too late to turn the course of the war in Germany's favor, but devastating enough to get the full attention of the U.S. government.

Many of the German V-2 rocket scientists, led by Werhner von Braun, surrendered to the Americans and were brought to the United States, where they became the core of the American rocket science program in the late-1940s, 1950s, and 1960s. Von Braun and his team took the basic technology of the V-2 missiles and scaled it up to build larger and larger rockets fueled by liquid hydrogen and liquid oxygen.

The Weapons Race Leads to a Space Race

In the beginning, the focus of the American rocket program was on national defense. After World War II, the United States and the Soviet Union emerged as the world's two most formidable powers. Each country was determined to keep its arsenal of weapons at least one step ahead of the other's. Chief among those arsenals would be intercontinental ballistic missiles, huge rockets that could carry nuclear warheads to the cities and defense installations of the enemy. To reach a target on the other side of the world, these rockets had to make suborbital flights, soaring briefly into space and back down again. In the 1950s, as rocket engines became more and more powerful, both nations realized that it would soon be possible to send objects into orbit around Earth and eventually to the Moon and other planets. Thus, the Cold War missile race gave birth to the space race and the possibility of eventual space flight by humans.

Werhner von Braun and his team of rocket scientists were sent to the Redstone Army Arsenal near Huntsville, Alabama and given the task of designing a super V-2 type rocket, which would be called the Redstone, named for its home base. The first Redstone rocket was launched from Cape Canaveral, Florida on August 20, 1953. Three years later, on September 20, 1956, the Jupiter C Missile RS-27, a modified Redstone, became the first missile to achieve deep space penetration, soaring to an altitude of 680 miles above Earth's surface and traveling more than 3,300 miles.

Meanwhile, in the Soviet Union, rocket scientists and engineers, led by Sergei Korolyev, were working to develop and build their own intercontinental ballistic missile system. In August of 1957, the Soviets launched the first successful Intercontinental Ballistic Missile (ICBM), called the R-7. However, the shot heard round the world would be fired two months later, on October 4, 1957, when the Soviets launched a modified R-7 rocket carrying the world's first man-made satellite, called *Sputnik*. *Sputnik* was a small, 23-inch aluminum sphere carrying two radio transmitters, but its impact on the world, and especially on the United States, was enormous. The "space race" had begun in earnest—there would be no turning back.

One month later, the Soviets launched *Sputnik 2*, which carried Earth's first space traveler, a dog named Laika. The United States government and

its people were stunned by the Soviet successes and the first attempt by the United States to launch a satellite was pushed ahead of schedule to December 6, 1957, just a month after Laika's journey into space. The Vanguard launch rocket, which was supposed to propel the satellite into orbit, exploded shortly after lift-off. It was one of a series of U.S. failures, and dealt a serious blow to Americans' confidence in the country's science and engineering programs, which were previously thought to be the best in the world. Success finally occurred on January 31, 1958 with the launch of the first U.S. satellite, *Explorer 1*, which rode into space atop another Jupiter C rocket.

The Race to the Moon

Both the Soviet Union and the United States wanted to be the first to send a satellite to the Moon. In 1958, both countries attempted several launches targeting the Moon, but none of the spacecraft managed to reach the 25,000 miles per hour speed necessary to break free of Earth's gravity.

On January 2, 1959, the Soviet spacecraft *Luna 1* became the first artificial object to escape Earth's orbit, although it did not reach the Moon as planned. However, eight months later, on September 14, 1959, *Luna 2* became the first man-made object to strike the lunar surface, giving the Soviets yet another first in the space race. A month later, *Luna 3* flew around the Moon and radioed back the first pictures of the far side of the Moon, which is not visible from Earth.

cosmonaut the term used by the Soviet Union and now used by the Russian Federation to refer to persons trained to go into space; synonymous with astronaut

The United States did not resume its attempts to send an object to the Moon until 1962, but by then, the space race had taken on a decidedly human face. On April 12, 1961, an R-7 rocket boosted a spacecraft named *Vostok* that carried 27-year-old Soviet **cosmonaut** Yuri Gagarin into Earth orbit and into history as the first human in space. Gagarin's 108-minute flight and safe return to Earth placed the Soviet Union clearly in the lead in the space race. Less than a month later, on May 5, 1961, a Redstone booster rocket sent the U.S. Mercury space capsule, *Freedom* 7, which carried American astronaut Alan Shepard, on a 15-minute suborbital flight. The United States was in space, but just barely. An American would not orbit Earth until February of 1962, when astronaut John Glenn's *Mercury* capsule, *Friendship* 7, would be lifted into orbit by the first of a new generation of American rockets known as Atlas.

Just weeks after Shepard's suborbital flight in 1961, U.S. President John F. Kennedy announced that a major goal for the United States was to send a man to the Moon and return him safely to Earth before the end of the decade. The fulfillment of that goal was to be the result of one of the greatest scientific and engineering efforts in the history of humanity. The project cost a staggering $25 billion, but Congress and the American people had become alarmed by the Soviets' early lead in the space race and were more than ready to fund Kennedy's bold vision.

The Mercury and Vostok programs became the first steps in the race to the Moon, showing that humans could survive in space and be safely brought back to Earth, but the *Mercury* and *Vostok* spacecraft were not designed to take humans to the Moon. By 1964, the Soviet Union was ready to take the next step with a spacecraft named *Voskhod*. In October of 1964,

Voskhod 1 carried three cosmonauts, the first multi-person space crew, into orbit. They stayed in orbit for a day and then returned safely to Earth. In March of 1965, the Soviets launched *Voskhod 2* with another three-man crew. One of the three, Alexei Leonov, became the first human to "walk" in space.

Within weeks of Leonov's walk, the United States launched the first manned flight of its second-generation *Gemini* spacecraft. The *Gemini* was built to carry two humans in considerably more comfort than the tiny Mercury capsule. Although *Gemini* was not the craft that would ultimately take men to the Moon, it was the craft that would allow American astronauts to practice many of the maneuvers they would need to use on lunar missions. Ten Gemini missions were flown in 1965 and 1966. On these missions, astronauts would walk in space, steer their spacecraft into a different orbit, rendezvous with another Gemini craft, dock with an unmanned spacecraft, reach a record altitude of 850 miles above Earth's surface, and set a new endurance record of 14 days in space. When the Gemini program came to an end in November of 1966, the United States had taken the lead in the race to the Moon. The Soviets had abandoned their Voskhod program after just two missions to concentrate their efforts on reaching the Moon before the United States. Both countries developed a third-generation spacecraft designed to fly to the Moon and back. The Soviet version was called *Soyuz*, while the American version was named Apollo. Both could accommodate a three-person crew.

In November of 1967, Werhner von Braun's giant Saturn V rocket was ready for its first test flight. The three-stage behemoth stood 363 feet high, including the Apollo command module perched on top. Its first stage engines delivered 7.5 million pounds of thrust, making it the most powerful rocket ever to fly. On its first test flight, the mighty Saturn launched an unmanned *Apollo* spacecraft to an altitude of 11,000 miles above Earth's surface. On December 21, 1968, the Saturn V boosted astronauts Frank Borman, Jim Lovell, and Bill Anders inside their *Apollo 8* spacecraft into Earth orbit. After two hours in orbit, the Saturn's third stage engines fired one more time, increasing *Apollo 8's* velocity to 25,000 miles per hour. For the first time in history, humans had escaped the pull of Earth's gravity and were on their way to the Moon.

On December 24, *Apollo 8* entered lunar orbit, where it would stay for the next twenty hours mapping the lunar surface and sending back television pictures to Earth. *Apollo 8* was not a landing mission, so on December 25, the astronauts fired their booster rockets and headed back to Earth, splashing down in the Pacific Ocean two days later. The last year of the decade was about to begin and the stage was set to fulfill President Kennedy's goal. Two more preparatory missions, *Apollo 9* and *Apollo 10*, would be used to do a full testing of the Apollo command module and the lunar module, which would take the astronauts to the surface of the Moon.

The astronauts chosen to ride *Apollo 11* to the Moon were Neil Armstrong, Edwin "Buzz" Aldrin, and Michael Collins. Liftoff occurred on July 16, 1969, with *Apollo 11* reaching lunar orbit on July 20. Armstrong and Aldrin squeezed into the small spider-like lunar module named *Eagle*, closed the hatch, separated from the command module, and began their descent

to the lunar surface. Back on Earth, the world watched in anticipation as Neil Armstrong guided his fragile craft toward the gray dust of the Moon's Sea of Tranquility. Armstrong's words came back across 239,000 miles of space: "Houston. Tranquility Base here. The *Eagle* has landed." At 10:56 P.M., eastern standard time, July 20, 1969, Armstrong became the first human to step on the surface of another world. "That's one small step for a man," he proclaimed, "and one giant leap for mankind." Aldrin followed Armstrong onto the lunar surface, where they planted an American flag and then began collecting soil and rock samples to bring back to Earth. The *Apollo 11* astronauts returned safely to Earth on July 24, fulfilling President Kennedy's goal with five months to spare.

Five more teams of astronauts would walk and carry out experiments on the Moon through 1972. A sixth team, the crew of *Apollo 13*, was forced to abort their mission when an oxygen tank exploded inside the service module of the spacecraft. On December 14, 1972, after a stunning three-day exploration of the lunar region known as Taurus-Littrow, astronauts Eugene Cernan and Harrison Schmitt fired the ascent rockets of the Lunar module Challenger to rejoin crewmate Ron Evans in the *Apollo 17* command module for the journey back to Earth.

Apollo 17 was the final manned mission to the Moon of the twentieth century. The American people and their representatives in Congress had exhausted their patience with paying the huge sums necessary to send humans into space. The Soviets never did send humans to the Moon. After the initial *Soyuz* flights, their Moon program was plagued by repeated failures in technology, and once the United States had landed men on the Moon, the Soviet government called off any additional efforts to achieve a lunar landing.

Although the Apollo program did not lead to an immediate establishment of scientific bases on the Moon or human missions to Mars as was once envisioned, human spaceflight did not end. The three decades following the final journey of *Apollo 17* have seen the development of the space shuttle program in the United States, as well as active space programs in Russia, Europe, and Japan. Scientific work is currently under way aboard the International Space Station, with space shuttle flights ferrying astronauts, scientists, and materials to and from the station. Deep spaceflights by humans to the outer planets and the stars await significant breakthroughs in rocket propulsion. SEE ALSO Astronaut; Space, Commercialization of; Space Exploration; Space, Growing Old in; Spaceflight, Mathematics of.

Stephen Robinson

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Spaceflight, Mathematics of

The mathematics of spaceflight involves the combination of knowledge from two different areas of physics and mathematics: rocket science and celestial mechanics. Rocket science is the study of how to design and build rockets and how to efficiently launch a vehicle into space. Celestial mechanics is the study of the mathematics of orbits and trajectories in space. Celestial mechanics uses the laws of motion and the law of gravity as first stated by Isaac Newton.

A Short History of Rockets

People have wondered about the possibility of space flight since ancient times. Centuries ago the Chinese used rockets for ceremonial and military purposes. In the latter half of the twentieth century, we have been able to develop rockets powerful enough to launch vehicles into space with sufficient energy to achieve or exceed **orbital velocities**.

orbital velocities the speed and direction necessary for a body to circle a celestial body, such as the Earth, in a stable manner

The rockets mentioned in the American national anthem, the "Star Spangled Banner," were British rockets fired at the American fortress in Baltimore. However, these so-called "Congreves," named for their inventor, Sir William Congreve, were not the first war rockets. Some six centuries before, the Chinese connected a tube of primitive gunpowder to an arrow and created an effective rocket weapon to defend against the Mongols.

The Physics Behind Rockets and Their Flight

The early Chinese "fire arrows," the British Congreves, and even our familiar Fourth of July skyrockets, are propelled by exactly the same principles of physics that propelled the Delta, Atlas, Titan, and Saturn rockets, as well as the modern space shuttle. In each case, some kind of fuel is burned with an **oxidant** in the **combustion** chamber, creating hot gases accompanied by high pressure. In the combustion chamber, these gases try to expand, pushing equally in all directions. At the bottom of the combustion chamber, there is an exhaust nozzle that flares out like a bell. The expanding gases push on all sides of the combustion chamber and the nozzle, except at the bottom. This results in a **net force** on the rocket, accelerating it in a direction opposite to the nozzle.

oxidant a chemical reagent that combines with oxygen

combustion chemical reaction combining fuel with oxygen accompanied by the release of light and heat

net force the final, or resultant, influence on a body that causes it to accelerate

This idea is summarized in Newton's Third Law of Motion: "To every action, there is an equal and opposite reaction." The gases "thrown" out of the bottom of the nozzle constitute the action. The reaction is the rocket's acceleration in the opposite direction. This system of propulsion does not depend on the presence of an atmosphere—it works just as well in empty space.✶

✶One of the early astronauts answered a question from mission control asking who was flying with the words, "Isaac Newton is flying this thing."

According to Newton's First Law of Motion, "A body at rest will remain at rest unless acted on by an unbalanced force." The unbalanced force is provided by the rocket motor, so the rocket motor and attached spacecraft are accelerated. Because Newton's laws were postulated in the seventeenth century, some people have argued, "All the basic scientific problems of spaceflight were solved 300 years ago. Everything since has been engineering." However, many difficult problems of mathematics, chemistry, and engineering had to be solved before humans could apply Newton's basic principles to achieve controlled spaceflight.

Using Rockets in Spaceflight

Robert Goddard was an early American rocket pioneer, who initially encountered resistance to his ideas. Some people, not understanding Newton's laws very well, insisted rockets would not operate in the vacuum of space because there would be no air for them to push against. They were thinking in terms of a propeller-driven airplane. But a rocket needs no air to push against; action-reaction forces propel it. Nor does a rocket need air to burn the fuel. An **oxidizer** is mixed with the fuel in the engine.

oxidizer the chemical that combines with oxygen or is made into an oxide

In 1926 in Massachusetts, Goddard successfully launched the world's first liquid-fuel rocket. By modern standards it was very rudimentary: It flew 184 feet in two and a half seconds. But the mighty Atlas, Titan, and Saturn liquid-fueled rockets are all direct descendents of Goddard's first rocket—even the Space Shuttle uses liquid fuel rockets as its main engine.

Now rockets are used to launch vehicles into space, but in World War II the Germans developed them for use as long-range weapons.

At the same time that Goddard was doing his pioneering work on rockets in the United States, two other pioneers were beginning their work in other countries. In the early Soviet Union, Konstantin Tsiolkovski was conducting experiments very similar to Goddard's. In Germany, Hermann Oberth was studying the same ideas. Military leaders in Nazi Germany recognized the possibilities of using long-range rockets as weapons. This led to the German V-2 missiles that could be launched from Germany to a height of 100 kilometers across the English Channel. These missiles could reach speeds greater than 5,600 kilometers per hour.

After World War II, the United States and the Soviet Union used captured German V-2 rockets, as well as the knowledge of German rocket scientists, to start their own missile programs. In October 1957, the Soviets propelled a satellite named Sputnik into space. Four years later, the first human orbited the Earth.

The first U.S. satellite, Explorer I, was launched into orbit in January of 1958. On May 5, 1961, Alan Shepard became the first American to fly into space with a fifteen-minute flight of the Freedom 7 capsule. Four years later, on February 20, 1962, John Glenn became the first American to orbit Earth in the historic flight of the *Friendship* 7 capsule.

In 1961, President John F. Kennedy had set a national goal of sending and landing a man on the Moon and then returning him safely to Earth within the decade. In July 1969, Astronaut Neil Armstrong took "a giant step for mankind" as he stepped onto the surface of the Moon. It was an event few Americans living at the time will ever forget.✶

✶In total, six Apollo expeditions landed on the Moon and returned to Earth between 1969 and 1972.

Navigating in Space

Getting to the Moon, Mars, or Venus is not simply a matter of aiming a rocket in the desired direction and hitting a launch button. The process of launching a spacecraft on a **trajectory** that will take it to another planet has been compared to trying to shoot at a flying duck from a moving speedboat. Launching a spacecraft into orbit is relatively simpler. The spacecraft is launched toward the east so that it can take advantage of Earth's rotation, and gain some extra speed. For the same reason, satellites are usually launched from sites as close to the equator as possible.

trajectory the path followed by a projectile; in chaotic systems, the trajectory is ordered and unpredictable

On the Apollo missions to the Moon, the spacecraft was boosted into an Earth orbit where it coasted until it was in the proper position to start

the second leg of the trip. At the appointed time, the Apollo spacecraft was boosted out of its parking orbit and into a trajectory or path to the Moon. This meant firing the spacecraft's rocket engines when it was at a point in orbit on the opposite side of Earth from the Moon. Orbital paths are **ellipses**, **parabolas**, or some other **conic** section.

ellipse one of the conic sections, it is defined as the locus of all points such that the sum of the distances from two points called the foci is constant

parabola a conic section; the locus of all points such that the distance from a fixed point called the focus is equal to the perpendicular distance from a line

conic of or relating to a cone, that surface generated by a straight line, passing through a fixed point, and moving along the intersection with a fixed curve

In the case of a spacecraft launched from Earth into orbit, the spacecraft follows an **elliptical orbit** with the center of Earth at one focus. The point in the orbit closest to Earth is known as the "perigee" and the point on the elliptical orbit farthest from Earth is called the "apogee." The Apollo's engines were fired when it was at perigee.

For Apollo to reach the Moon, the point of maximum altitude above Earth, the apogee, had to intersect the orbit of the Moon and the arrival had to be timed so that the Moon would arrive at the same position when the spacecraft appeared.

When the Moon's gravitational pull on the spacecraft was greater than Earth's pull, it made sense to shift the points of reference. Thus, the position of the Apollo spacecraft began to be considered in terms of its where it was relative to the Moon. At that point, the spacecraft was more than half way to the Moon. Its orbit was now calculated in relation to the Moon's orbit. As the spacecraft approached the Moon, it swung around the Moon in an elliptical orbit, with the Moon at one focus. Apollo then fired its engines again to slow down and enter orbit.

Orbital Velocity

The velocities needed for specific space missions were calculated long before space flight was possible. To put an object into orbit around the Earth, for instance, a velocity of at least 7.9 kilometers/second (km/s) (about 18,000 miles per hour), depending on the precise orbit desired, must be achieved. This is called "orbital velocity."

In effect, an object in Earth orbit is falling around the Earth. A satellite or missile with a speed less than orbital velocity moves in an elliptical orbit with the center of Earth at one focus. However, the orbit intersects Earth's surface. With a slightly increased speed, the satellite still falls back toward Earth. In simply moves sideways fast enough that it does not actually hit the Earth. It keeps missing.

To leave Earth orbit for distant space missions, a velocity greater than 11.2 km/s (25,000 miles per hour) is required. This is called "escape velocity." In Earth orbit, the downward force of Earth's gravity maintains the satellite's acceleration towards Earth, but Earth keeps curving away. At still higher speeds, the orbit becomes more elliptical and the apogee point is at a greater distance. If the launch speed reaches escape velocity, the apogee is infinitely far away and the ellipse has become a parabola in Earth's frame of reference. At even greater speed, the orbit is a **hyperbola** in Earth's frame of reference. SEE ALSO Solar System Geometry, Modern Understandings of; Space, Commercialization of; Space Exploration; Space, Growing Old in; Spaceflight, History of.

Elliot Richmond

WHAT EXACTLY IS "FREE-FALL"?

A satellite in orbit around Earth is falling towards Earth; it is accelerating downward. Since there is nothing to stop its fall, it is said to be in "free-fall." Astronauts on board the satellite are also falling downward, and this produces a sensation of weightlessness.

However, it would be incorrect to say there is no gravity or that the astronauts are weightless, since the satellite is only a few hundred kilometers above Earth's surface. NASA likes to use the term "microgravity," which refers to the fact that the objects in the spacecraft actually attract each other. Astronauts feel weightless not because they have "escaped gravity" but because they are freely falling toward the surface of Earth.

elliptical orbit a planet, comet, or satellite follows a curved path known as an ellipse when it is in the gravitational field of the Sun or another object; the Sun or other object is at one focus of the ellipse

hyperbola a conic section; the locus of all points such that the absolute value of the difference in distance from two points called foci is a constant

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Sports Data

Who was the greatest home run hitter: Babe Ruth or Roger Maris? How does Mark McGwire compare? What professional football team won the most games in one season? Sports fans argue such questions and use data collected about players and teams to try to answer them.

Every sport from baseball to golf, from tennis to rodeo competitions keep statistics about individual and team performances. These statistics are used by coaches to help plan strategies, by team owners to match salaries with performance, and by fans who simply enjoy knowing about their favorite sports and players. Most statistics are either about players or teams. There are many different statistical measures used, but the most common ones are maxima and minima, rankings, averages, ratios, and percentages.

Maximum or Minimum

Perhaps the simplest of all statistics answers the question, "Who won the most competitions?" or "Who made the fewest errors?" In all professional sports, data are kept on the number of games won and lost by teams and the number of errors made by players. A listing of the number of wins quickly shows which team won the most games. Mathematically, one is seeking the largest number (or the smallest number) in a list. These numbers are called the maximum (or the minimum) values of the list.

Rankings

Another common way to compare players or teams is to rank them. The number of games won each season often ranks teams. Some sports rank individual players as well. Tennis, for example, ranks players (called their seed) based on the number of tournaments they have completed, the strength of their opponents, and their performance in each. Wins against higher-ranked opponents affect a player's seed more than wins against lower-ranked opponents.

Average

Averaging is one of the most well known ways to create a statistic from data, and one of the most well known of all sports statistics is a baseball player's batting average. To average a series of numbers, add them up and divide by the number of numbers; a batting average is no different. Each time a player comes to the plate, he either gets 1 hit or he does not hit. To compute the batting average, add up all the numbers (that is, the total number of hits in a season) and divide by the total number of times he was at bat.

Averages are common in other sports as well. In football, for example, one can compute a player's average punt returns (total number of yards / number of attempts), his average yards rushing (total number of yards /

In addition to mathematics being involved in technical aspects of snowboarding, it is also relevant to the score a participant receives in competition.

number of attempts), and his average yards receiving (total number of yards / number of times received).

In basketball, players are compared by their scoring average (total number of points scored divided by number of games played), their rebound average (total number of rebounds / number of games played), and their stealing average (total number of balls stolen / number of games played). In other sports, the scores are based on averages. In snowboarding, for example, each competitor is given a score from five different judges who look at different characteristics of a run. The average of these numbers determines a snowboarder's score on a scale of 1.0–10.0.

Averages sometimes need to be adjusted. One may want to compare the performance of two pitchers, for example, who played a different number of innings. To compute the pitcher's earned run average, divide the total number of earned runs by the total number of innings pitched and then multiply by 9. This gives a number that represents the number of runs that would have been earned if the player had completed nine innings of play.

Ratios and Percentages

Often interesting information cannot be obtained from a simple average. A baseball coach, for example, might want his next batter to get a walk and he must decide which of his pinch hitters is most likely to get a walk. In this case the most useful statistic is a percentage called the base on balls percentage. It is computed by dividing the number of times a player is at bat by the number of times he was walked. This decimal is then written as a percent. Similarly, the stolen base percentage is computed by dividing the number of stolen bases by the number of attempts at a stolen base.

Percentages are a commonly used statistic in other sports as well. In basketball, for example, a player's field-goal percentage is the ratio of the number of field goals attempted to the number of field goals made; a player's free-throw percentage and three-point field-goal percentage is calculated

similarly. In football, a quarterback's efficiency is measured by the percentage of passes that are completed (number of passes completed / number of passes attempted).

Weighted Average

The batting average compares the number of hits to the number of times a player was at bat. This allows a comparison of players who have been at bat a different number of times. In this average, however, both singles and home runs are counted the same—as hits. A player who hits mainly singles could have a higher batting average than a player who mostly belts home runs. In order to average numbers where some of the numbers are more important than others, a statistician will often prefer to use a weighted average.

A baseball player's slugging average is a weighted average of all of the player's hits. To compute a weighted average, each number is assigned a weight. For the slugging average, a home run is assigned a weight of 4, a triple 3, a double 2, a single 1, and an out 0 points. The slugging average is computed as follows.

Multiply 4 (number of home runs).

Multiply 3 (number of triples).

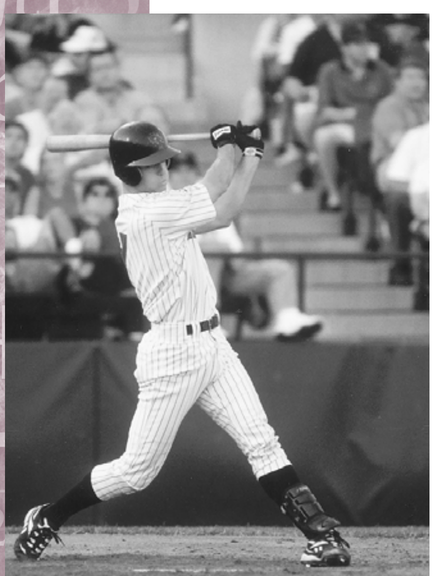
Multiply 2 (number of doubles).

Multiply 1 (number of singles).

Multiply 0 (number of outs).

Add these numbers and divide by the number of at-bats.

For example, a player who has 80 at-bats and got 20 hits would have a batting average of 20 / 80 = .250 whether those hits were home runs or singles. Suppose those 20 hits included 4 home runs, 3 triples, 8 doubles, and 5 singles. To compute the player's slugging average, first compute 4(4) + 3(3) + 8(2) + 1(5) = 46 and then divide by the number of at-bats to get SA = 46 / 80 = .575. Another player who had 15 singles and 5 doubles in 80 times at bat would have the same batting average but a slugging average of only [2(5) + 1(15)] / 80 = .3125.



A player's batting average is found by dividing the total number of hits by the total times at bat. A pitcher's earned run average is the ratio of earned runs allowed to innings pitched, scaled to the level of a game.

Threshold Statistics

In many situations there is no need for any computation. Often there is an interest in knowing whether a player has reached some mark of excellence. These are called threshold statistics. For example, pitchers who have pitched a perfect game (no one gets on base) or bowlers who have scored 300 (a perfect score) have reached a threshold of excellence. Other examples of threshold statistics are pitchers who have pitched a no-hitter, pitchers who have attained 3,000 strikeouts in their careers, and batters who have hit "for the cycle" (a single, double, triple, and home run in the same game).

Algebraic Formula

Many sports enthusiasts and coaches have come up with other, sometimes quite complex measures of individual and team performance. These are often expressed as algebraic formulas that combine other more elementary statistics. A simple example from hockey, for example, is a player's +/−

TOTAL TURNOVERS GAINED IN THE BIG TEN 2001

		Gained			Lost			
Team	**Games Played**	**Fumbles**	**Interceptions**	**Total**	**Fumbles**	**Interceptions**	**Total**	**Margin**
Indiana	11	31	4	35	13	15	28	7
Ohio State	12	10	18	28	6	12	18	10
Wisconsin	13	6	20	26	11	9	20	6
Michigan	12	12	14	26	10	5	15	11
Northwestern	12	13	12	25	6	7	13	12
Penn State	12	7	17	24	11	9	20	4
Illinois	11	8	12	20	12	10	22	–2
Minnesota	12	13	6	19	9	8	17	2
Purdue	12	9	10	19	7	12	19	0
Michigan State	11	6	12	18	9	16	25	–7
Iowa	12	8	9	17	9	11	20	–3

score. This is computed from the formula PM = (T+) − (T−) where T+ stands for the number of minutes the player was on the ice when his team scored a goal and T− stands for the number of minutes he was on the ice when a goal was scored against his team. The difference in these numbers is the +/− score or PM. This gives a measure of the efficiency of different players in both offensive and defensive play.

A similar computation is used in football to determine a team's total turnovers (called the margin in the table.) It is computed by subtracting the number of times a team lost the ball through fumbles or interceptions from the number of the times the team gained the ball through fumbles or interceptions.

A more complex algebraic computation is a baseball player's adjusted production statistic. It is computed from the formula APRO = OBP / LOBP + SA / LSA − 1. That is, a player's On-Base Percentage is compared to the League's On-Base Percentage and his Slugging Average is compared to the League's Slugging Average. These two ratios are then added and 1 is subtracted.

Estimation

The distance of a home run may be estimated using math. At Wrigley Field in Chicago, there is a person responsible for estimating the distance of each home run. When one occurs, about 10 seconds are needed to determine the distance and relay the information to the scoreboard operator for display. The dimensions of Wrigley are 355 feet down the left-field line, 353 feet down the right-field line, and 400 feet to dead center. The power alleys are both marked at 368 feet.

When a home run is hit and lands inside the park, these dimensions are used to estimate the distance. If the ball lands in the stands, 3 feet are added for each row of bleacher seats. For example, if a ball is hit into the left side of the field between the foul pole and left center, four rows into the bleachers, then the total distance is estimated as 362 feet (the midpoint of 355 and 368) plus 12 feet (4 rows multiplied by 3 feet per row), which equals 374 feet. Of course, it gets even harder to estimate the distance when a ball leaves the park! SEE ALSO Athletics, Technology in.

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Standardized Tests

Standardized tests are administered in order to measure the aptitude or achievement of the people tested. A distribution of scores for all test takers allows individual test takers to see where their scores rank among others. Well-known examples of standardized tests include "IQ" (Intelligence Quota) tests, the PSAT (Preliminary Scholastic Achievement Test) and SAT (Scholastic Achievement Test) tests taken by high school students, the GRE (Graduate Requirements Examination) test taken by college students applying to graduate school, and the various admission tests required for business, law, and medical schools.

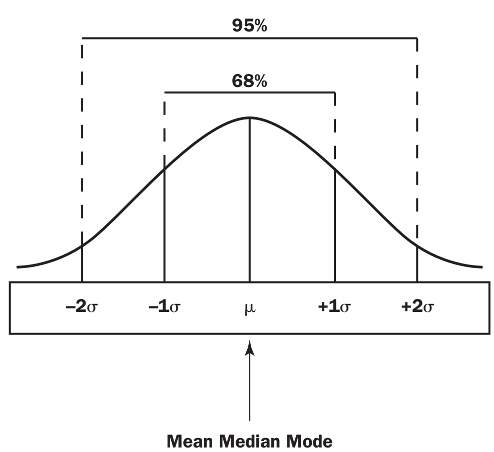
The "Normal" Curve

The mathematics behind the distribution of scores on standardized tests comes from the fields of probability theory and mathematical statistics. A cornerstone of this mathematical theory is the "Central Limit Theorem," which states that for large samples of observations (or scores in the case of standardized tests), the distribution of the observations will follow the bell-shaped normal probability curve illustrated below. This means that most of the observations will cluster symmetrically around the mean or average value of all the observations, with fewer observations farther away from the **mean** value.

mean the arithmetic average of a set of data

standard deviation a measure of the average amount by which individual items of data might be expected to vary from the arithmetic mean of all data

One measure of the spread or dispersion of the observations is called the **standard deviation**. According to statistical theory illustrated above, about 68 percent of all observations will lie within plus or minus one standard deviation of the mean; 95 percent will lie within plus or minus two standard deviations of the mean (see graph below); and 99.7 percent will lie within plus or minus three standard deviations of the mean. Standardized test scores are examples of observations that have this property.



Consider, for example, a standardized test for which the mean score is 500 and the standard deviation is 100. This means that about 68 percent of all test takers will have scores that fall between 400 and 600; 95 percent will have scores between 300 and 700; and virtually all of the scores will fall between 200 and 800. In fact, many standardized tests, including the PSAT and SAT, have just such a scale on which 200 and 800 are the minimum and maximum scores, respectively, that will be given.

Scaled Scores

The "standardized" in standardized tests means that similar scores must represent the same level of performance from year to year. Statisticians and test creators work together to ensure that, for example, if a student scores 650 on one version of the SAT as a junior and 700 on a different version as a senior, that this truly represents a gain in achievement rather than one version of the test being more difficult than the other.

By "embedding" some questions that are identical in all versions of a test and analyzing the performance of each group on those common questions, test creators can ensure a level of standardization. If one group scores significantly lower on the common questions, this is interpreted to mean that the lower scoring group is not as strong as the higher scoring group.

If group A scores higher than group B on questions identical to both their tests but then scores the same or lower than group B on the complete test, it would be assumed that the test given to group A was more difficult than that given to group B. Statisticians can develop a mathematical formula that will correct for such a variance in the difficulty of tests.

Such a formula would be applied to the "raw" scores of the test takers in order to obtain "scaled" scores for both groups. These scaled scores could then be compared. A scaled score of 580 on version A means the same thing as a scaled score of 580 on version B, even though the raw scores may be different. In this sense the scores are said to have been "standardized."

Statistical Scores

A second meaning of "standardized" is more subtle, more mathematically involved, and not well understood by the general public. This meaning has to do with the bell-shaped normal probability curve mentioned at the beginning of this article. Theoretically, there are an infinite number of normal curves—one for each different set of observations that might be made. Mathematicians would say that there is an entire "family" of normal curves, and, the members of the normal curve family share similarities as well as differences.

All bell-shaped curves are high in the middle and slope down to long "tails" to the right and left. Although different types of observations will have different mean values, those mean values will always occur at the middle of the distributions. They may also have different standard deviations as discussed earlier, but the percentage of values lying between plus or minus one of those standard deviations will still be about 68 percent, the percentage of values lying between plus or minus two standard deviations will still be about 95 percent, and so on.

In order to make the analysis of normal distributions simpler, statisticians have agreed upon one particular normal curve that will represent all the rest. This special normal curve has a mean of 0 and a standard deviation of 1 and is called the "standard normal curve." A "standardized" test result, therefore, is one based on the use of a standard normal curve as its reference.

The advantage of having the standard normal curve represent all the other normal curves is that statisticians can then construct a single table of probabilities that can be applied to all normal distributions. This can be done by "mapping" those distributions onto the standard normal curve and making use of its probability table. The term "mapping" in mathematics refers to the transformation of one set of values to a different set of values.

To illustrate, consider the test with a mean of 500 and a standard deviation of 100. The mean of this set of scores lies 500 units to the right of the standard normal distribution's mean of 0. So to "map" the mean of the test scores onto the standard normal mean, 500 is subtracted from all the test scores. Now there is a new distribution with the correct mean but the wrong standard deviation.

To correct this, all of the scores in the new distribution are divided by 100, since $\frac{100}{100} = 1$, which is the standard deviation of the standard normal distribution. The two distributions are now identical. In mathematical terms the test scores have been "mapped" onto the standard normal values.

This mapping is composed of two transformations: a translation of 500 to the left and a scale change of 1/100. This composition can be represented by $\frac{(x - 500)}{100}$, where x is any test score.

Building on this example, suppose one wants to know the percentage of test takers who scored 650 or above. First, compute $\frac{(650 - 500)}{100} = 1.5$. Then go to a standard normal table, look up a standard score of 1.5, and see that about 6.88 percent of standard normal scores are at 1.5 or above. This means that about 6.88 percent of the test scores are 650 or higher. This procedure may be used with any normally distributed data set for which the mean and standard deviation are known. SEE ALSO Central Tendency, Measures of; Mapping, Mathematical; Statistical Analysis; Transformations.

Stephen Robinson

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Statistical Analysis

You may have heard the saying "You can prove anything with statistics," which implies that statistical analysis cannot to be trusted, that the conclu-

sions that can be drawn from it are so vague and ambiguous that they are meaningless. Yet the opposite is also true. Statistical analysis can be reliable and the results of statistical analysis can be trusted if the proper conditions are established.

What Is Statistical Analysis?

Statistical analysis uses **inductive reasoning** and the mathematical principles of probability to assess the reliability of a particular experimental test. Mathematical techniques have been devised to allow measurement of the reliability (or fallibility) of the estimate to be determined from the data (the sample, or "N") without reference to the original population. This is important because researchers typically do not have access to information about the whole population, and a sample—a subset of the population— is used.

inductive reasoning drawing general conclusions based on specific instances or observations; for example, a theory might be based on the outcomes of several experiments

Statistical analysis uses a sample drawn from a larger population to make **inferences** about the larger population. A population is a well-defined group of individuals or observations of any size having a unique quality or characteristic. Examples of populations include first-grade teachers in Texas, jewelers in New York, nurses at a hospital, high school principals, Democrats, and people who go to dentists. Corn plants in a particular field and automobiles produced by a plant on Monday are also populations. A sample is the group of individuals or items selected from a particular population. A random sample is taken in such a way that every individual in the population has an equal opportunity to be chosen. A random sample is also known as an unbiased sample.

inferences the act or process of deriving a conclusion from given facts or premises

Most mail surveys, mall surveys, political telephone polls, and other similar data gathering techniques generally do not meet the proper conditions for a random, unbiased sample, so their results cannot to be trusted. These are "self-selected" samples because the subjects choose whether to participate in the survey and the subjects may be picked based on the ease of their availability (for example, whoever answers the phone and agrees to the interview).

Selecting a Random Sampling

The most important criterion for trustworthy statistical analysis is correctly choosing a random sample. For example, suppose you have a bucket full of 10,000 marbles and you want to know how many of the marbles are red. You could count all of the marbles, but that would take a long time. So, you stir the marbles thoroughly and, without looking, pull out 100 marbles. Now you count the red marbles in your random sample. There are 10. Thus you could conclude that approximately 10 percent of the original marbles are red. This is a trustworthy conclusion, but it is not likely to be exactly right. You could improve your accuracy by counting a larger sample; say 1,000 marbles. Of course if you counted all the marbles, you would know the exact percentage, but the point is to pick a sample that is large enough (for example, 100 or 1,000) that gives you an answer accurate enough for your purposes.

Suppose the 100 marbles you pulled out of the bucket were all red. Would this be proof that all 10,000 marbles in the bucket were red? In science, statistical analysis is used to test a **hypothesis**. In the example we are testing, the hypothesis would be "all the marbles in the bucket are red."

hypothesis a proposition that is assumed to be true for the purpose of proving other propositions

Statistical inference makes it possible for us to state, given a sample size (100) and a population size (10,000), how often false hypotheses will be accepted and how often true hypotheses are rejected. Statistical analysis cannot conclusively tell us whether a hypothesis is true; only the examination of the entire population can do that. So "statistical proof" is a statement of just how often we will get "the right answer."

Using Basic Statistical Concepts

Statistics is applicable to all fields of human endeavor, from economics to education, politics to psychology. Procedures worked out for one field are generally applicable to the other fields. Some statistical procedures are used more often in some fields than in others.

Example 1. Suppose the Wearemout Pants Company wants to know the average height of adult American men, an important piece of information for a clothing manufacturer producing pants. The population is all men over the age of 25 who live in the United States. It is logistically impossible to measure the height of every man who lives in the United States, so a random sample of around 1,000 men is chosen. If the sample is correctly chosen, all ethnic groups, geographic regions, and socioeconomic classes will be adequately represented. The individual heights of these 1,000 men are then measured. An average height is calculated by dividing the sum of these individual heights by the total number of subjects (N = 1,000). By doing so, imagine that we calculate an average height is 1.95 meters (m) for this sample of adult males in the United States. If a representative sample was selected, then this figure can be generalized to the larger population.

The random sample of 1,000 men probably included some very short men and some very tall men. The difference between the shortest and the tallest is known as the "range" of the data. Range is one measure of the "dispersion" of a group of observations. A better measure of dispersion is the "standard deviation." The standard deviation is the square root of the sum of the squares of the differences divided by one less than the number of observations.

$$S = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \overline{x})}{n - 1}}$$

In this equation, x_i is an observed value and $\overline{x}$ is the arithmetic mean.

In our example, if a smaller height interval is used (1.10 m, 1.11 m, 1.12 m, 1.13 m, and so on) and the number of men in each height interval plotted as a function of height a smooth curve can be drawn which would have a characteristic shape, known as a "bell" curve or "normal frequency distribution." A normal frequency distribution can be stated mathematically as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

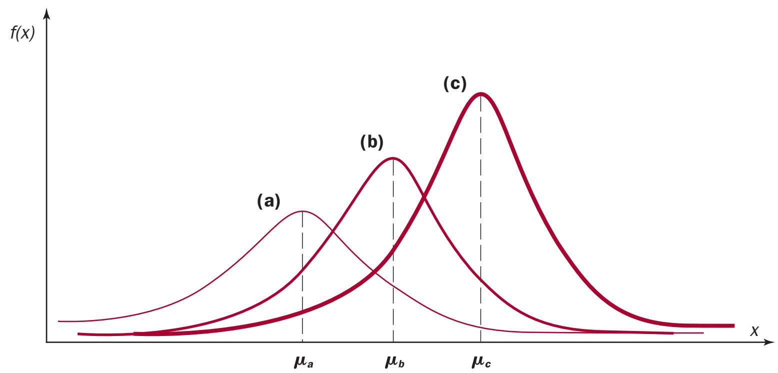
The value of sigma (σ) is a measure of how "wide" the distribution is. Not all samples will have a normal distribution, but many do, and these distributions are of special interest.

The following figure shows three normal probability distributions. Because there is no skew, the mean, median, and mode are the same. The mean

MEAN, MEDIAN, AND MODE

The average in the clothing example is known as the "arithmetic mean," which is one of the measures of central tendency. The other two measures of central tendency are the "median" and the "mode." The median is the number that falls in the mid-point of an ordered data set, while the mode is the most frequently occurring value.

of curve (a) is less than the mean of curve (b), which in turn is less than the mean of (c). Yet the standard deviation, or spread, of (c) is least, whereas that of (a) is greatest. This is just one illustration of how the parameters of distributions can vary.



Example 2. One of the most common uses of statistical analysis is in determining whether a certain treatment is efficacious. For example, medical researchers may want to know if a particular medicine is effective at treating the type of pain resulting from extraction of third molars (known as "wisdom" teeth). Two random samples of approximately equal size would be selected. One group would receive the pain medication while the other group received a "placebo," a pill that looked identical but contained only inactive ingredients. The study would need to be a "double-blind" experiment, which is designed so that neither the recipients nor the persons dispensing the pills knew which was which. Researchers would know who had received the active medicines only after all the results were collected.

Example 3. Suppose a student, as part of a science fair project, wishes to determine if a particular chemical compound (Chemical X) can accelerate the growth of tomato plants. In this sort of experiment design, the hypothesis is usually stated as a **null hypothesis**: "Chemical X has no effect on the growth rate of tomato plants." In this case, the student would *reject* the null hypothesis if she found a significant difference. It may seem odd, but that is the way most of the statistical tests are set up. In this case, the **independent variable** is the presence of the chemical and the **dependent variable** is the height of the plant.

The next step is experiment design. The student decides to use height as the single measure of plant growth. She purchases 100 individual tomato plants of the same variety and randomly assigns them to 2 groups of 50 each. Thus the population is all tomato plants of this particular type and the sample is the 100 plants she has purchased. They are planted in identical containers, using the same kind of potting soil and placed so they will receive the same amount of light and air at the same temperature. In other words, the experimenter tries to "control" all of the variables, except the single variable of interest. One group will be watered with water containing a small amount of the chemical while the other will receive plain water. To make the experiment double-blind, she has another student prepare the watering cans each day, so that she will not know until after the experiment is complete which group was receiving the treatment. After 6 weeks, she plans to measure the height of the plants.

null hypothesis the theory that there is no validity to the specific claim that two variations of the same thing can be distinguished by a specific procedure

independent variable in the equation $y = f(x)$, the input variable is x (or the independent variable)

dependent variable in the equation $y = f(x)$, if the function f assigns a single value of y to each value of x, then y is the output variable (or the dependent variable)

The next step is data collection. The student measures the height of each plant and records the results in data tables. She determines that the control group (which received plain water) had an average (arithmetic mean) height of 1.3 m (meters), while the treatment group had an average height of 1.4 m.

Now the student must somehow determine if this small difference was significant or if the amount of variation measured would be expected under no treatment conditions. In other words, what is the probability that 2 groups of 50 tomato plants each, grown under identical conditions would show a height difference of 0.1 m after 6 weeks of growth? If this probability is less than or equal to a certain predetermined value, then the null hypothesis is rejected. Two commonly used values of probability are 0.05 or 0.01. However, these are completely arbitrary choices determined mostly by the widespread use of previously calculated tables for each value. Modern computer analysis techniques allow the selection of any value of probability.

The simplest test of significance is to determine how "wide" the distribution of heights is for each group. If there is a wide variance (σ) in heights (say, $\sigma = 25$), then small differences in mean are not likely to be significant. On the other hand, if the dispersion is narrow (for example, if all the plants in each group were close to the same height, so that $\sigma = 0.1$) then the difference would probably be significant.

chi-square test a generalization of a test for significant differences between a binomial population and a multinomial population

nominal scales a method for sorting objects into categories according to some distinguishing characteristic, then attaching a label to each category

There are several different tests the student could use. Selecting the right test is often a tricky problem. In this case, the student can reject several tests outright. For example, the **chi-square test** is suitable for **nominal scales** (yes or no answers are one example of nominal scales), so it does not work here. The *F*-test measures variability or dispersion within a single sample. It too is not suitable for comparing two samples. Other statistical tests can also be rejected as inappropriate for various reasons.

In this case, since the student is interested in comparing means, the best choice is a *t* test. The *t*-test compares two means using this formula:

$$t = \frac{M_1 - M_2 - (\mu_1 - \mu_2)}{S_{M_1 - M_2}}$$

In this case, the null hypothesis assumes that $\mu 1 - \mu 2 = 0$ (no difference in the sample groups), so that we can say:

$$t = \frac{M_1 - M_2}{S_{M_1 - M_2}}$$

The quantity $S_{M_1-M_2}$ is known as the standard error of the mean difference. When the sample sizes are the same, $S_{M_1-M_2} = \sqrt{S_{M_1}{}^2 + S_{M_2}{}^2}$. The standard error of the mean difference is the square root of the sums of the squares of the standard errors of the means for each group. The standard error of the mean for each group is easily calculated from $S_m = \frac{S}{\sqrt{N}}$. N is the sample size, 50, and the student can calculate the standard deviation by the formula for standard deviation given above.

The final experimental step is to determine sensitivity. Generally speaking, the larger the sample, the more sensitive the experiment is. The choice of 50 tomato plants for each group implies a high degree of sensitivity.

Students, teachers, psychologists, economists, politicians, educational researchers, medical researchers, biologists, coaches, doctors and many others use statistics and statistical analysis every day to help them make decisions. To make trustworthy and valid decisions based on statistical information, it is necessary to: be sure the sample is representative of the population; understand the assumptions of the procedure and use the correct procedure; use the best measurements available; keep clear what is being looked for; and to avoid statements of **causal relations** if they are not justified. SEE ALSO Central Tendency, Measures of; Data Collection and Interpretation; Graphs; Mass Media, Mathematics and the.

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causal relation a response to an input that does not depend on values of the input at later times

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Step Functions

In mathematics, functions describe relationships between two or more quantities. A step function is a special type of relationship in which one quantity increases in steps in relation to another quantity.

For example, postage cost increases as the weight of a letter or package increases. In the year 2001 a letter weighing between 0 and 1 ounce required a 34-cent stamp. When the weight of the letter increased above 1 ounce and up to 2 ounces, the postage amount increased to 55 cents, a step increase.

A graph of a step function f gives a visual picture to the term "step function." A step function exhibits a graph with steps similar to a ladder.

The domain of a step function f is divided or partitioned into a number of intervals. In each interval, a step function $f(x)$ is constant. So within an interval, the value of the step function does not change. In different intervals, however, a step function f can take different constant values.

One common type of step function is the greatest-integer function. The domain of the greatest-integer function f is the real number set that is divided into intervals of the form . . .$[-2, -1)$, $[-1, 0)$, $[0, 1)$, $[1, 2)$, $[2, 3)$,. . . The intervals of the greatest-integer function are of the form $[k, k + 1)$, where k is an integer. It is constant on every interval and equal to k.

$$f(x) = 0 \text{ on } [0, 1), \text{ or } 0 \le x < 1$$

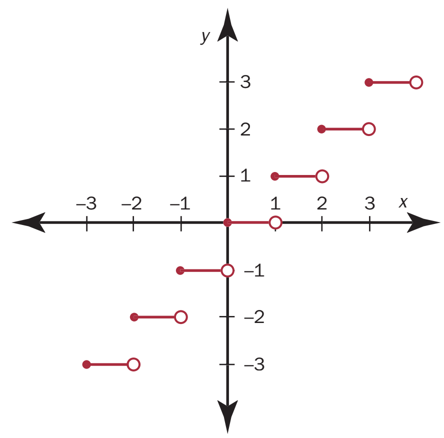
$$f(x) = 1 \text{ on } [1, 2), \text{ or } 1 \le x < 2$$

$$f(x) = 2 \text{ on } [2, 3), \text{ or } 2 \le x < 3$$

For instance, in the interval $[2, 3)$, or $2 \le x < 3$, the value of the function is 2. By definition of the function, on each interval, the function equals the greatest integer less than or equal to all the numbers in the interval. Zero,

1, and 2 are all integers that are less than or equal to the numbers in the interval [2, 3), but the greatest integer is 2.

Therefore, in general, when the interval is of the form $[k, k + 1)$, where k is an integer, the function value of greatest-integer function is k. So in the interval [5, 6), the function value is 5. The graph of the greatest integer function is similar to the graph shown below.



There are many examples where step functions apply to real-world situations. The price of items that are sold by weight can be presented as a cost per ounce (or pound) graphed against the weight. The average selling price of a corporation's stock can also be presented as a step function with a time period for the domain.

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Stock Market

The term "stock market" refers to a large number of markets worldwide in which shares of stock are bought and sold. In 1999 there were 142 such markets in major cities around the world, where hundreds of millions of shares of stock changed hands daily.

Some stock trading takes place in traditional markets, where representatives of buyers and sellers meet on a trading floor to bid on and sell stocks. The stocks traded in these markets are sold to the highest bidder by auction. Increasingly, however, stocks are traded in markets that have no physical trading floor, but are bought or sold electronically.

Electronic stock trading involves networks of computers and/or telephones used to link brokers. Stock price is determined by negotiation between representatives of buyers and sellers, rather than by auction. Hence, these markets are called "negotiated markets." All the stock exchanges around the world, including auction markets at centralized locations and electronic negotiated markets, are collectively called the stock market.

Shares of stock are the basic elements bought and sold in the stock market. In order to raise working capital, a company may offer for sale to the public "shares" or "portions of ownership" in the company. These shares are represented by a certificate stating that the individual owns a given number of shares. Revenue from this initial public offering of stock goes to the issuing corporation.

Investors buy stock hoping that its value will increase. If it does, the shareholder may sell the stock at a profit. If the stock price declines, selling the stock results in a loss for the stockholder. The issuing company does not receive any income when stocks are traded after the initial public offering.

The Price of Stocks

A stock is worth whatever a buyer will pay for it. Because stock market prices reflect the varying opinions and perceptions of millions of traders each day, it is sometimes difficult to pinpoint why a certain stock rises or falls in price. A wide range of factors affects stock prices. These may include the company's actual or perceived performance, overall economic climate and market conditions, philosophy and image projected by the firm's corporate management, political action around the world, the death of a powerful world leader, a change in interest rates, and many other influences.

Investors bid on a stock if they believe it is a good value for the current price. If there are more buyers than sellers, the price of the stock rises. When investors collectively stop buying a given stock or begin selling the stock from their portfolios, the stock price falls.

Understanding Stock Tables

Perceptions of a stock's value are further shaped by information that is available in the financial press, at financial web sites, and from brokers either online or in person. Most major newspapers print this information in stock tables. Some stock charts are printed daily; others contain weekly or quarterly summaries of trading activity.

Because each stock table reports data from only one stock exchange, many sources print more than one table. Although the format varies, most stock tables contain, at a minimum, certain important figures.

52 Week										
High	Low	Stock	Div	Yld	PE	Vol	High	Low	Close	Change
45 1/4	21	Goodrch	1.10	25	18	1953	28 1/4	27 1/16	27 1/2	–1 3/4
66 3/4	20 3/8	Goodyr	1.20	20.1	16	3992	24 1/2	23	24 1/8	+11/16

The high and low prices for the stock during the last 52 weeks, shown in the first two columns of the table above, are indicators of the volatility of the stock. Volatility is a measure of risk. The more a stock's price varies over a short time, the more potential there is to make or lose money trading that stock.

Although the prices of all stocks go up and down with market fluctuations, some stocks tend to rise more quickly and fall more sharply than the overall market. Other stocks generally move up and down more slowly than the market as a whole.

✶**Stock exchanges are increasingly moving to decimal notation.**

In the United States, stock prices are traditionally reported in eighths, sixteenths, or even thirty-seconds of a dollar.✶ This practice dates from the time when early settlers cut European coins into pieces to use as currency. Goodrich's 52-week high of $45\frac{1}{4}$ in the above example means the highest trading price for this stock in the last year was $45.25.

The third column in the table holds an abbreviated name for the stock's issuing corporation. Some financial papers also print the ticker symbol, a very short code name for the company. Brokers often use ticker symbols when ordering or selling stock. Ticker symbols are abbreviated names; for example, "GM" is used for General Motors or "INTC" for Intel, depending on which stock market exchange is being consulted.

The total cash dividend per share paid by the company in the last year is found in the fourth column of the table. The company's board of directors determines the size and frequency of dividends. Traditionally, only older and more mature companies pay dividends, whereas smaller, new companies tend to reinvest profits to finance continued growth. Some dividends are paid in the form of additional stock given to shareholders rather than in cash. Stock tables do not report the value of these non-cash dividends.

The dividend-to-price ratio, or yield, is shown in the table's fifth column. This is the annual cash dividend per share divided by the current price of the stock and is expressed as a percentage. In this example, Goodrich's closing stock price of $27.50 divided into the annual dividend of $1.10 shows

The New York Stock Exchange trading floor became jammed with traders as stocks plunged in the biggest one-day sellout in history on October 19, 1987.

a current yield of 4 percent. Yield can be used to compare income from a particular stock with income from other investments. It does not reflect the gain or loss realized if the stock is sold.

The price-to-earnings ratio (P/E ratio) is the price of the stock divided by the company's earnings per share for the last twelve months. A P/E ratio of 18 means that the closing price of a given stock was eighteen times the amount per share that the firm earned in the preceding twelve months.

Volume in a stock table is the total number of shares traded in the reported period. Daily volumes are usually reported in hundreds. This chart shows a trading volume of 195,300 shares of Goodrich stock. Unusually high volumes can indicate either that investors are enthusiastic about a stock and likely to bid it up, or that wary stockholders are selling their holdings.

The next three columns show the high, low, and closing price for a particular trading period, such as a day or a week. The final column reports the net change in price since the last reporting period. In this daily chart, the closing price for Goodrich stock was $1\frac{3}{4}$ ($1.75) lower than the previous day's close, whereas Goodyear closed $\frac{11}{16}$ of a dollar ($0.6875) higher.

UNPRECEDENTED CLOSURES OF FINANCIAL STOCK MARKETS

On Tuesday, September 11, 2001, terrorist assaults demolished the World Trade Center in New York City and damaged the Pentagon in Washington, D.C. These attacks on the structure of the U.S. financial and governmental systems led to the closing of the U.S. stock markets, including the New York Stock Exchange (NYSE), the National Association of Securities Dealers Automated Quotations (NASDAQ), and the American Stock Exchange (AMEX).

These financial stock exchanges reopened the following Monday after remaining shut down for the longest period of time since 1929.

Stock Exchanges

The New York Stock Exchange, sometimes called the "Big Board," is the largest and oldest auction stock market in the United States. Organized in 1792, it is a traditional market, where representatives of buyers and sellers meet face to face. More than 3,000 of the largest and most prestigious companies in the United States had stocks listed for sale on the NYSE in early 2000, and it is not unusual for a billion shares of stock to change hands daily, making it by far the busiest traditional market in the world.

Requirements for a corporation to list its stock on the NYSE are stringent. They include having at least 1.1 million publicly traded shares of stock, with an initial offering value of at least 40 million dollars. The American Stock Exchange (AMEX) in New York and several regional stock exchanges are traditional markets with less stringent listing requirements.

The NASDAQ is an advanced telecommunications and computer-based stock market that has no centralized location and no single trading floor. However, more shares trade daily on the NASDAQ than on any other U.S. exchange. The NASDAQ is a negotiated market, rather than an auction market. Nearly 5,000 companies had stocks listed on the NASDAQ in 1999. Companies listing on the NASDAQ range from relatively small firms to large corporations such as Microsoft, Intel, and Dell.

Stocks not listed on any of the exchanges are traded over the counter (OTC). They are bought and sold online or through brokers. These are generally stocks from smaller or newer companies than those listed on the exchanges. Stocks in about 11,000 companies are traded in the OTC market.

Stock Market Indexes

A large number of indexes, or summaries of stock market activity, track and report stock market fluctuations. The most widely known and frequently quoted index is the Dow Jones Industrial Average (DJIA). The DJIA is an adjusted average of the change in price of thirty major industrial stocks listed on the NYSE.

Another widely used stock market index is the Standard and Poor's 500 (S&P 500). It measures performance of stocks in 400 industrial, 20 transportation, 40 utility, and 40 financial corporations listed on the NYSE, AMEX, and NASDAQ. Because of this, many analysts feel that the S&P 500 gives a much broader picture of the stock market than does the Dow Jones.

Indexes for the NASDAQ and AMEX track the performance of stocks listed on those exchanges. Lesser known indexes monitor specific sectors of the market (transportation or utilities, for example). The *Wall Street Journal* publishes twenty-nine different stock market indexes each day. SEE ALSO Economic Indicators.

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Stone Mason

Masonry is the building material used for the construction of brick, concrete, and rock structures. A stone mason constructs, erects, installs, and repairs structures using brick, concrete block insulation, and other masonry units. Walls, arches, floors, and chimneys are some of the structures that a stone mason builds or repairs.

algebra the branch of mathematics that deals with variables or unknowns representing the arithmetic numbers

geometry the branch of mathematics that deals with the properties and relationships of points, lines, angles, surfaces, planes, and solids

Part of the responsibility of a stone mason is to cut and trim masonry materials to certain specifications. A stone mason's work tools include a framing square for setting project outlines, levels for setting forms, a line level for making layouts and setting slope, and a tape measure.

Masons need a working knowledge of ratios for mixing concrete and mortar. A foundation in **algebra** and **geometry** is helpful for laying out a

Stone masons rely on mathematics and specially designed tools to calculate and measure exact dimensions.

building site, and for building the frames that will contain the object being built. An ability to calculate slope, volume, and area is important for a stone mason. A basic knowledge of trade math is required for most stone mason training programs and apprenticeships.

Marilyn Schwader

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Sun

The Sun is located in the "suburbs" of the Milky Way galaxy, around 30,000 light-years from the center and within one of its spiral arms. It revolves around the galaxy's center at an average speed of 155 miles per second, taking 225 million years to complete each circuit.

Although the Sun is just one star among an estimated 400 billion stars in the Milky Way galaxy, it is the closest star to Earth. More importantly, it is the only star that provides Earth with enough light, heat, and energy to sustain life. Also, the strong gravitational pull of the Sun holds the Earth in orbit within its solar system.

Many people forget that the Sun is a star because it looks so big and different when compared to other stars and because the Sun appears in the sky during the day, whereas other stars only appear at night. Because the Sun is so close to the Earth, its luminosity (brightness) overwhelms the brightness of other stars, drowning out their daytime light.✶

✶**Because of its relative nearness, the Sun appears about 10 billion times brighter than the next brightest star.**

Size and Distance

Archimedes (287 B.C.E.–212 B.C.E.) placed the Sun at the center of the solar system. Observations of Galileo (1564–1642) supported this heliocentric theory, nullifying the ancient belief that the Earth was the center of the solar system.

Before the Sun's distance was known, Aristarchus (310 B.C.E.–230 B.C.E.) knew that the Moon shines by reflected sunlight. Aristarchus decided that if he measured the angle between the Moon and the Sun when the Moon is half-illuminated, he could then compute the ratio of their distances from the Earth. Aristarchus estimated that this angle was 87°, resulting in the ratio of their distances at sin 3°.

Before the invention of trigonometry, Aristarchus used a similar method to calculate the inequality $\frac{1}{18} > \sin 3° > \frac{1}{20}$, reasoning that the Sun was 18 to 20 times farther away from the Earth than the Moon. As calculations were refined, the angle between Moon and Sun was shown to be 89°50′. Astronomers learned that the Sun is actually 400 times farther away from the Earth than the Moon.

Scientists now know that the Earth-Sun distance is 93 million miles. This distance was discovered when radar signals were bounced off Venus's surface to determine the Earth-to-Venus distance. At a speed of 500 mph, a journey from the Earth to the Sun would take 21 years.

Ancient civilizations thought that the Sun and Moon were the same size. Yet the Sun's diameter is really 864,338 miles across, which is more than

400 times the Moon's diameter and about 109 times the Earth's diameter. The Sun has a volume 1.3 million times the Earth's volume. Thus, a million Earths could be packed within the Sun. Although enormous compared to Earth, the Sun is an average-sized star.

German mathematician Johannes Kepler (1531–1630) devised his laws of planetary motion while studying the motion of Mars around the Sun. The Sun's mass can be calculated from his third law with the equation $T^2 = \frac{(\pi)r^3}{(GM_s)}$, where T is the period of Earth's revolution (3.15×10^7 seconds), r is the radius of Earth's revolution (1.5×10^{11}m), and G is the planetary constant (6.67×10^{-11} Newton's m^2/kg). To find M_s (Sun's mass) insert these values into the equation and solve

$$\begin{aligned} M_s &= (\pi)(1.5 \times 10^{11}\text{m})^3/(6.67 \times 10^{-11}N\text{m}^2/\text{kg})(3.15 \times 10^7 s)^2 \\ &= 2.0 \times 10^{30} \text{ kilograms.} \end{aligned}$$

Therefore, the Sun's mass is about 300,000 times the Earth's mass. Yet with respect to other stars, the Sun's mass is just average.

Time and Temperature

The Sun's rotation is similar to the Earth's rotation (one rotation every day), but because the Sun is gaseous not all parts rotate at the same speed. Galileo first noticed that the Sun rotates when he observed sunspots moving across the disk. He found that a particular spot took 27 days to make a complete circuit. Later observations found that the Sun at its equator rotates in slightly over 24.5 days. At locations two-thirds of the distance above and below the equator, the rotation take nearly 31 days.

In order to support its large mass, the Sun's interior must possess extremely large pressures and temperatures. The force of gravity at the core's surface is about 250 million times as great as Earth's surface gravity. No solids or liquids exist under these conditions, so the Sun's body primarily consists of the gases hydrogen (73 percent) and helium (25 percent).

Within the Sun's core, nuclear fusion reactions release huge amounts of energy. About 5 billion kilograms of hydrogen convert to helium, releasing energy each second. The core temperature is about 15 million kelvin, with a density of 160 grams per cubic centimeter. Based on mathematical calculations, the solar core is approximately the size of Jupiter, or approximately 75,000 to 100,000 miles in diameter. The amount of hydrogen within the Sun's core should sustain fusion for another 7 billion years.

For now, the Sun is bathing the Earth with just the right amount of heat. High-energy gamma rays (the particles created by fusion reactions) travel outward from the core and ultimately through the outer layers of the **photosphere**, **chromosphere**, and **corona**, losing most of their energy in the process. Along the way to the Sun's surface, the temperature of the gamma rays has dropped from 15 million to 6,000 kelvin. Yet even at these temperatures, the Sun is in the middle range of stellar surface temperatures.

photosphere the very bright portion of the Sun visible to the unaided eye; the portion around the Sun that marks the boundary between the dense interior gases and the more diffuse cooler gases in the outer realm of the Sun

chromosphere the transparent layer of gas that resides above the photosphere in the atmosphere of the Sun

corona the upper, very rarefied atmosphere of the Sun that becomes visible around the darkened Sun during a total solar eclipse

Measuring the Sun

Information available about the Sun has increased with revolutionary scientific discoveries. Early telescopic observations allowed scientific study to begin, showing that the Sun is a dynamic, changing body. Later develop-

ments within spectroscopy, and the discovery of elementary particles and nuclear fusion, allowed scientists to further understand its composition and the processes that fuel it.✶

✶Astronomers now believe that the Sun is about 4.6 billion years old and will shine for another 7 billion years.

Recent developments of artificial satellites and other spacecraft now allow scientists to continuously study the Sun. Among the advances that have significantly influenced solar physics are the spectroheliograph, which measures the spectrum of individual solar features; the coronagraph, which permits study of the solar corona without an eclipse; and the magnetograph, which measures magnetic-field strength over the solar surface. Space instruments have revolutionized solar study and continue to add to increased, but still incomplete, knowledge about the Sun. SEE ALSO Archimedes; Galileo Galilei; Solar System Geometry, History of; Solar System Geometry, Modern Understandings of.

William Arthur Atkins (with Philip Edward Koth)

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Superconductivity

In the age of technology, with smaller and smaller electronic components being used in a growing number of applications, one pertinent application of mathematics and physics is the study of superconductivity. All elements and compounds possess **intrinsic** physical properties including a melting and boiling point, **malleability**, and conductivity. Conductivity is the measure of a substance's ability to allow an electrical current to pass from one end of a sample to the other. The measurement of resistivity (the inverse of conductivity) is called resistance and is measured in the unit ohms (Ω).

intrinsic of itself; the essential nature of a thing; originating within the thing

malleability the ability or capability of being shaped or formed

Ohm's Law

An important and useful formula in science is called Ohm's Law, $E = IR$. E is voltage in volts, I is current in amperes, and R is resistance in ohms. To visualize this, imagine a wire conducting electrons along its length, like a river flowing on the surface of the wire. The resistance acts like rocks in the river, slowing the flow. Current is equivalent to the amount of water flowing (or the number of electrons per unit time), and voltage is equivalent to the slope of the river. As the water hits the rocks, it splashes up and away. This is equivalent to resistance generating heat in a circuit. All normal materials have some sort of resistance, thus all circuits generate heat to a greater or lesser degree.

An interesting thing happens as the temperature of the wire changes. As the temperature elevates, the resistance increases; as the temperature lowers, the resistance decreases, but only to a point, then it goes back up again. In 1911 Kamerlingh Onnes discovered that mercury cooled to 4 kelvin (4 K) (that is $-269.15°$ C , about $-453°$ F) suddenly loses all resistance. He called this phenomenon superconductivity. Superconductivity is the ability of a substance to conduct electricity without resistance. If applied to Ohm's Law, a voltage (E) is applied, the current (I) should continue on its own if the voltage is then removed and the resistance is zero. This makes sense in terms of Ohm's Law, as $E\ (0) = I\ (X) \times R\ (0)$. When tested, it was found that this does indeed take place, with the current value (X) dropping over time as a function of the voltage applied, with this current being referred to as a Josephson current.

superconduction the flow of electric current without resistance in certain metals and alloys while at temperatures near absolute zero

✶The zero point of the kelvin scale is the temperature at which all molecular motion theoretically stops, sometimes called absolute zero.

At the time Onnes discovered **superconduction**, it was believed that superconductivity was simply an intrinsic property of a given material.✶ However, Onnes soon learned that he could turn superconduction on and off with the application of a large current, or strong magnetic field. Other than a lack of resistance, it was believed for many years that superconducting materials possessed the same properties as their normal counterparts. Then in 1933, it was discovered that superconducting materials are highly diamagnetic (that is, highly repelled by and exerting a great influence on magnetic fields), even if their normal counterparts were not. This led one of the discoverers, W. Meissner, to make scientific predictions regarding the electromagnetic properties of superconductors and have his name assigned to the effect, the Meissner effect.

Meissner's predictions were confirmed in 1939, paving the way for further discoveries. In 1950 it was demonstrated for the first time that the movement of electrons in a superconductor must take atomic vibrational effects into account. Finally in 1957 a fundamental theory presented by physicists J. Bardeen, Leon Cooper, and J. R. Schrieffer, called BCS theory, allowed predictions of possible superconducting materials, and the behavior of these materials.

BCS Theory

BCS theory explains superconductivity in a manner similar to the river of electrons example. When the material becomes superconducting, the electrons become grouped into pairs called Cooper pairs. These pairs dance around the rocks (resistance), like two people holding hands around a pole. This symmetry of movement allows the electrons to move without resistance.

The first superconductors were experimental rarities, for research only. During the first 75 years of research, the temperature at which materials could be made to superconduct did not rise very much. Before 1986 the highest temperature superconductor worked at a temperature of 23 K. Karl Muller and Johannes Bednorz found a material that had a transition temperature (the temperature where a material becomes superconducting) of nearly 30 K, in 1986. Their research and discovery allowed even higher temperature superconductors to be made of ceramics containing various ratios of, usually, barium or strontium, copper, and oxygen. These ceramics allowed superconductivity to be done at liquid nitrogen temperatures (77 K)—

a much more obtainable temperature than 4.2 K for liquid helium. However, ceramics are difficult to produce, break easily, and do not readily lend themselves to mass production. The newest generation of superconductors are nearing −148° C (125 K), which is the high temperature record as of 1998.

Superconductivity already touches the world, with its use in MRI (magnetic resonance imaging) magnets, chemical analytical tools such as NMR (nuclear magnetic resonance) spectroscopy, and unlimited electrical and electronic uses. If a high temperature superconductor could be mass produced cheaply, it would revolutionize the electronics industry. For example, one battery could be made to last years. In the future, people may look back at this basic research and compare it to the first discovery of fire. SEE ALSO Absolute Zero; Temperature, Measurement of.

Brook E. Hall

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Surveyor *See Cartographer.*

Symbols

Symbols are part of the language of mathematics. The use of symbols gives mathematics the power of generality: The ability to focus not only on a specific object, but also on a collection of objects. For instance, variables—symbols with the ability to vary in value used in place of numbers—make it possible to write general equations. Consider the equation $3 + 4 = 7$. This equation is specific. Suppose we are interested in all pairs of numbers whose sum is equal to 7. Using variables, x and y, to denote the two numbers, we would write the general equation $x + y = 7$. The variables, x and y, stand for many different pairs of numbers; x can be 5 and y can be 2 or x can be 9 and y can be -2. The only condition is that the sum of x and y is 7.

Symbols also include special notation for mathematical operations, like $+$ for addition and $\sqrt{\ }$ for taking the square root. Some common symbols and their uses are explained in the accompanying table.

Basic Mathematical Symbols

The Set. The foundation of mathematics rests on the concept of a set, which is a collection of objects that share a well-defined characteristic. The elements of a set are written in braces {}. An element x that belongs to a set A is written as $x \in A$. An element y that does not belong to A is written as $y \notin A$. {0} is a set that contains zero as its only element. This set has one element. {} is the empty set and does not contain any element. The empty set is also called a null set and sometimes written with the symbol $\varnothing$.

Operations. Numbers are the building blocks of mathematics. They can be combined by four operations: addition and its inverse operation,

MATHEMATICAL SYMBOLS

General Arithmetic and Algebra	
Symbol	**Meaning**
+	Plus (also indicates positive sign)
−	Minus (also indicates negative sign)
±	Plus or minus
x or • or *	Multiplied by (also called times)
/ or ÷	Divided by
:	Ratio
=	Equal to
≠	Does not equal
≈ or ≐	Approximately to
: : or α	Proportional to
∼	Similar to
≅	Congruent to (also means defines)
⇒	Implies
→	Approaches
<	Less than
<<	Much less than
>	Greater than
>>	Much greater than
≤	Less than or equal to
≥	Greater than or equal to
$\sqrt{x}$	Square root of x
$\sqrt[n]{x}$	n^{th} root of x
Σ	Summation of
Π	Product of
!	Factorial (e.g., 3! = 3*2*1)
|x|	Absolute value of x (e.g., |−3| = 3)
∞	Infinity
i	Imaginary number
{ }	Set
%	Percent
$\log_a x$	Logarithm of x (base a)
f : A → B	Mapping or map, where x ∈ A and y ∈ B.
4^a	Exponent (a is the exponent)

Differential Calculus and Integral Calculus	
Symbol	**Meaning**
∫x	Integral of x
f(x) or Φ(x)	Function of x
dx	Differential of x
dx/dy or x′	First derivative of x with respect to y
d^nx/dy^n	nth derivative of x with respect to y
∂x/∂y	Partial derivative of x with respect to y
$\dot{x}$	Partial derivative of x with respect to time
Δx	Increment of x
Γ (n)	Gamma function of n

Trigonometry and Geometry	
Symbol	**Meaning**
⊥	Perpendicular to
∥	Parallel to
∠ or θ	Angle
∟	Right angle

Constants	
Symbol	**Meaning**
e	Base of natural logarithm system = 2.71828…
π	pi = 3.14159…
ϒ	Euler-Mascheroni constant = 0.577216…

subtraction, and multiplication and its inverse operation, division. The additive inverse of a number b is $-b$ and the sum of a number and its additive inverse is always 0, that is, $b + (-b) = 0$. The additive inverse of 4 is -4. Similarly, the multiplicative inverse of a nonzero number b is b^{-1}, or $\frac{1}{b}$, and the product of a number and its multiplicative inverse is always 1, that is, $bb^{-1} = b(\frac{1}{b}) = 1$. The multiplicative inverse of 7 is $\frac{1}{7}$.

Size. Numbers can be compared according to their magnitude, or size. Given any two numbers, say x and y, there are three possibilities: x is *equal* to y ($x = y$), or x is *less* than y ($x < y$), or x is *greater* than y ($x > y$). Two numbers, x and y, can also be compared by taking their ratio, x:y, or x/y. A number percent, say 5 percent denotes the ratio of 5 to 100 or 5/100. In some problems, it is not important to know if a given number, say x, is positive or negative. The only important thing is its magnitude and it is represented by its absolute value, $|x|$. For instance, $|5| = 5$ and $|-5| = 5$.

Exponents. If a number is multiplied by itself two or more times, it can be written more compactly using an exponent that appears as a superscript to the right of the number. For example, if 4 is multiplied by itself three times, then it can be written as $4 \times 4 \times 4$, or 4^3, where 3 is the exponent. An exponent is also called a power. In general, x^n is read as "x raised to the power n." Raising a number to powers that are fractional, like $\frac{1}{2}$ or $\frac{1}{3}$, is called a radical, or root of the number. $\sqrt[n]{b}$ denotes the nth root of the number b; where n is called the index and b is called the radicand. For $n = 2$, the index is omitted and $\sqrt{b}$ is the square root of b. For $n = 3$, $\sqrt[3]{b}$ is called the cube root of b.

Functions. A fundamental idea in mathematics is that of a function that describes a relationship between two quantities. In symbols, it is written as $y = f(x)$ and read as "y is the function of x," which means that the value of y depends on x. A function is sometimes called a map, or mapping. Logarithm, $y = \log_a x$, is a particular type of function. By definition, a logarithm is an exponent. If $10^x = y$, then $\log_{10} y = x$.

Algebra. In algebra, numbers are represented as both variables and constants. A variable is a number whose value can change. Variables are mostly denoted by lowercase, terminating letters of the alphabet like x, y, or z. A constant is a fixed number, denoted mostly by first few lower case letters a, b, and c, or k. Algebraic expressions called polynomials contain one or more variable and/or constant terms. For example, $3x$, $2x + \frac{5}{2}$, and $x^2 + 3x + 2$ are all polynomials: $3x$ is a monomial (one term), $2x + \frac{5}{2}$ is a binomial (two terms), and $x^2 + 3x + 2$ is a trinomial (three terms). A constant number that multiplies a variable in an equation, or polynomial, is called a coefficient. For example, 2 is a coefficient of $2x$ in the equation $2x + 3 = 0$. In ax^2, a is the coefficient of x^2.

Equations. Solving different types of equations appears in many situations in mathematics. The simplest equation is linear, in which the highest power of the variable, x, is 1, for example, $3x + 2 = 1$. If the highest power of the variable is 2, then the equation is a quadratic equation. For example, $x^2 = 4$ and $2x^2 + 3x + 1 = 0$ are quadratic equations.

Calculus. Calculus is a branch of mathematics that partly deals with rate of changes. For a variable x, Δx denotes a small change in x. Further, $\frac{\Delta y}{\Delta x}$ measures the change in variable y with respect to change in x. When Δx becomes increasingly smaller and approaches 0, the rate of change $\frac{\Delta y}{\Delta x}$ equals the instantaneous rate of change and is denoted by $\frac{dy}{dx}$ and is called the derivative of y with respect to x. Also, calculus deals with finding the area of a region under the curve, or graph, of a function $f(x)$. If the region is bounded by lines $x = a$ and $x = b$, then the area is denoted by the integral of the function f. In symbols, it is written as $\int_a^b f(x)dx$. SEE ALSO Algebra; Calculus; Functions and Equations; Number System, Real.

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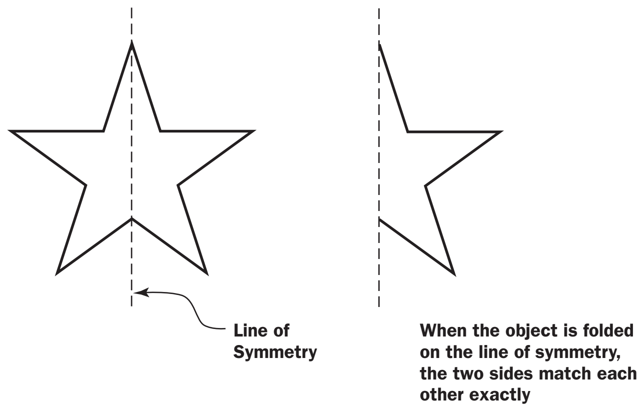
Symmetry

Symmetry is a visual characteristic in the shape and design of an object. Take a starfish for an example of an object with symmetry. If the arms of the starfish are all exactly the same length and width, then the starfish could be folded in half and the two sides would exactly match each other. The

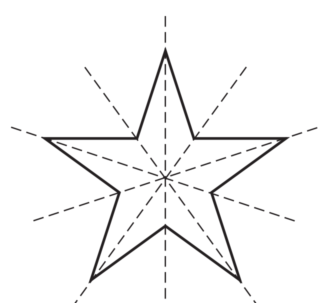
line on which the starfish was folded is called a line of symmetry. Any object that can be folded in half and each side matches the other is said to have line symmetry.

Line Symmetry



When an object has line symmetry, one half of the object is a reflection of the other half. Just imagine that the line of symmetry is a mirror. The actual object and its reflection in the "mirror" would exactly match each other. A human face, if vertically bisected into two halves (left and right), would reveal its bilateral symmetry: that is, the left half ideally matches the right half.

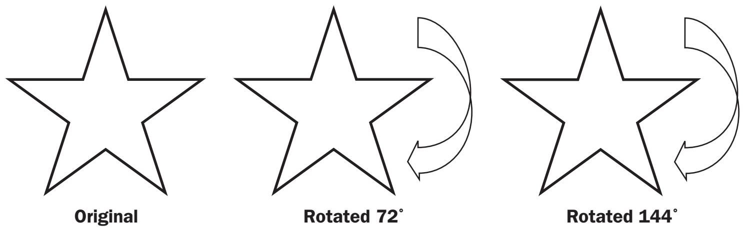
It is possible for an object to have more than one line of symmetry. Imagine a square. A vertical line of symmetry could be drawn down the middle of the square and the left and right sides would be symmetrical. Also, a horizontal line could be drawn across the middle of the square and the top and bottom would be symmetrical. As shown in the figure below, a star or a starfish has five lines of symmetry.



Five lines of symmetry

Alternately, the ability of an object to match its original figure with less than a full turn about its center is called rotational or point symmetry. For example, if a starfish embedded in the sand on a beach is picked up, rotated less than 360°, and then set back down exactly into its original imprint in the sand, the starfish would be said to have rotational symmetry.

Rotational Symmetry



The design of this starfish demonstrates both line symmetry and rotational symmetry.

Many natural and manmade items have reflection and/or rotation symmetry. Some examples are pottery, weaving, and quilting designs; architectural features such as windows, doors, roofs, hinges, tile floors, brick walls, railings, fences, bridge supports, or arches; many kinds of flowers and leaves; honeycomb; the cross section of an apple or a grapefruit; snowflakes; an open umbrella; letters of the alphabet; kaleidoscope designs; a pinwheel, windmill, or ferris wheel; some national flags (but not the U.S. flag); a ladder; a baseball diamond; or a stop sign.

Mathematicians use symmetries to reduce the complexity of a mathematical problem. In other words, one can apply what is known about just one of multiple symmetric pieces to other symmetric pieces and thereby learn more about the whole object. SEE ALSO Transformations.

Sonia Woodbury

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Telescope

There is much confusion and debate concerning the origin of the telescope. Many notable individuals appear to have simultaneously and independently discovered how to make a telescope during the last months of 1608 and the early part of 1609. Regardless of its origins, the invention of the telescope has led to great progress in the field of astronomy.

The Origin of the Telescope

Contrary to popular belief, Galileo Galilei (1564–1642) did not invent the telescope, and he was probably not even the first person to use this instrument in astronomy. Instead, the latter honor may be attributed to Thomas Harriot (1560–1621). Harriot developed a map of the Moon several months before Galileo began observations. Nevertheless, Galileo distinguished himself in the field through his patience, dedication, insight, and skill.

The actual inventor of the telescope may never be known with certainty. Its invention may have been by a fortuitous occurrence when some spectacle maker happened to look through two lenses at the same time. Several accounts report that Hans Lipperhey of Middelburg in the Netherlands had two lenses set up in his spectacle shop to allow visitors to look through them and see the steeple of a distant church. However, this story cannot be verified.

It is known that the first telescopes were shown in the Netherlands. Records show that in October 1608, the national government of the Netherlands examined the patent application of Lipperhey and a separate application by Jacob Metius of Alkmaar. Their devices consisted of a **convex** and **concave** lens mounted in a tube. The combination of the two lenses magnified objects by 3 or 4 times. However, the government of the Netherlands considered the devices too easy to copy to justify awarding a patent. The government did vote a small award to Metius and employed Lipperhey to devise binocular versions of his telescope. Another citizen of Middelburg, Zacharias Janssen, had also made a telescope at about the same time but was out of town when Lipperhey and Matius made their applications.

convex curved outward, bulging

concave hollowed out or curved inward

News of the invention of the telescope spread rapidly throughout Europe. Within a few months, simple telescopes, called "spyglasses," could be purchased at spectacle-maker's shops in Paris. By early 1609, four or five

telescopes had made it to Italy. By August of 1609, Thomas Harriot had observed and mapped the Moon with a six-power telescope.

What Galileo Discovered

Despite Harriot's honor as the first telescopic astronomer, it was Galileo who made the telescope famous. At the time, Galileo was Professor of Mathematics at the University of Padua in Italy. Somehow, he learned of the new instrument that had been invented in Holland, although there is no evidence that he actually saw one of the telescopes known to be in Italy. Over the next several months in 1609 and 1610, Galileo made several progressively more powerful and optically superior telescopes using lenses he ground himself. Galileo used these instruments for a systematic study of the night sky. He saw mountains and craters on the Moon, discovered four satellites of Jupiter, viewed sunspots, observed and recorded the phases of Venus, and found that the Milky Way galaxy consisted of clouds of individual stars.

Galileo summarized his discoveries in the book *Sidereus Nuncius* (*The Starry Messenger*) published in March of 1610. Others working at around the same time claimed to have made similar discoveries—others certainly observed sunspots—but Galileo gathered all of his observations together and wrote about them first. Consequently, he is generally credited with their discovery.

Ptolemaic theory the theory that asserted Earth was a spherical object at the center of the universe surrounded by other spheres carrying the various celestial objects

The observation of Venus's phases was especially important to Galileo. According to **Ptolemaic theory**, Venus would show crescent and "new" phases, but it would not go through a complete cycle of phases. The Ptolemaic model never placed Venus on the opposite side of the Sun as seen from Earth, so Venus would never appear "full." Yet Galileo clearly observed a nearly full Venus. He also observed that the four satellites of Jupiter orbited the planet, conclusively demonstrating that there was at least one instance of an orbital center other than Earth, a clear contradiction to the Ptolemaic model.

Adapting the Telescope

Galileo apparently had no real knowledge of how the telescope worked but he immediately recognized its military value, as well as its entertainment value. He set about building a version that is commonly known as a "Galilean telescope." It had a convex lens as a primary objective (the lens in front) and a concave lens as the eyepiece. The focal point of the objective lens was behind the eyepiece, and the eyepiece served primarily to form the upright image desired for terrestrial observation.

Johannes Kepler was arguably the first person to give a concise theory of how light passed through the telescope and formed an image. Kepler also discussed the various ways in which the lenses could be combined in different optical systems, improving on Galileo's design. Kepler's design used convex lenses for both the primary objective and the eyepiece. However, in spite of his theoretical understanding, there is no evidence that Kepler ever actually tried to put together a telescope.

Telescopes built following Kepler's design were not practical for military applications or everyday use because they inverted and reversed the images and showed people upside-down. However, their greater magnification,

brighter image, and wider angle of view made them best for astronomical observations where the inverted image made no difference. The telescope rapidly came into common astronomical use during the 20 years after it was invented.

Unfortunately, it soon became evident that the refracting telescope had a great disadvantage. The main problem with early telescopes was the low quality of their glass and the poor manner in which the lenses were ground. However, even the best lenses had two inherent defects. One defect resulted because the objective lens did not bend all wavelengths equally, and this resulted in the red part of the light-beam being brought to a focus at a greater distance from the objective. An image of a star viewed through an astronomical telescope from this period seemed to be surrounded by colored fringes. This defect is known as "chromatic aberration."

The other problem resulted when the surface of the lens was ground to a spherical shape. Rays passing through the edge of the lens were brought to a focus at a different distance than rays passing near the center of the lens. This defect is called "spherical aberration." A lens can be ground to a different shape (all modern optical instruments use "aspheric" lenses) but the lens grinders of Galileo's time did not possess the technology to do this.

One remedy for both chromatic and spherical aberrations was to make telescopes with extremely long focal length lenses (so that the lenses did not have much curvature), but this required telescopes several meters long. These telescopes were cumbersome and difficult to use. Another solution for chromatic aberration, unknown at the time, was to use an objective lens made from two different kinds of glass glued together. This method greatly reduces chromatic aberration.

Development of the Reflecting Telescope

During the 1680s, Cambridge University was often closed for fear of the Plague. When this occurred, physicist Isaac Newton would retreat to his country home in Lincolnshire. During one of these intervals, Newton began trying to unravel the problem of chromatic aberration.

Newton allowed a beam of sunlight to pass through a glass prism and observed that the beam was split into a rainbow of colors. On the basis of this and other experiments, he decided (incorrectly, it turns out) that the refractor could never be cured of chromatic aberration. Newton consequently developed a new type of telescope, "the reflector," in which there is no objective lens. The light from the object under observation is collected by a curved mirror, which reflects all wavelengths equally.

Newton likely did not originate the idea of a reflecting telescope. Earlier, in 1663, Scottish mathematician James Gregory had suggested the possibility of a reflector. However, Newton was apparently the first person to actually build a working reflector.

Newton created a reflecting telescope with a 2.5-cm (centimeter) metal mirror and presented it to the Royal Society in 1671. But using a reflecting mirror instead of a reflecting lens created another problem. The light beam is reflected back up the tube but it cannot be observed without blocking the light entering the tube. Gregory had suggested inserting a curved secondary mirror that would reflect the light back through a hole in the center of the

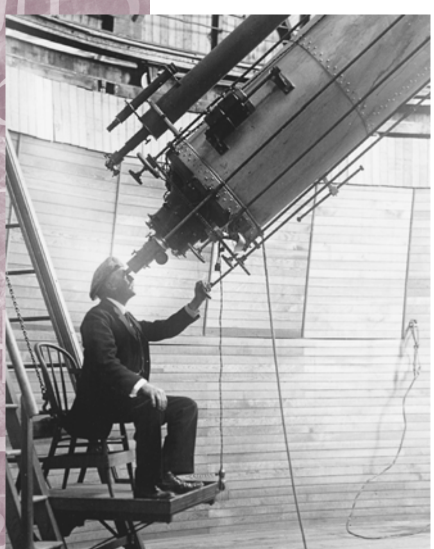
primary mirror. However, the technology needed to grind the complex mirror surfaces required for Gregory's design did not exist at the time. Newton solved this problem by introducing a second flat mirror in the middle of the tube mounted at a 45-degree angle so that the light beam is reflected out the side of the tube. The eyepiece was then mounted to the side of the tube. This design is still known as a "Newtonian reflector." Since Newton's time, several other reflecting telescope designs have been developed.

Newtonian reflectors were not free of problems. Metal mirrors were hard to grind. The mirror surface tarnished quickly and had to be polished every few months. These problems kept the Newtonian reflector from being widely accepted until after 1774, when new designs, polishing techniques, the use of silvered glass, and other innovations were developed by William Herschel. Herschel discovered the planet Uranus in 1781 using a telescope he had made. He continued to build reflecting telescopes over the next several years, culminating in an enormous 122-cm instrument completed in 1789.

Herschel's 122-cm telescope remained the largest in the world until 1845, when the Irish astronomer, William Parsons, the third Earl of Rosse, completed an instrument known as the Leviathan, which had a mirror diameter of 180 cm. Lord Rosse used this instrument to observe "spiral nebulae," which are now known to be other galaxies. Throughout the eighteenth and nineteenth centuries, telescopes of ever-increasing size were built.

Modern Telescopes

The twentieth century saw continued improvement in telescope size and design. Larger telescopes are preferred for two reasons. Larger instruments gather more light. Astronomical distances are so great that most objects are not visible to the unaided eye. The Andromeda galaxy (M31) is generally considered the most distant object that can be seen with the naked eye, and it is the closest galaxy to Earth outside of the Milky Way. To see very far out into space requires large telescope objectives. This is another reason for the general preference of astronomers for reflecting telescopes. It is easier to build large mirrors than it is to build large lenses.



Percival Lowell peers through a telescope at his namesake observatory in Flagstaff, Arizona. The historic observatory would be the site of a number of significant findings, among them Clyde Tombaugh's discovery of the planet Pluto in 1930.

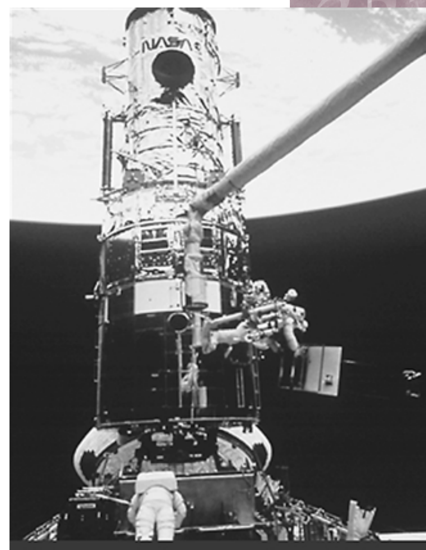
A second reason for the general trend toward large instruments is resolving power. The ability of a telescope to separate two closely spaced stars (or see fine detail in general) is known as resolving power. If R is the resolving power (in arc seconds), λ is the wavelength (in micrometers) and d is the diameter of the objective (in meters) then: $R = 0.25\,\frac{\lambda}{d}$. As d gets larger, R gets smaller (smaller is better).

During most of the twentieth century, astronomical images were recorded on photographic film. Later in the century, most observatories and research astronomers switched to solid state devices called CCDs (charge-coupled devices). These devices are much more sensitive to low light levels, and they also have the advantage of creating an electronic image that can be fed directly into a computer where it can be immediately processed.

Earth's atmosphere continues to challenge the progress of astronomy. Infrared and ultraviolet wavelengths do not pass through the atmosphere,

so astronomy in those parts of the spectrum is generally done by balloon-based telescopes or by satellites. A bigger problem with the atmosphere is its inherent instability. Even on the clearest of nights, images jiggle and quiver due to small variations in the atmosphere. One way to get around this problem is to get above the atmosphere.

The Hubble Space Telescope (named after Edwin Hubble, who discovered the galactic **redshift**) is a satellite based optical and infrared observatory. The spectacular images from "Hubble" have pushed back the frontiers of astronomical knowledge, while raising many new questions.



The Hubble Space Telescope was launched in 1990 as part of NASA's Great Observatories program. It is capable of recording images in the visible, near-ultraviolet, and near-infrared.

redshift motion-induced change in the frequency of light emitted by a source moving away from the observer

Ground-based large telescope design also continues to evolve. Very large mirrors suffer from many problems. To make them stiff, they must be very thick, which makes them very heavy. Modern telescopes use thin mirrors with many different supports that can be independently controlled by a computer. As the mirror is moved, the supports continually adjust to compensate for gravity, keeping the shape of the mirror within precise tolerances.

As of 2001, the largest telescope in the world is the twin mirror Keck Telescope. The Keck consists of two matched mirrors housed in separate buildings. Each telescope housing stands eight stories tall and each mirror weighs 300 tons. The mirrors are not made of one solid piece of glass. Instead, each mirror combines thirty-six 1.8-m (meter) hexagonal cells combined to form a collecting area equivalent to one 10-m mirror. Since the twin mirrors are separated by a wide distance, the Keck telescope has a much greater resolving power than either mirror alone would have.

The other advantage of the Keck telescope is its position on the top of an extinct volcano, Mauna Kea, in Hawaii. The atmosphere is very stable and the mountaintop, at 4 km (kilometers), is so high that the telescopes are above most of Earth's atmosphere. The European Southern Observatory in Chile is constructing a similar telescope that will combine light from four 8.2-m mirrors working as a single instrument. These and similar instruments around the world promise to reveal even more about our universe. SEE ALSO Astronomer.

Elliot Richmond

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Television Ratings

Suppose your favorite television program has just been cancelled because of bad ratings. In the world of television ratings, the number of people watching a show is the important factor, not the intrinsic, qualitative value of the show. So who is rating television shows, and how is it done?

Nielsen Media Research was founded in 1923 to measure the number of people listening to radio programs. Early on, the number of people lis-

tening to the radio was low enough to actually count. However, with the advent of television, cable, satellite, and digital television, the number of viewers has greatly increased. In fact, counting the number of viewers of a television show is impossible, so the Nielsen company has devised a way to estimate the number of viewers.

How Ratings Are Determined

Nielsen Media Research measures a sample of television viewers to determine when and what they are watching. Nielsen uses a sample of more than 5 thousand households, which is over 13 thousand people. With close to 100 million households in the United States, one might wonder whether the sample of 5,000 households is large enough to provide a cross-section of the whole market. The company illustrates the sampling procedure with this example:

> . . .You don't need to eat an entire pot of vegetable soup to know what kind of soup it is. A cup of soup is more than adequate to represent what is in the pot. If, however, you don't stir the soup to make sure that all of the various ingredients have a good chance of ending up in the cup, you might just pour a cup of vegetable broth. Stirring the soup is a way to make sure that the sample you draw represents all the different parts of what is in the pot.

A truly random list of households will statistically provide a variety of demographic groups. Using demographic information from the U.S. Census Bureau, Nielsen Media Research is able to compare characteristics of the entire population with the characteristics of the sample population. Their findings indicate that the two samples compare favorably. In addition, Nielsen Media Research occasionally completes special tests in which thousands of randomly selected telephone numbers are called, and the people are asked if their televisions are on and who is watching. This information provides a check on television usage and viewing.

The Nielsen company provides information on the amount of television set usage, the programs viewed, the commercials watched, and the people watching. To measure television set usage in sample homes, the Nielsen company installs meters to televisions which automatically log when the sets are on and to which programs the sets are tuned. To measure which programs are being watched, the Nielsen company collects network and local programming information in order to know what programs are being shown on any given television set in a sample home. Each commercial aired is marked with an invisible "fingerprint" that enables trackers to determine when and where commercials are aired. Combining this information with the programming information allows trackers to note which commercials aired during which television programs. The Nielsen company provides each sample home with what is known as a "people meter" in addition to the television meter. The people meter is a box with a button for each member of the household and one for visitors. When a viewer begins watching television, he or she presses a button that logs in the television usage. When the program is completed, the viewer again presses the button to register the end the viewing time.

In addition to measuring national network programs, Nielson Media Research also measures local viewership. The Nielsen company uses pro-

gram diaries to measure the local audiences. These diaries are records of who in the sample group watches what programs and what channels during a one-week period. Diaries are collected in 210 television markets. In addition, in the 48 largest television markets, there is a sample of homes with set meters that record when the television is on and the channel to which it is tuned. Homes with set meters installed on television sets do not have people meters installed.

Using Ratings

This information regarding what television shows are being watched and by whom is used by two industries: marketing companies and television production companies. Marketing companies use the information to determine the demographic characterization of the viewers. The information generates questions such as: Are men or women watching the show? Is the audience mainly older adults or teenagers? Does this show appeal to minorities?

The television show "The Crocodile Hunter" relies on the outlandish adventures of its host to attract viewers and boost ratings.

Marketing companies want this information in order to produce commercials that will appeal to the group that is watching a given program. The television production companies cannot pay more for a program than they earn from selling advertising during the program. The larger the audience of a particular program, the more the commercial time during that program is worth. The basic formula for commercial time is a negotiated rate per thousand viewers, multiplied by the Nielsen audience estimate. Nielsen ratings are also used by public broadcasting stations, enabling them to learn who their audience is and, therefore, to make more informed programming decisions.

All that is left to understand, is who are the Nielsen families and how are they selected? The sample population includes homes in all fifty states—in cities, towns, suburbs, and rural areas. Some households include renters; others own their home. These households may or may not include children. Various ethnic and income groups are represented. Overall, the sample population is very similar to the true population of the United States. The selection of the sample families is random; therefore it is not possible to increase one's chances. In fact, including all those who ask to be included would skew the sample, making it nonrepresentative of the population at large. Only by chance can a household become a member of the Nielsen family. SEE ALSO Randomness; Statistical Analysis.

Susan Strain Mockert

Internet Resources

Who We Are and What We Do. Nielsen Media Research. <http://www.nielsenmedia.com>.

Temperature, Measurement of

From the Eskimo in Alaska to the Nigerian living near the equator, people of various cultures are exposed to variations in temperature. Our early experiences help us to develop the concept of temperature as a measure of how hot or cold something is. Warmth may be associated with a season of the year or an object such as a stove or a fireplace. The measurement of

temperature is important to everyday life, providing information about our health and regulating our outdoor activities with accurate weather reports.

Defining and Measuring Temperature

Temperature is the number assigned to an object to indicate its warmth. The concept of temperature came about because people wanted to quantify and measure differences in warmth. When an object with a higher temperature comes in contact with a cooler object, a transfer of heat occurs until the two objects are the same temperature. When the heat transfer is complete, it can be said that that the two objects are in thermal equilibrium. Temperature can hence be defined as the quantity of warmth that is the same for two or more objects that are in thermal equilibrium. The temperature 0° C, 273.16 K (kelvin), is the point at which ice, water, and water vapor are all present and in thermal equilibrium. This is known as the triple point of water. Absolute zero temperature occurs when the motion of atoms and molecules practically stops, which occurs at −273° C, or 0 K.

Accurate temperature readings are necessary in maintaining meteorological records. Meteorology is the science that deals with the study of weather, which is the condition of the atmosphere at a specific time and place. A meteorologist is a professional who studies and forecasts the weather. The accuracy of weather forecasts is dependent upon collecting data that includes temperature. Air temperature is measured by using a thermometer.

In this modern replica of a Galilean thermometer (also known as a slow thermometer), differently sized weights float within a liquid. Because a cool liquid is denser than a warm one, it supports more weight and more floats will rise to the top of the cylinder. The lowest of the floating weights indicates the room temperature.

Evolution of the Thermometer

Thermometers are instruments that are used to measure degrees of heat. Ferdinand II, Grand Duke of Tuscany, used the first sealed alcohol-in-glass thermometer in 1641. Robert Hook used a thermometer containing red dye in alcohol in 1664. Hook's thermometer used water as the fixed freezing point and became the standard thermometer that was used by Gresham College and the Royal Society until 1709.

In 1702, Ole Roemer of Copenhagen developed a thermometer that used a scale with two fixed points: snow and boiling water. This thermometer became a calibrated instrument that contained a visible substance, either mercury or alcohol, that traveled along a narrow passageway in a tube that used two-fixed points on a scale. The passageway through the tube is called a bore. At the bottom of the bore is a bulb that contains the liquid. The thermometer operates on the principle that these fluids expand (swell) when heated and contract (shrink) when cooled. When the liquid is warmed, it expands and moves up the bore, but when the liquid cools, it contracts and moves down the bore in the opposite direction. When a thermometer is placed in contact with an object and reaches thermal equilibrium, we have a quantitative measure for the temperature of the object. For instance, when a thermometer is placed under the arm of an infant, heat is transferred until thermal equilibrium is reached. Thus, when you observe how much the mercury or alcohol has expanded in the bore, you can find the baby's temperature by reading the scale on the thermometer.

Other thermometers besides alcohol and mercury thermometers are metal thermometers, digital thermometers, and strip thermometers. Rather than containing liquid, the metal thermometer uses a strip of metal made

up of two different heat-sensitive metals that have been welded together. A thermograph is a metal thermometer that records the temperature continuously all day. However, today's meteorologists use electronic computers rather than thermographs to obtain permanent records. Digital thermometers are highly sensitive and give precise readings on electronic meters. The strip thermometer is a celluloid tape made with heat-sensitive liquid crystal chemicals that react by changing the color of the tape according to the temperature. Strip thermometers are useful in taking the temperature of infants because one only need put the strip on the baby's forehead for a few seconds to obtain an accurate temperature reading.

Thermometer Scales

The scale on the thermometer tells us how high the liquid is in the thermometer and gives the temperature of whatever is around the bulb. Thus, the thermometer can be used to measure heat in any object including solids, liquids, and gases. The scale on a thermometer is divided equally with divisions called degrees. The most commonly used scales are Fahrenheit (F) and Celsius (C). The Fahrenheit scale (English system of measurement) is used in the United States while the Celsius scale (metric system of measurement) is used by most other countries.

Gabriel Fahrenheit (1685–1736), a German physicist, developed the Fahrenheit scale, which was first used in 1724. In 1745, Carolus Linnaeus of Sweden described a scale of a hundred steps or centigrade, which became known as the Centigrade scale. In 1948, the term "centigrade" was changed to Celsius. The Celsius scale was named after the Swedish scientist, Anders Celsius (1701–1744).

The Celsius scale was used to develop the Kelvin scale. William Thomson (1824–1907), also known as Lord Kelvin, was a British physicist and proposed the Kelvin scale in 1848. The scale is based on the concept that a gas will lose 1/273.16 of its volume for every one-degree drop in Celsius temperature. Thus, the volume would become zero at -273.16 degrees Celsius—a temperature known as absolute zero. Thus, the Kelvin scale is used primarily to measure gases.

Mathematics is used to convert temperatures from one scale to another. To convert from Celsius to Fahrenheit, multiply by 1.8 and add 32 degrees ($°F = 1.8°C + 32$). To convert from Fahrenheit to Celsius, subtract 32 degrees and divide by 1.8 ($°C = \frac{(°F - 32)}{1.8}$). To convert from Celsius to Kelvin, simply add 273 degrees to the Celsius temperature ($K = °C + 273$). SEE ALSO ABSOLUTE ZERO; MEASUREMENT, TOOLS OF; METEROLOGY, MEASUREMENTS IN.

Jacqueline Leonard

WATER AND TEMPERATURE SCALES

Water is the medium that is used as an international standard for determining temperature scales. On the Fahrenheit scale, water freezes at 32 degrees and boils at 212 degrees. Water freezes at 0 degrees and boils at 100 degrees Celsius.

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Tessellations

plane generally considered an undefinable term, a plane is a flat surface extending in all directions without end, and that has no thickness

For centuries, mathematicians and artists alike have been fascinated by tessellations, also called tilings. Tessellations of a **plane** can be found in the regular patterns of tiles in a bathroom floor, or flagstones in a patio. They are also widely used in the design of fabric or wallpaper. Tessellations of three-dimensional space play an important role in chemistry, where they govern the shapes of crystals.

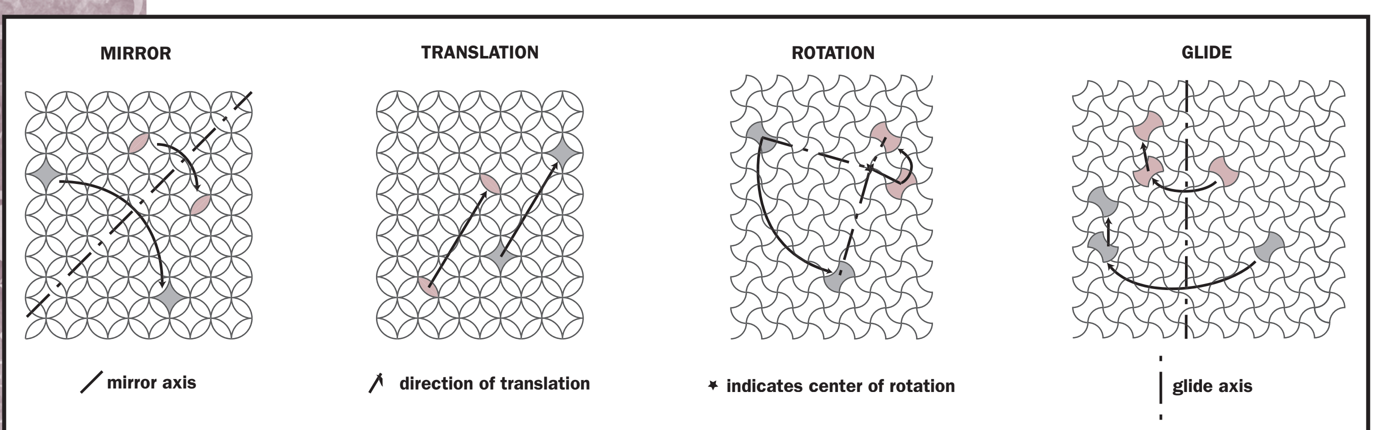
translate to move from one place to another without rotation

A tessellation of a plane (or of space) is any subdivision of the plane (or space) into regions or "cells" that border each other exactly, with no gaps in between. The cells are usually assumed to come in a limited number of shapes; in many tessellations, all the cells are identical. Tessellations allow an artist to **translate** a small motif into a pattern that covers an arbitrarily large area. For mathematicians, tessellations provide one of the simplest examples of symmetry.

In everyday language, the word "symmetric" normally refers to an object with dihedral or mirror symmetry. That is, a mirror can be placed exactly in the middle of the object and the reflection of the mirrored half is the same as the half not mirrored. Such is the case with the leftmost tessellation in the figure. If an imaginary mirror is placed along the axis shown, then every seed-shaped cell, such as the one shown in color, has an identical mirror image on the other side of the axis. Likewise, every diamond-shaped cell has an identical diamond-shaped mirror image.

Mirror symmetry is not the only kind of symmetry present in tessellations. Other kinds include *translational* symmetry, in which the entire pattern can be shifted; *rotational* symmetry, in which the pattern can be rotated about a central point; and *glide* symmetry, in which the pattern can first be reflected and then shifted (translated) along the axis of reflection. Examples of these three kinds of symmetry are shown in the other three blocks of the figure. In each case, the tessellation is called *symmetric* under a transformation only if that transformation moves every cell to an exactly matching cell.

Examples of symmetry in tessellations show that the transformation moves each cell to an identical cell.



In the rightmost block of the figure, the tessellation has glide symmetry but does not have mirror symmetry because the mirror images of the shaded cells overlap other cells in the tessellation.

The collection of all the transformations that leave a tessellation unchanged is called its *symmetry group.* This is the tool that mathematicians traditionally use to classify different types of tilings. The classification of patterns can be further refined according to whether the symmetry group contains translations in one dimension only (a frieze group), in two dimensions (a wallpaper group), or three dimensions (a crystallographic group). Within these categories, different groups can be distinguished by the number and kind of rotations, reflections, and glides that they contain. In total, there are seven different frieze groups, seventeen wallpaper groups, and 230 crystallographic groups.

Exploring Tessellations

To an artist, the design of a successful pattern involves more than mathematics. Nevertheless, the use of symmetry groups can open the artist's eyes to patterns that would have been hard to discover otherwise. A stunning variety of patterns with different kinds of symmetries can be found in the decorations of tiles at the Alhambra in Spain, built in the thirteenth and fourteenth centuries.

In modern times, the greatest explorer of tessellation art was M. C. Escher. This Dutch artist, who lived from 1898 to 1972, enlivened his woodcuts by turning the cells of the tessellations into whimsical human and animal figures. Playful as they appear, such images were based on a deep study of the seventeen (two-dimensional) wallpaper groups.

One of the most fundamental constraints on classical wallpaper patterns is that their rotational symmetries can only include half-turns, one-third turns, quarter-turns, or one-sixth turns. This constraint is related to the fact that regular triangles, squares, and hexagons fit together to cover the plane, whereas regular pentagons do not.

However, one of the most exciting developments of recent years has been the discovery of simple tessellations that do exhibit five-fold rotational symmetry. The most famous are the Penrose tilings, discovered by English mathematician Roger Penrose in 1974, which use only two shapes, called a "kite" and a "dart." Three-dimensional versions of Penrose tilings* have been found in certain metallic alloys. These "non-periodic tilings," or "quasicrystals," are not traditional wallpaper or crystal patterns because they have no translational symmetries. That is, if the pattern is shifted in any direction and any distance, discrepancies between the original and the shifted patterns appear.

*A Penrose tiling was used to design a mural in the science building of Denison University in Granville, Ohio.

Nevertheless, Penrose tilings have a great deal of long-range structure to them, just as ordinary crystals do. For example, in any Penrose tiling the ratio of the number of kites to the number of darts equals the "golden ratio," 1.618. . .. Mathematicians are still looking for new ways to explain such patterns, as well as ways to construct new non-periodic tilings. SEE ALSO Escher, M. C.; Tessellations, Making; Transformations.

Dana Mackenzie

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Tessellations, Making

The Dutch artist Maurits Cornelius Escher was, more than anyone else, responsible for bringing the art of tessellations to the public eye. True, tessellations have been used for decorative purposes for centuries, but the cells were usually very bland: squares, equilateral triangles, regular hexagons, or **rhombuses**. The visual interest of such a tessellation lies in the pattern as a whole, not in the individual cells. But Escher discovered that the cells themselves could become an interesting part of the pattern. His tessellations feature cells in the shape of identifiable figures: a bird, a lizard, or a rider on horseback.

rhombus a parallelogram whose sides are equal

To create an Escher-style tessellation, start with one of the polygons mentioned above. (This is not essential, but it is easiest.) Next, choose two sides and replace them with any curved or polygonal path. The only rule is that anything that is added to the polygon on one side has to be taken away from the other. If a protruding knob has been added to one side, the other side must have an identically shaped dimple.

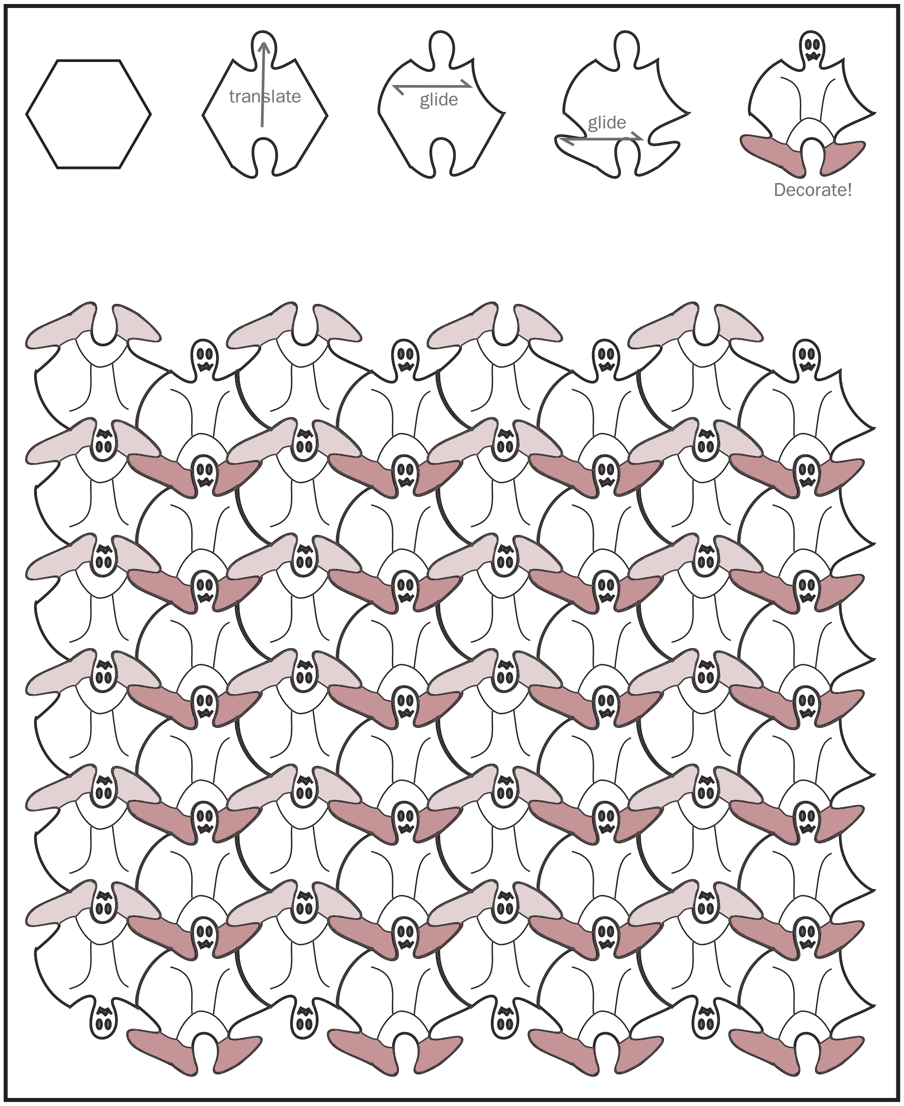
This procedure can be repeated with other pairs of sides. Usually, the result will vaguely resemble something, perhaps a lizard. The outline can then be revised so that the result becomes more recognizable. But every time one side is modified, it is imperative that its corresponding side is modified in precisely the same way. This makes it nearly impossible to draw a convincing lizard, but by adding a few embellishments—eyes, scales and paws—a reasonable caricature can be produced. Escher, thanks to his years of experience, was a master at this last step.

The figure illustrates the process of making tessellations. Starting with a hexagon, the top and bottom sides are replaced with identical curves that are related by a **translation**. The upper curve suggests a head, and the bottom curve suggests two heels. Next the top left and top right sides are replaced with two identical curves that are related by a **glide reflection**. (Note: A translation will only work if the two sides are parallel.) One curve bulges outward and the other inward, in accordance with the rule that anything removed from one side must be given back to the other.

translation to move from one place to another without rotation

glide reflection a rigid motion of the plane that consists of a reflection followed by a translation parallel to the mirror axis

Then the bottom left and bottom right sides are replaced with S-shaped curves that represent the feet and legs. Again, these two curves are identical and related by a glide reflection. Finally, the figure is decorated so that it looks a little bit like a ghost wearing boots. (The complete tessellation is shown here, but obviously Escher did it better.)



The figure shows how to transform a hexagon into a pattern that will tile the plane. The completed tessellation is decorated to suggest ghosts with boots.

Another way of creating interesting tessellations has been called the kaleidoscope method. In this method, one begins with a triangular "primary cell," which may have angles of 60°, −60°, −60°; 30°, −60°, −90°; or 45°, −45°, −90°. After it is decorated in any way, the triangle should be reflected through one side after another until a hexagon or a square is completed. Finally, this figure can be used as a tile to cover the plane.

The kaleidoscope method can be modified by using **rotations** about the midpoint of a side or about a vertex, instead of **reflections**, to generate the additional copies of the decorated primary cell. Different patterns of rotation or reflection can yield a dazzling variety of different tessellations, all arising from the same primary cell. SEE ALSO Escher, M. C.; Tessellations.

Dana Mackenzie

rotation a rigid motion of the plane that fixes one point (the center of rotation) and moves every other point around a circle centered at that point

reflection a rigid motion of the plane that fixes one line (the mirror axis) and moves every other point to its mirror image on the opposite side of the line

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Time, Measurement of

The history of time measurement is the story of the search for more consistent and accurate ways to measure time. Early human groups recorded the phases of the Moon some 30,000 years ago, but the first minutes were counted accurately only 400 years ago. The atomic clocks that allowed mankind to track the approach of the third millennium (in the year 2001) by a billionth of a second are less than 50 years old. Thus, the practice and accuracy of time measurement has progressed greatly through humanity's history.

The study and science of time measurement is called horology. Time is measured with instruments such as a clock or calendar. These instruments can be anything that exhibits two basic components: (1) a regular, constant, or repetitive action to mark off equal increments of time, and (2) a means of keeping track of the increments of time and of displaying the result.

Imagine your daily life—getting ready in the morning, driving your car, working at your job, going to school, buying groceries, and other events that make up your day. Now imagine all the people in your neighborhood, in your city, and in your country doing these same things. The social interaction that our existence involves would be practically impossible without a means to measure time. Time can even be considered a common language between people, and one that allows everybody to proceed in an orderly fashion in our complicated and fast-paced world. Because of this, the measurement of time is extremely important to our lives.

The Origins of Modern Time Measurement

The oldest clock was most likely Earth as it related to the Sun, Moon, and stars. As Earth rotates, the side facing the Sun changes, leading to its apparent movement from one side of Earth, rising across the sky, reaching a peak, falling across the rest of the sky, and eventually disappearing below Earth on the opposite side to where it earlier appeared. Then, after a period of darkness, the Sun reappears at its beginning point and makes its journey again. This cyclical phenomenon of a period of brightness followed by a period of darkness led to the intervals of time now known as a day. Little is known about the details of timekeeping in prehistoric eras, but wherever records and artifacts are discovered, it is found that these early people were preoccupied with measuring and recording the passage of time.

A second cyclical observation was likely the repeated occurrence of these days, followed by the realization that it took about 30 of these days (actually, a fraction over 27 days) for the Moon to cycle through a complete set of its shape changes: namely, its main phases of new, first quarter, full, and last quarter. European ice-age hunters over 20,000 years ago scratched lines and gouged holes in sticks and bones, possibly counting the days between phases of the Moon. This timespan was eventually given the name of "month." Similarly, the sum of the changing seasons that occur as Earth orbits around the Sun gave rise to the term "year."

For primitive peoples it was satisfactory to divide the day into such things as early morning, mid-day, late afternoon, and night. However, as societies became more complex, the need developed to more precisely divide the day.

The modern convention is to divide it into 24 hours, an hour into 60 minutes, and a minute into 60 seconds.

The division into 60 originated from the ancient Babylonians (1900 B.C.E.–1650 B.C.E.), who attributed mystical significance to multiples of 12, and especially to the multiple of 12 times 5, which equals 60. The Babylonians divided the portion that was lit by the Sun into 12 parts, and the dark interval into 12 more, yielding 24 divisions now called hours.

Ancient Arabic navigators measured the height of the Sun and stars in the sky by holding their hand outstretched in front of their faces, marking off the number of spans. An outstretched hand **subtends** an angle of about 15 degrees at eye level. With 360 degrees in a full circle, 360° divided by 15° equals 24 units, or 24 hours. Babylonian mathematicians also divided a complete circle into 360 divisions, and each of these divisions into 60 parts. Babylonian astronomers also chose the number 60 to subdivide each of the 24 divisions of a day to create minutes, and each of these minutes were divided into 60 smaller parts called seconds.

MARKING CELESTIAL TIME

Some archeologists believe the complex patterns of lines on the desert floor in Nazca, Peru were used to mark celestial time. Medicine Wheels—stone circles found in North America dating from over 1,000 years ago—also may have been used to follow heavenly bodies as they moved across the sky during the year.

subtend to extend past and mark off a chord or arc

Early Time-Measuring Instruments

The first instrument to measure time could have been something as simple as a stick in the sand, a pine tree, or a mountain peak. The steady shortening of its shadow would lead to the noon point when the Sun is at its highest position in the sky, and would then be followed by shadows that lengthen as darkness approaches. This stick eventually evolved into an obelisk, or shadow clock, which dates as far back as 3500 B.C.E. The Egyptians were able to divide their day into parts comparable to hours with these slender, four-sided monuments (which look similar to the Washington Monument). The moving shadows formed a type of sundial, enabling citizens to divide the day into two parts.

Sundials evolved from flat horizontal or vertical plates to more elaborate forms. One version from around the third century B.C.E.) was the hemispherical dial, or hemicycle, a half-bowl-shaped depression cut into a block of stone, carrying a central vertical pointer and marked with sets of hour lines for different seasons.

Apparently 5,000 to 6,000 years ago, civilizations in the Middle East and North Africa initiated clock-making techniques to organize time more efficiently. Ancient methods of measuring hours in the absence of sunlight included fire clocks, such as the notched candle and the Chinese practice of burning a knotted rope. All fire clocks were of a measured size to approximate the passage of time, noting the length of time required for fire to travel from one knot to the next. Devices almost as old as the sundial include the hourglass, in which the flow of sand is used to measure time intervals, and the water clock (or "clepsydra"), in which the water flow indicates the passage of time.

The Mechanization of Clocks

By about 2000 B.C.E., humans had begun to measure time mechanically. Eventually, a weight falling under the force of gravity was substituted for the flow of water in time devices, a precursor to the mechanical clock. The first recorded examples of such mechanical clocks are found in the four-

This sundial in a Bolivian town in 1986 illustrates the endurance and utility of ancient time-measuring devices.

teenth century. A device, called an "escapement," slowed down the speed of the falling weight so that a cogwheel would move at the rate of one tooth per second.

The scientific study of time began in the sixteenth century with Italian astronomer Galileo Galilei's work on pendulums. He was the first to confirm the constant period of the swing of a pendulum, and later adapted the pendulum to control a clock. The use of the pendulum clock became popular in the 1600s when Dutch astronomer Christiaan Huygens applied the pendulum and balance wheel to regulate the movement of clocks. By virtue of the natural period of oscillation from the pendulum, clocks became accurate enough to record minutes as well as hours.

Although British physicist Sir Isaac Newton continued the scientific study of time in the seventeenth century, a comprehensive explanation of time did not exist until the early twentieth century, when Albert Einstein proposed his theories of relativity. Einstein defined time as the fourth dimension of a four-dimensional world consisting of space (length, height, depth) and time.

Quartz-crystal clocks were invented in the 1930s, improving timekeeping performance far beyond that of pendulums. When a quartz crystal is placed in a suitable electronic circuit, the interaction between mechanical stress and electric field causes the crystal to vibrate and generate a constant frequency that can be used to operate an electronic clock display.

But the timekeeping performance of quartz clocks has been substantially surpassed by atomic clocks. An atomic clock measures the frequency of electromagnetic radiation emitted by an atom or molecule. Because the atom or molecule can only emit or absorb a specific amount of energy, the radiation emitted or absorbed has a regular frequency. This allowed the National Institute of Standards and Technology to establish the second as the amount of time radiation would take to go through 9,192,631,770 cycles at the frequency emitted by cesium atoms making the transition from one state

to another. Cesium clocks are so accurate that they will be off by only one second after running for 300 million years.

Time Zones

In the 1840s, the Greenwich time standard was established with the center of the first time zone set at the Royal Greenwich Observatory in England, located on the 0-degree **longitude meridian.** In total, twenty-four time zones—each fifteen degrees wide—were established **equidistant** from each other east and west of Greenwich's prime meridian. Today, when the time is 12:00 noon in Greenwich, it is 11:00 A.M. inside the next adjoining time zone to the west, and 1:00 P.M. inside the next adjoining time zone to the east.

The term "A.M." means *ante meridiem* ("before noon"), while "P.M." means *post meridiem* ("after noon"). But military time is measured differently. The military clock begins its day with midnight, known as either or 0000 hours ("zero hundred hours") or 2400 hours ("twenty-four hundred hours"). An early-morning hour such as 1:00 A.M. is known in military time as 0100 hours (pronounced "oh one hundred hours"). In military time, 12:00 noon is 1200 ("twelve-hundred hours"). An afternoon or evening hour is derived by adding 12; hence, 1:00 p.m. is known as 1300 ("thirteen-hundred hours"), and 10:00 p.m. is known as 2200 ("twenty-two hundred hours"). SEE ALSO Calendar, Numbers in the.

William Arthur Atkins (with Philip Edward Koth)

ATOMIC TIME?

Variations and vagaries in the Earth's rotation eventually made astronomical measurements of time inadequate for scientific and military needs that required highly accurate timekeeping. Today's standard of time is based on atomic clocks that operate on the frequency of internal vibrations of atoms within molecules. These frequencies are independent of the Earth's rotation, and are consistent from day to day within one part in 1,000 billion.

longitude one of the imaginary great circles beginning at the poles and extending around Earth; the geographic position east or west of the prime meridian

meridian a great circle passing through Earth's poles and a particular location

equidistant at the same distance

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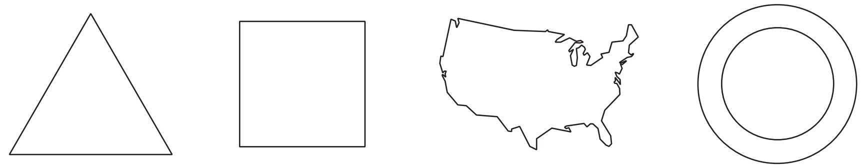
Topology

Topology is sometimes called "rubber-sheet geometry" because if a shape is drawn on a rubber sheet (like a piece of a balloon), then all of the shapes you can make by stretching or twisting—but never tearing—the sheet are considered to be topologically the same.

Topological properties are based on *elastic* motions rather than rigid motions like rotations or inversions. Mathematicians are interested in the qualities of figures that remain unchanged even when they are stretched and distorted. The qualities that are unchanged by such transformations are said to be topologically invariant because they do not vary, or change, when stretched.

A MUSICAL COMPARISON

A soprano and a baritone can sing the same song even though one sings high notes and the other low notes. Shifting a song up or down the musical scale changes some qualities of the music but not the pattern of notes that creates the song. Similarly, topology deals with variations that occur without changing the underlying "melody" of the shape.



As an example, the figure shows a triangle, a square, a rough outline of the United States, and a ring. The first three shapes are topologically equivalent; we can stretch and pull the boundary of the square until it becomes a circle or the U.S. shape. But no matter how much we pull or stretch this basic outline we cannot make it look like a ring.

Since a triangle is topologically the same as a square, and a sphere is the same as a cone, the idea of angle, length, perimeter, area, and volume play no role in topology. What remains the same is the number of boundaries that a shape has. A triangle has an inside and an outside separated by a closed boundary line. Every possible distortion of a triangle will also have an inside and an outside separated by a boundary. The ring, on the other hand, has two boundaries forming an inside region separated from two disconnected outside regions. No matter how you transform a ring it will always have two boundaries, one inside region, and two outside regions.

It can be quite challenging and surprising to discover whether two shapes are topologically the same. For example, a soda bottle is the same as a soup bowl, which is the same as a dinner plate. A coffee cup and a donut are topologically the same. But a coffee cup is topologically different from a soup bowl because the hole in the cup's handle does not occur in the bowl.

Because topology treats shapes so differently from the way we are accustomed to thinking about them, some of the most interesting objects studied in topology may seem very strange. One of the most well known objects is called the Möbius Strip, named for the German mathematician August Ferdinand Möbius who first studied it. This curious object is a two-dimensional surface that has only one side. A Möbius Strip can be easily constructed by taking the two ends of a long, rectangular strip of paper, giving one end a half twist, and gluing the two ends together. The Klein Bottle, which theoretically results from sewing together two Möbius Strips along their single edge, is a bottle with only one side. In our three-dimensional world, it is impossible to construct because a true Klein Bottle can exist only in four dimensions.

The concepts of sidedness, boundaries, and invariants have been generalized by topologists to higher dimensions. Although difficult to visualize, topologists will talk about surfaces in four, five, and even higher dimensions. While much of the study of topology is theoretical, it has deep connections to relativity theory and modern physics which also imagine our universe as having more than three dimensions. SEE ALSO Dimensions; Mathematics, Impossible; Möbius, August Ferdinand.

Alan Lipp

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Toxic Pollutants, Measuring

The amount of pollution in our environment continues to pose a challenge for industry, business, and decisionmakers. Some of the major chemicals of concern are those that harm the environment on a large scale, such as chlorofluorocarbons, which destroy the atmospheric ozone layer. Other chemicals that are harmful to human health are lead in water, cyanide (a deadly poison) leaching from landfills into water supplies, ammonia from car exhaust, and a long list of carbon-based chemicals that are byproducts of industrial processes.

Pollutants may start out as a solid, liquid, or gas, but eventually are counted as individual molecules within another substance. For instance, if a contaminant contains many toxic chemicals, each one will be measured separately. Some chemicals are more toxic to living organisms than others, so they are measured independently to discover their presence in the environment.

A toxic substance is usually measured in "parts per million" or "ppm." This measure states that the number of units of the particular chemical under survey occurs in comparison to one million units of the surrounding natural chemicals. When amounts of toxic chemicals are measured over large areas over a specific period of time, the amounts of pollutants can be measured in pounds.

An environmental technician monitors the underground water level at an abandoned gas station. These sites may contain underground tanks of hazardous petroleum products.

Categories of Toxic Pollutants

Identifying dangerous wastes can be a complicating factor in measuring certain pollutants. There are four major categories of toxic pollutants that cover a wide range of materials.

The first major grouping of pollutants is based on ignitability, or its ability to catch on fire or explode. These are wastes and pollutants that may be a fire hazard if they collect in landfills or other storage sites. Liquids and solids are not usually ignitable, but the vapors or fumes they make can be very dangerous. Each type of ignitable pollutant has a flash point. The flash point is the temperature at which the flammable vapors will interact with a spark or flame to create an explosion. Each pollutant has its own flash point which has been determined in a laboratory.

Another measure of toxic chemicals is their corrosivity, which is a chemical process in which metals and minerals are converted into undesirable byproducts. One of the ways in which corrosiveness is measured is by the pH, which is an indicator of the acidity or alkalinity of solution.

Mathematically, pH is the negative logarithm of the hydrogen ion (a single proton with no electron) concentration in water. The pH scale ranges from 1 to 14. A pH of between 1 and 3 is highly acidic while values between 11 and 14 indicate a highly alkaline solution. A pH of 7 is considered neutral.

The third general category of pollutants is based on the reactivity of the substance. Reactive chemicals are identified as pollutants because they may explode or make toxic fumes that pose a threat to human health and the environment. Soils are often acidic or basic, so the potential reactivity of chemicals in soils is of great concern. Among other properties, reactive pollutants:

- readily undergo violent chemical change;
- react violently or form potentially explosive mixtures with water; and
- generate toxic fumes when exposed to mildly acidic or basic conditions.

The last of the four general categories of toxic pollutants is based on the toxicity of substances. Toxicity is the ability of a substance to produce adverse health effects by exposure to the substance.

Measuring Toxic Pollutants

An example of how some pollutants are measured, including ignitable and toxic substances, is the Toxicity Characteristic Leaching Procedure (TCLP). This procedure and others, sometimes called Extraction Procedures (EP), are based on how a leaching pollutant may decompose and distribute to the water supply or soil from a landfill containing city solid waste. The major concern about the leaching process is the possibility that toxic chemicals may reach the groundwater and migrate to other sources of drinking water.

There are numerous chemical processes used to obtain a sample for analysis. These are many stepped (phased) processes and all have very clear guidelines on how they are to be performed. Once the TCLP extract has been obtained it is immediately tested for pH. All individual chemicals are analyzed separately. The initial analysis starts with the following formula:

$$\text{Final Analyte Concentration} = \frac{(\text{V1})(\text{C1}) + (\text{V2})(\text{C2})}{\text{V1} + \text{V2}}$$

where

V1 = volume of the first phase (L)

C1 = concentration of the analyte of concern in the first phase (mg/L)

V2 = volume of the second phase (L)

and

C2 = concentration of the analyte of concern in the second phase (mg/L).

To assure quality control over of initial measurements there are many additional mathematical and chemical measurement techniques that are performed after the first analysis. In one type of quality control one in every 20 samples is remeasured. This particular sample is called spike. It is measured against a sample without the potential toxin in it (the control). The spikes are calculated by the following formula:

$$\text{Percent R (percent recovery)} = 100\ (X_s - X_u)/K$$

where

X_s = measured value for the spiked sample

X_u = measured value for the unspiked sample

and

K = known of the spike in the sample (this comes from the first or known measurement of the sample).

SEE ALSO Logarithms.

Brook E. Hall

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Transformations

A transformation is a mathematical function that repositions points in a one-dimensional, two-dimensional, three-dimensional, or any other *n*-dimensional space. In this article, only transformations in the familiar two-dimensional rectangular coordinate plane will be discussed.

Transformations map one set of points onto another set of points, generally with the purpose of changing the position, size, and/or shape of the figure made up by the first set of points. The first set of points, from the **domain** of the transformation, is called the set of pre-images, whereas the second set of points, from the **range** of the transformation, is called the set of images. Therefore, a transformation maps each pre-image point to its image point.

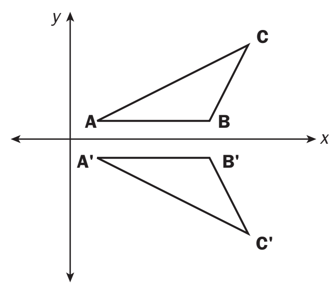
domain the set of all values of a variable used in a function

range the set of all values of a variable in a function mapped to the values in the domain of the independent variable

Reflections

Reflections are transformations that involve "flipping" points over a given line; hence, this type of transformation is sometimes called a "flip." When a figure is reflected in a line, the points on the figure are mapped onto the points on the other side of the line which form the figure's mirror image.

For example, in the first figure below, the triangle ABC, the pre-image figure, is reflected in the x-axis to produce the image triangle A'B'C'. Note that if triangle ABC is traversed from A to B to C and back to A, the direction of this movement is counterclockwise. If triangle A'B'C' is traversed from A' to B' to C' and back to A', the direction is clockwise. This "reversal of orientation" is similar to the way images in a mirror are reversed, and is a fundamental property of all reflections.

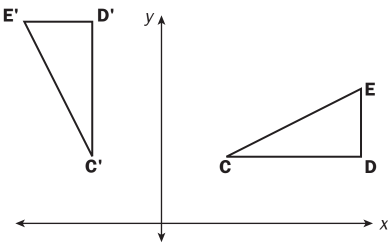


In the case of the reflection in the x-axis, as seen in the first figure, the first coordinate of each image point is the same as the first coordinate of its pre-image point, but the second coordinate of any image point is the opposite, or negative, of the second coordinate of its pre-image point. Mathematically, we say that for a reflection in the x-axis, pre-image points of the form (x, y) are mapped to image points of the form $(x, -y)$, or, more compactly, $r(x, y) = (x, -y)$, where r represents the reflection.

Such a formula is sometimes called an image formula, since it shows how a transformation acts on a pre-image point to produce its image point. A reflection in the y-axis leaves all second coordinates the same but replaces each first coordinate with its opposite; therefore, an image formula for a reflection in the y-axis may be written $r(x, y) = (-x, y)$. The image formula $r(x, y) = (y, x)$ represents a reflection in the line $y = x$. When this reflection is done, the coordinates of each pre-image point are reversed to give the image.

Rotations

A second type of transformation is the rotation, also known as a "turn." A rotation, as its name suggests, takes pre-image points and rotates them by some specified angle measure about some fixed point. In the figure below, the pre-image triangle CDE has been rotated 90° about the origin of the coordinate system to produce the image triangle C'D'E'.

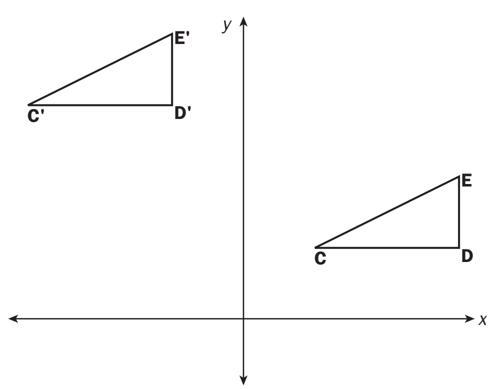


The image formula for a 90° rotation is $R(x, y) = (-y, x)$. It can be shown that, in general, the image formula for a rotation of angle measure t about the origin has image formula $R(x, y) = [x\cos(t) - y\sin(t)], [x\sin(t) + y\cos(t)]$. If t is positive, the direction of the rotation is counterclockwise; if t is negative, then the rotation is clockwise.

Translations

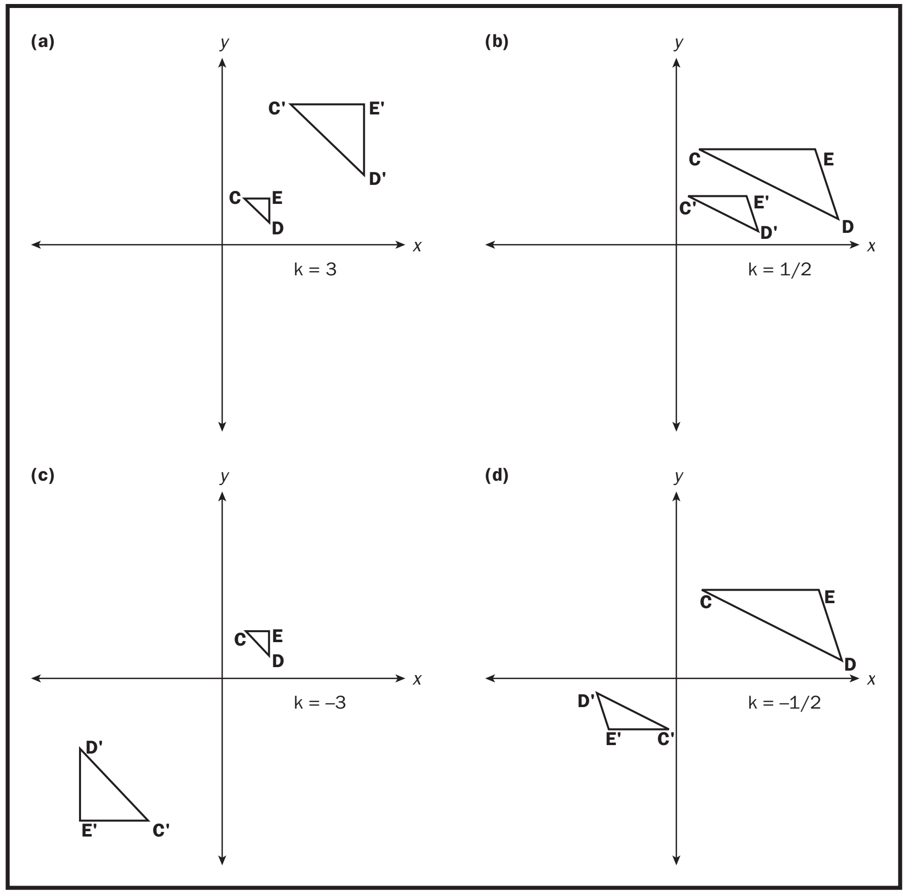
Another type of transformation is the translation or "slide." Translations take pre-image points and move them a given number of units horizontally and/or a given number of units vertically. A translation image formula has the form $T(x, y) = (x + a, y + b)$, where a is the number of units moved horizontally and b is the number of units moved vertically. If a is positive, the horizontal shift is to the right. If a is negative, then it is to the left. Similarly, if b is positive, the vertical shift is upward; but if b is negative, the vertical shift is downward.

In the figure below, the pre-image triangle CDE has been translated 4 units to the left and 2 units upward to give the image triangle C'D'E'. The image formula would be $T(x, y) = (x + (-4), y + 2) = (x - 4, y + 2)$.



Reflections, rotations, and translations change only the location of a figure. They have no effect on the size of the figure or on the distance between points in the figure. For this reason they are called "isometries" from the Greek words meaning "same measure." An isometry is also known as a "distance-preserving transformation." The next transformation discussed—the dilation—is not an isometry; that is, it does not preserve distance.

In transformations known as dilations, the resulting image may be larger or smaller than its original pre-image.



Dilations

A dilation is also known as a "stretch" or "shrink" depending on whether the image figure is larger or smaller than the pre-image figure. The image formula for a dilation is $d(x, y) = (kx, ky)$, where k is a real number, called the magnitude or scale factor, and where the center of the dilation is the origin. If $k > 1$, the image of the dilation is an enlargement of the pre-image on the same side of the center as the pre-image. Part (a) of the boxed figure illustrates this for $k = 3$.

If $0 < k < 1$, the image of the dilation is a reduction in size of the pre-image on the same side of the center as the pre-image. Part (b) of the figure illustrates this for $k = \frac{1}{2}$.

If $k < -1$, the image of the dilation is an enlargement of the pre-image on the opposite side of the center from the pre-image. Part (c) of the figure illustrates this for $k = -3$.

If $-1 < k < 0$, the image of the dilation is a reduction in size of the pre-image on the opposite side of the center from the pre-image. Part (d) of the figure illustrates this for $k = -\frac{1}{2}$.

Notice that the effect of a negative value for k is equivalent to that of a dilation with a magnitude equal to the absolute value of k followed by a rotation of 180° about the center of the dilation.

All of the transformations discussed above can easily be extended into 3-dimensional space. For example, a dilation in the 3-dimensional *xyz*-coordinate system would have an image formula of the form d(x, y, z) = (kx, ky, kz). When transformations—reflections, rotations, translations, and dilations—are expressed as **matrices** (as is taught in linear algebra), they can then be combined to create the movement of figures in computer animation programs. SEE ALSO Mapping, Mathematical; Tessellations.

matrix a rectangular array of data in rows and columns

Stephen Robinson

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Triangles

A triangle is a closed three-sided, three-angled figure, and is the simplest example of what mathematicians call polygons (figures having many sides). Triangles are among the most important objects studied in mathematics owing to the rich mathematical theory built up around them in **Euclidean geometry** and **trigonometry**, and also to their applicability in such areas as astronomy, architecture, engineering, physics, navigation, and surveying.

Euclidean geometry the geometry of points, lines, angles, polygons, and curves confined to a plane

trigonometry the branch of mathematics that studies triangles and trigonometric functions

In Euclidean geometry, much attention is given to the properties of triangles. Many theorems are stated and proved about the conditions necessary for two triangles to be similar and/or congruent. Similar triangles have the same shape but not necessarily the same size, whereas congruent triangles have both the same shape and size.

One of the most famous and useful theorems in mathematics, the Pythagorean Theorem, is about triangles. Specifically, the Pythagorean Theorem is about right triangles, which are triangles having a 90° or "right" angle. The theorem states that if the length of the sides forming the right angle are given by a and b, and the side opposite the right angle, called the **hypotenuse**, is given by c, then $c^2 = a^2 + b^2$. It is almost impossible to overstate the importance of this theorem to mathematics and, specifically, to trigonometry.

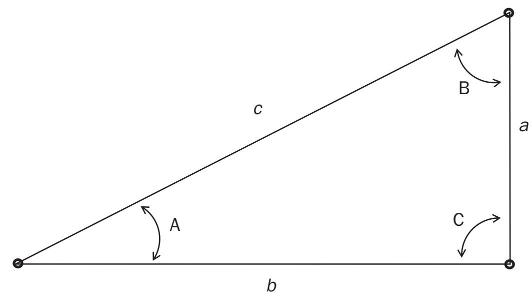
hypotenuse the long side of a right triangle; the side opposite the right angle

sine if a unit circle is drawn with its center at the origin and a line segment is drawn from the origin at angle theta so that the line segment intersects the circle at (x, y), then y is the sine of theta

Triangles and Trigonometry

Trigonometry literally means "triangle measurement" and is the study of the properties of triangles and their ramifications in both pure and applied mathematics. The two most important functions in trigonometry are the **sine** and **cosine** functions, each of which may be defined in terms of the sides of right triangles, as shown below.

cosine if a unit circle is drawn with its center at the origin and a line segment is drawn from the origin at angle theta so that the line segment intersects the circle at (x, y), then x is the cosine of theta



$$\sin(A) = \frac{a}{c}$$

$$\cos(A) = \frac{b}{c}$$

These functions are so important in calculating side and angle measures of triangles that they are built into scientific calculators and computers. Although the sine and cosine functions are defined in terms of right triangles, their use may be extended to any triangle by two theorems of trigonometry called the Law of Sines and the Law of Cosines (see page 108). These laws allow the calculation of side lengths and angle measures of triangles when other side and angle measurements are known.

Ancient and Modern Applications

triangulation the process of determining the distance to an object by measuring the length of the base and two angles of a triangle

parallax the apparent motion of a nearby object when viewed against the background of more distant objects due to a change in the observer's position

Perhaps the most ancient use of triangles was in astronomy. Astronomers developed a method called **triangulation** for determining distances to far away objects. Using this method, the distance to an object can be calculated by observing the object from two different positions a known distance apart, then measuring the angle created by the apparent "shift" or **parallax** of the object against its background caused by the movement of the observer between the two known positions. The Law of Sines may then be used to calculate the distance to the object.

The Greek mathematician and astronomer, Aristarchus (310 B.C.E.–250 B.C.E.) is said to have used this method to determine the distance from the Earth to the Moon. Eratosthenes (c. 276 B.C.E.–195 B.C.E.) used triangulation to calculate the circumference of Earth.

algorithm a rule or procedure used to solve a mathematical problem

Modern Global Positioning System (GPS) devices, which allow earth-bound travelers to know their longitude and latitude at any time, receive signals from orbiting satellites and utilize a sophisticated triangulation **algorithm** to compute the position of the GPS device to within a few meters of accuracy. SEE ALSO Astronomy, Measurements in; Global Positioning System; Polyhedrons; Trigonometry.

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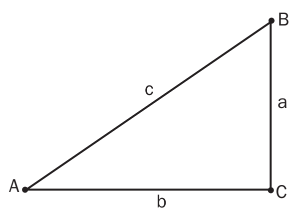
Trigonometry

The word "trigonometry" comes from two Greek words meaning "triangle measure." Trigonometry concerns the relationships among the sides and angles of triangles. It also concerns the properties and applications of these relationships, which extend far beyond triangles to real-world problems.

Evidence of a knowledge of elementary trigonometry dates to the ancient Egyptians and Babylonians. Led by Ptolemy, the Greeks added to this field of knowledge during the first millennium B.C.E.; simultaneously, similar work was produced in India. Around 1000 C.E., Muslim astronomers made great advances in trigonometry. Inspired by advances in astronomy, Europeans contributed to the development of this important mathematical area from the twelfth century until the days of Leonhard Euler in the eighteenth century.

Trigonometric Ratios

To understand the six trigonometric functions, consider right triangle *ABC* with right angle *C*. Although triangles with identical angle measures may have sides of different lengths, they are similar. Thus, the ratios of the corresponding sides are equal. Because there are three sides, there are six possible ratios.



Working from angle *A*, label the sides as follows: side *c* represents the **hypotenuse**; leg *a* represents the side opposite angle *A*; and leg *b* is adjacent to angle *A*. The definitions of the six trigonometric functions of angle *A* are listed below.

hypotenuse the long side of a right triangle; the side opposite the right angle

Sine	$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$	**Cotangent**	$\cot A = \frac{\text{adjacent}}{\text{opposite}} = \frac{b}{a}$
Cosine	$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$	**Secant**	$\sec A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{b}$
Tangent	$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}$	**Cosecant**	$\csc A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{a}$

For any angle **congruent** to angle *A*, the numerical value of any of these ratios will be equal to the value of that ratio for angle *A*. Consequently, for any given angle, these ratios have specific values that are listed in tables or can be found on calculators.

congruent exactly the same everywhere, superposable

Basic Uses of Trigonometry. The definitions of the six functions and the Pythagorean Theorem provide a powerful means of finding unknown sides and angles. For any right triangle, if the measures of one side and either another side or angle are known, the measures of the other sides and angles can be determined.

For example, suppose the measure of angle A is 36° and side c measures 12 centimeters (and angle C measures 90°). To determine the measure of angle B, subtract 36 from 90 because the two non-right angles must sum to 90°. To determine sides a and b, solve the equations $\sin 36° = \frac{a}{12}$ and $\cos 36° = \frac{b}{12}$, keeping in mind that sin 36° and cos 36° have number values. The results are $a = 12\sin 36° \approx 7.1$ cm and $b = 12\cos 36° \approx 9.7$ cm.

Two theorems that are based on right-triangle trigonometry—the Law of Sines and the Law of Cosines—allow us to solve for the unknown parts of any triangle, given sufficient information. The two laws, which can be expressed in various forms, follow.

Law of Sines: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

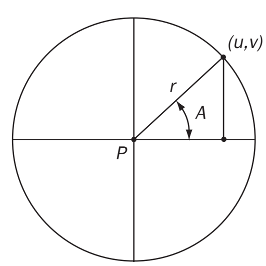
Law of Cosines: $a^2 = b^2 + c^2 - 2bc \cos A$

Expanded Uses of Trigonometry

The study of trigonometry goes far beyond just the study of triangles. First, the definitions of the six trigonometric functions must be expanded. To accomplish this, establish a rectangular coordinate system with P at the origin. Construct a circle of any radius, using point P as the center. The positive horizontal axis represents 0°. As one moves counter-clockwise along the circle, a positive angle A is generated.

Consider a point on the circle with coordinates (u, v). (The reason for using the coordinates (u, v) instead of (x, y) is to avoid confusion later on when constructing graphs such as $y = \sin x$.) By projecting this point onto the horizontal axis as shown below, a direct analogy to the original trigonometric functions can be made. The length of the adjacent side equals the u-value, the length of the opposite side equals the v-value, and the length of the hypotenuse equals the radius of the circle. Thus the six trigonometric functions are expanded because they are no longer restricted to **acute** angles.

acute sharp, pointed; in geometry, an angle whose measure is less than 90 degrees



$\sin A = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{a}{c} = \dfrac{v}{r}$ $\qquad$ $\cot A = \dfrac{\text{adjacent}}{\text{opposite}} = \dfrac{b}{a} = \dfrac{u}{v}$

$\cos A = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{b}{c} = \dfrac{u}{r}$ $\qquad$ $\sec A = \dfrac{\text{hypotenuse}}{\text{adjacent}} = \dfrac{c}{b} = \dfrac{r}{u}$

$\tan A = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{a}{b} = \dfrac{v}{u}$ $\qquad$ $\csc A = \dfrac{\text{hypotenuse}}{\text{opposite}} = \dfrac{c}{a} = \dfrac{r}{v}$

For any circle, similar triangles are created for equal central angles. Consequently, one can choose whatever radius is most convenient. To simplify calculations, a circle of radius 1 is often chosen. Notice how four of the functions, especially the sine and cosine functions, become much simpler if the radius is 1.

These expanded definitions, which relate an angle to points on a circle, allow for the use of trigonometric functions for any angle, regardless of size. So far the angles discussed have been measured in degrees. This, however, limits the applicability of trigonometry. Trigonometry is far less restricted if angles are measured in units called **radians**.

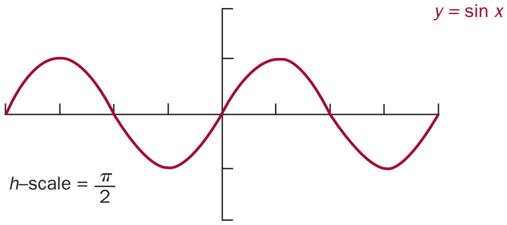
radian an angle measure approximately equal to 57.3 degrees, it is the angle that subtends an arc of a circle equal to one radius

Using Radian Measure. Because all circles are similar, for a given central angle in any circle, the ratio of an intercepted arc to the radius is constant. Consequently, this ratio can be used instead of the degree measure to indicate the size of an angle.

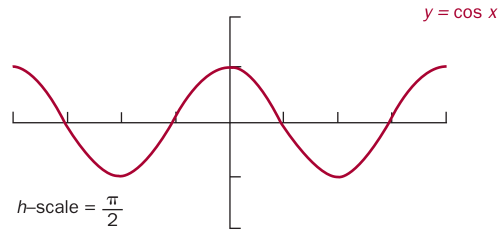
Consider for example a semicircle with radius 4 centimeters. The arc length, which is half of the circumference, is exactly 4π centimeters. In radians, therefore, the angle is the ratio 4π centimeters to 4 centimeters, or simply π. (There are no units when radian measure is used.) This central angle also measures 180°. Recognizing that 180° is equivalent to π (when measured in radians), there is now an easy way of converting to and from degrees and radians. This can also be used to determine that an angle of 1 radian, an angle which intercepts an arc that is precisely equal to the radius of the circle, is approximately 57.3°.

Now the domain for the six trigonometric functions may be expanded beyond angles to the entire set of real numbers. To do this, define the trigonometric function of a number to be equivalent to the same function of an angle measuring that number of radians. For example, an expression such as sin 2 is equivalent to taking the sine of an angle measuring 2 radians. With this freedom, the trigonometric functions provide an excellent tool for studying many real-world phenomena that are periodic in nature.

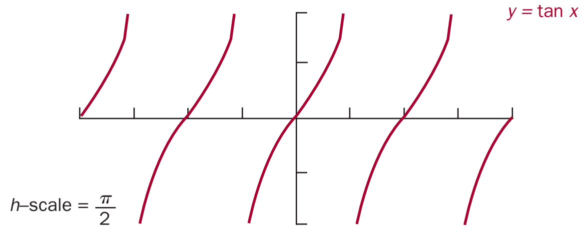
The figure below shows the graphs of the sine, cosine, and tangent functions, respectively. Except for values for which the tangent is undefined, the domain for these functions is the set of real numbers. The domain for the parts of the graphs that are shown is $-2\pi \leq x \leq 2\pi$. Each tick mark on the x-axis represents $\frac{\pi}{2}$ units, and each tick mark on the y-axis represents one unit.



Sine curve



Cosine curve



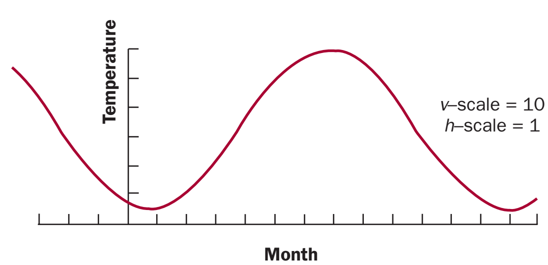
Tangent curve

To understand the graphs, think back to a circle with radius 1. Because the radius is 1, the sine function, which is defined as $\frac{v}{r}$, simply traces the vertical value of a point as it moves along the circumference of the circle. It starts at 0, moves up as high as 1 when the angle is $\frac{\pi}{2}$ (90°), retreats to 0, goes down to -1, returns to 0, and begins over again. The graph of the cosine function is identical except for being $\frac{\pi}{2}$ (90°) out of phase. It records the horizontal value of a point as it moves along the unit circle.

The tangent is trickier because it concerns the ratio of the vertical value to the horizontal value. Whenever the vertical component is 0, which happens at points along the horizontal axis, the tangent is 0. Whenever the horizontal component is 0, which happens at points on the vertical axis, the tangent is not defined—or infinite. Thus, the tangent has a vertical **asymptote** every π units.

asymptote the line that a curve approaches but never reaches

A Practical Example. By moving the sine, cosine, and tangent graphs left or right and up or down and by stretching them horizontally and vertically, these trigonometric functions serve as excellent models for many things. For example, consider the function, in which x represents the month of the year and $y = 54 + 22 \cos(\frac{\pi}{6}(x - 7))$, in which x represents the average monthly temperature measured in Fahrenheit.



Graph of $y = 54 + 22 \cos\left[\frac{\pi}{6}(x - 7)\right]$

The "parent" function is the cosine, which intercepts the vertical axis at its maximum value. In our model, we find the maximum value shifted 7 units to the right, indicating that the maximum temperature occurs in the seventh month, July.

The normal period of the cosine function is 2π units, but our transformed function is only going $\frac{\pi}{6}$ as fast, telling us that it takes 12 units, in this case months, to complete a cycle.

The amplitude of the parent graph is 1; this means that its highest and lowest points are both 1 unit away from its horizontal axis, which is the mean functional (vertical) value. In our example, the amplitude is 22, indicating that its highest point is 22 units (degrees) above its average and its lowest is 22 degrees below its average. Thus, there is a 44-degree difference between the average temperature in July and the average temperature in January, which is half a cycle away from July.

Finally, the horizontal axis of the parent function is the x-axis; in other words, the average height is 0. In this example, the horizontal average has been shifted up 54 units. This indicates that the average spring temperature—in April to be specific—is 54 degrees. So too is the average temperature in October. Combining this with the amplitude, it is found that the average July temperature is 76 degrees, and the average January temperature is 32 degrees.

Trigonometric equations often arise from these mathematical models. If, in the previous example, one wants to know when the average temperature is 65 degrees, 65 is substituted for y, and the equation is solved for x. Any of several techniques, including the use of a graph, can work. Similarly, if one wishes to know the average temperature in June, 6 is substituted for x, and the equation is solved for y. SEE ALSO Angles, Measurement of.

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Turing, Alan

British Mathematician and Cryptanalyst
1912–1954

In the October 1950 issue of *Mind*, the brilliant thinker Alan Turing wrote, "We may hope that machines will eventually compete with men in all purely intellectual fields." Turing believed that machines could mimic the processes of the human brain but acknowledged that people would have difficulty accepting such a machine—a problem that still plagues artificial intelligence today.

In a footnote to his 1936 paper on computable numbers, Alan Turing proposed a machine that could solve mathematical problems—a model for what later would become the modern computer.

Turing also proposed a test to measure whether a machine could be considered "intelligent." The widely acclaimed "Turing test" involved a connecting a human by teletype (later a computer keyboard) to either another human or a machine. The first human would then ask questions that are translated through these mechanical links. If the respondent on the other end was indeed a machine, but the human asking questions could not tell whether the responses came from a human or machine, then the machine should be regarded as intelligent.

While attending graduate school in mathematics, Turing wrote *On Computable Numbers* (1936), in which he described hypothetical devices (later dubbed "Turing machines") that presaged today's computers. A Turing machine could perform logical operations and systematically read, write, or erase symbols written on paper tape.

Upon completing his doctorate at Princeton University in 1938, Turing was invited to the Institute of Advanced Studies (also at Princeton) to become assistant to John von Neumann, the brilliant mathematician, synthesizer, and promoter of the stored program concept. Turing declined and instead returned to Cambridge, England. Turing began to work for the wartime cryptanalytic headquarters at Bletchley Park, halfway between Oxford and Cambridge, after the British declared war with Germany on September 3, 1939.

At Bletchley Park, Turing and his colleagues utilized electric and mechanical forms of decryption to break the code of the Enigma cipher machines, on which almost all German communications were enciphered (computed arithmetically). Turing helped construct decoders (called Bombes) that could more rapidly test key codes until correct combinations were found, cutting the time it took to decipher German codes from weeks to hours. As a result, many Allied convoys were saved, and it has been estimated that World War II might have lasted 2 more years if not for the work at Bletchley Park.

Turing's life was not without tumult or a sad conclusion. In 1952, Turing was tried and convicted of "gross indecency" for being homosexual. He was sentenced to probation and hormone treatments. In June 1954, he died from cyanide poisoning. Although he possessed the cyanide in connection with chemical experiments he was performing, it is widely believed he committed suicide.

Turing's life has inspired writers, artists, and sculptors. *Breaking the Code*, a play based on his life, opened in London in 1986. Although the dialogue and scenes are mostly invented, the play ably conveys the remarkable life of this extraordinary man. SEE ALSO Computers, Evolution of Electronic; Cryptology.

Marilyn K. Simon

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Internet Resources

The Alan Turing Home Page. Ed Andrew Hodges. <http://www.turing.org.uk/turing/index.html>.

Undersea Exploration

U

Over the last two centuries there has been an explosion in our knowledge of the global underwater environment. This understanding of the world's oceans and seas has been accomplished by a variety of methods, including the use of both manned and unmanned (i.e., robotic) vehicles. When people use any of these modern methods to explore the undersea realm, they must, in one way or another, use a mathematical description of the undersea environment in order to be successful. For example, scuba divers must be aware of such quantities as the depth below the surface and the total time they can remain submerged. Each of these quantities is expressed as a number: for example, depth in feet and total dive time in minutes. The use and application of mathematics is therefore an essential part of undersea exploration.

Oceanography is the branch of earth science concerned with the study of all aspects of the world's oceans, such as the direction and strength of currents, and variations in temperatures and depths. Oceanography also encompasses the study of the marine life inhabiting the oceans. As noted earlier, there are multiple ways to explore the oceans and oceanic life. When people and/or machines are sent to directly explore the ocean environment, several different properties of the oceans must be taken into account in order for the dive to be successful and safe. These properties include—but are not limited to—ocean depth and pressure, temperature, and illumination (visibility).

Ocean Properties

"Pressure" is a measurement related to force, and is especially useful when dealing with gases or liquids. When an object is immersed in a swimming pool, in a lake, or in the ocean, it experiences a force pushing inward over its entire surface. For instance, a balloon that is inflated, sealed, and then submerged will be pushed inward by water pressure. The pressure on the balloon is the force exerted by the water on every square inch of the balloon's surface. Pressure is therefore measured as the force-per-unit-area, such as pounds-per-square-inch. Due to Earth's atmosphere, all objects at sea level experience a pressure of approximately 14.7 pounds-per-square-inch (abbreviated 14.7 lbs/in^2) over their entire surface.

IS AN OCEAN DIFFERENT FROM A SEA?

The words "ocean" and "sea" are often used interchangeably to refer to any of the large bodies of salt water that together cover a majority of Earth's surface. The word "undersea" refers to everything that lies under the surface of the world's seas or oceans, down to the seafloor.

Perhaps the greatest limitation to underwater exploration is water pressure, which increases with depth (i.e., the deeper one goes, the greater the pressures encountered). Moreover, the pressures within the ocean depths can be immense. A balloon lowered deeper and deeper into the ocean will continuously shrink in size as the water pressure increases. Just under the ocean's surface the water pressure is 14.7 lbs/in^2, equal to one **atmosphere**. For every 10 meters (about 33 feet) of additional ocean depth, the pressure increases by one atmosphere.

atmosphere a unit of pressure equal to 14.7 lbs/in^2, which is the air pressure at mean sea level

"Visibility" refers to how well one can see through the water. As light from the Sun passes through seawater, it is both absorbed and scattered so that visibility due to sunlight decreases rapidly with increasing depth. Below about 100 meters (approximately 330 feet), most areas of the ocean appear dark.

"Temperatures" at the ocean surface vary greatly across the globe. Tropical waters can be quite comfortable to swim in, while polar waters are much colder. However, below a depth of about 250 meters most seawater across the world hovers around the freezing point, from 3° C to –1° C.

The Development of Diving Equipment

In the past, people who wished to dive down and explore the oceans were limited by the extremes of pressure, visibility and temperature found there. Nevertheless, for thousands of years people have dove beneath the waves to exploit the sea's resources for things like pearls and sponges. However, divers without the benefit of mechanical devices are limited to the amount of air stored in their lungs, and so the duration of their dives is quite restricted. In such cases, diving depths of around 12 meters (less than 40 feet) and durations of less than two minutes are normal.

Various devices have been invented to extend the depth and duration of divers. All these devices provide an additional oxygen supply. Over 2,300 years ago Alexander the Great (356 B.C.E–323 B.C.E) was supposedly lowered beneath the waves in a device that allowed him an undersea view. Alexander may have used an early form of the diving bell, which is like a barrel with one end open and the other sealed. With the closed end at the top, air becomes trapped inside the barrel, and upon being lowered into the water the occupant is able to breathe.

About 2,000 years after Alexander's undersea excursion, Englishman Edmund Halley (1656–1742) constructed a modified diving bell, similar to Alexander's apparatus, but with additional air supplied to the occupant through hoses. Modern diving stations, like diving bells, are containers with a bottom opening to the sea. Water is prevented from entering the station because, like a diving bell, the air pressure inside the station is equal to the water pressure outside. Researchers can live in a diving station for long periods of time, allowing them immediate access to the ocean environment at depths of up to 100 meters (328 feet).

The Emergence of Diving Suits

To allow greater freedom of movement when submerged, various underwater suits and breathing devices have been invented. In the early 1800s, diving suits were constructed, and modern versions are still in use. These suits possess a hard helmet with a transparent front for visibility; an airtight garment that keeps water out but still allows movement; and connecting hoses linked to a surface machine that feeds a continuous flow of fresh air to the suit. Over the years, additional improvements have been added to diving suits, such as telephone lines running between the surface and diver for communication.

From an exploratory point-of-view, diving suits are limited because of their bulkiness, the cumbersome connecting lines that run to the surface, and the restriction of the diver to the seafloor. Skin diving (or "free diving") overcomes the mobility problem by allowing the diver to freely swim underwater. As the name implies, most skin divers are not completely covered by a suit, but rather use minimal equipment such as a facemask, flippers, and often a snorkel for breathing. A snorkel has tubes that allow the diver to breathe surface air while underwater. This type of diving (called snorkeling) must be performed close to the surface.

As compared to snorkeling, scuba diving considerably increases the diving depth of snorkeling by employing an oxygen supply contained in tanks (scuba is derived from "*s*elf *c*ontained *u*nderwater *b*reathing *a*pparatus"). Varying versions of scuba gear have been invented. Probably the most popular is the aqualung, introduced by the undersea explorer Jacques Cousteau (1910–1997). When the diver begins to inhale, compressed air is automatically fed into her or his mouthpiece by valves designed to assure a constant airflow at a pressure that matches the outside water pressure. The compressed air is contained in cylinders that are carried on the diver's back. Using conventional mixtures of oxygen and compressed air, divers can safely submerge to around 75 meters (almost 250 feet). With special breathing mixtures, such as oxygen with helium, scuba dives of greater than 150 meters (about 500 feet) have been safely accomplished. To illuminate their surroundings, scuba divers sometimes carry battery-powered lights.

Station, suit, and scuba divers utilize compressed gases for breathing. As divers go deeper, the pressure on their lungs increases, requiring that they breathe higher-pressure air. This process forces gases such as nitrogen into the cells of the diver's body. Divers who do not follow the proper procedure upon returning to the surface risk experiencing a painful, possibly fatal condition known as "the bends." The bends are the result of the pressurized gases inside the diver's body being released quickly, like carbon dioxide being released from a shaken soda can. A diver can avoid the bends by ascending to the surface in a controlled and timed manner, so that the gas buildup inside his or her cells is slowly dissipated. This procedure is called decompression. A competent diver uses gauges to be aware at all times of the depth of the dive, and ascends to the surface in a timed manner.

It is recommended that divers ascend no faster than 30 feet per minute. For example, if a diver is at a depth of 90 feet, it should take 3 minutes to

USING PROPORTIONS WHEN DIVING

Scuba divers know that the deeper the dive, the greater the pressure—a direct proportion. Yet the deeper the dive, the less time they can safely stay underwater to avoid "the bends" —an inverse proportion.

Alvin, a manned submersible vehicle, explores the submerged hull of the *Titanic*.

ascend. The use of numbers and measurements are therefore of utmost importance in all types of diving.

Underseas Exploration Using Submersibles

Although scuba divers can explore some shallow parts of the ocean, especially environments such as coral reefs, submersibles allow exploration of every part of the ocean, regardless of depth. Submersibles are airtight, rigid diving machines built for underwater activities, including exploration. They may be classified as being either manned or remotely operated.

Manned submersibles require an onboard crew. Unlike scuba or suit divers, the people inside submersibles usually breathe air at sea-level pressure. Therefore, people in submersibles are not concerned with decompressing at the end of a dive. In 1960, the *Trieste* submersible made a record dive of 10,914 meters (35,800 feet) into an area of the Pacific Ocean known as Challenger Deep. Mathematics was essential for both the construction of *Trieste* and for its record-breaking dive. For instance, the maximum pressure anticipated for its dives (over 80 tons per-square-inch) had to be correctly calculated, and then the vessel walls had to be built accordingly, or the craft would be destroyed. Today there are many new and sophisticated manned submersibles possessing their own propulsion system, life support system, external lighting, mechanical arms for sampling, and various recording devices, such as video and still cameras.

Remotely operated submersibles do not need a human crew onboard and they are guided instead by a person at a different location (hence its name). These vehicles are equipped with video cameras from which the remote operator views the underwater scene. Remote submersibles can by-and-large duplicate the operations of a manned submersible but without endangering human life in the case of an accident.

Philip Edward Koth (with William Arthur Atkins)

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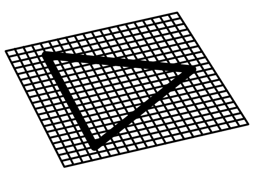
Universe, Geometry of

For centuries, mathematicians and physicists believed that the universe was accurately described by the axioms of Euclidean geometry, which now form a standard part of high school mathematics. Some of the distinctive properties of Euclidean geometry follow:

- Straight lines go on forever.
- Parallel lines are unique. That is, given a line and a point not on that line, there is one and only one parallel line that passes through the given point.
- Parallel lines are equidistant. That is, two points moving along parallel lines at the same speed will maintain a constant distance from each other.
- The angles of a triangle add to 180°.
- The circumference of a circle is proportional to its radius, r, ($C = 2\pi r$) and the area, A, of a circle is proportional to the square of the radius ($A = \pi r^2$), where pi (π) is defined to be approximately 3.14 ($\frac{c}{2r}$).

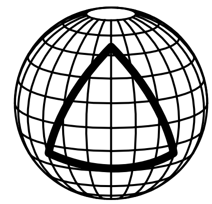
The last four properties are characteristic of a "flat" space that has no intrinsic curvature. (See figure below.)



Flat Geometry

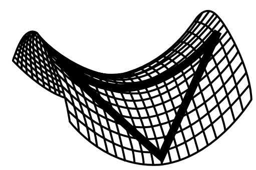
Other types of geometry, called non-Euclidean geometries, violate some or all of the properties of Euclidean geometry. For example, on the surface of a sphere (a positively curved space), the closest thing to a straight path is a great circle, or the path you would follow if you walked straight forward without turning right or left. But great circles do not go on forever: They loop around once and then close up.

If "lines" are interpreted to mean "great circles," the other properties of Euclidean geometry are also false in spherical geometry. Parallel great circles do not exist at all. The positive curvature causes great circles that start out in the same direction to converge. For example, the meridians on a globe converge at the north and south poles. The angles of a spherical triangle add up to more than 180°, and the circumference and area of spherical circles are smaller than $2\pi r$ and πr^2, respectively. (See figure on p. 120.)



Closed Geometry

In the early years of the nineteenth century, three mathematicians—Karl Friedrich Gauss, Janos Bolyai, and Nikolai Lobachevski—independently discovered non-Euclidean geometries with negative curvature. In these geometries, lines that start out in the same direction tend to diverge from each other. The angles of a triangle add to less than 180°. The circumference of a circle grows exponentially faster than the radius. Although negatively curved spaces are difficult to visualize and depict because we are accustomed to looking at the world through "Euclidean eyes," you can get some idea by looking at the figure directly below.



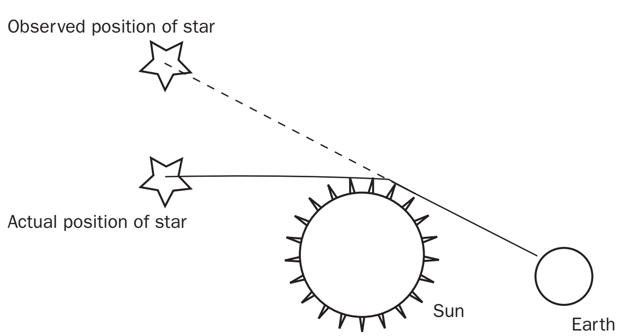
Open Geometry

Curved Space and General Relativity

For a long time curved geometries remained in the realm of pure mathematics. However, in 1915 Albert Einstein proposed the theory of general relativity, which stated that our universe is actually curved, at least in some places. Any large amount of matter tends to impart a positive curvature to the space around it. The force that we call gravity is a result of the curvature of space.✶

✶**An oft-repeated aphorism is: "Matter tells space how to curve, and space tells matter how to move."**

In general relativity, rays of light play the role of lines. Because space is positively curved near the Sun, Einstein predicted that rays of light from a distant star that just grazed the surface of the Sun would bend towards the Sun. (See figure below.) To observers on Earth, such stars would appear displaced from where they really were. Most of the time, of course, the light from the Sun completely obscures any light that comes from distant stars, so we cannot see this effect. But Einstein's prediction was dramatically confirmed by observations made during the total solar eclipse of 1919. Such "gravitational lensing" effects are now commonplace in astronomy.



Einstein, like many later scientists, was also very interested in the overall geometry of the universe. The stars are, after all, aberrations in a universe that is mostly empty space. If you distributed all the matter in the universe evenly, would the curvature be positive, negative, or zero? Interestingly, on this issue Einstein was not a revolutionary. He assumed, in his first model of the universe, that the overall geometry would be flat. It was a Russian physicist, Alexander Friedmann,* who first pointed out in 1922 that Einstein's equations could be solved just as well by positively or negatively curved universes.

***Alexander Alexandrovitch Friedmann (1888–1925) was a founder of dynamic meteorology.**

Friedmann also showed that the curvature was not just a mathematical curiosity: The fate of the entire universe depended on it. This dependence occurs because the curvature of the universe is closely related to the density of matter. Namely, if there is less than the critical density of matter in the universe, then the universe is negatively curved, but if there is more than the critical density of matter, then it is positively curved. If the universe is negatively curved, it keeps expanding forever because the gravitational attraction caused by the small amount of matter is not enough to rein in the universe's expansion. But if the universe is positively curved, the universe ends with a "Big Crunch" because the gravitational attraction causes the expansion of space to slow down, stop, and reverse. If the amount of matter is "just right," then the curvature is neither positive nor negative, and space is Euclidean. In a flat universe, the growth of space slows down but never quite stops.

Big Bang Theory

Both of Friedmann's models of the universe assume that it began with a bang. Although Einstein was skeptical of these theories at first, the **Big Bang** theory received powerful support in 1929, when Edwin Hubble showed that distant galaxies are receding from Earth at a rate proportional to their distance. For the first time, it was possible to determine the age of the universe. (Hubble's original estimate was 2 billion years; the currently accepted range is 10 to 15 billion years.)

big bang the singular event thought by most cosmologists to represent the beginning of our universe; at the moment of the big bang, all matter, energy, space, and time were concentrated into a single point

Another piece of evidence for the Big Bang theory came in 1964, in the form of the cosmic microwave background, a sort of afterglow of the Big Bang. The discovery of this radiation, which comes from all directions of the sky, confirmed that the early universe (about 300,000 years after the Big Bang) was very hot, and also very uniform. Every part of the cosmic fireball was at almost exactly the same temperature.

After 1964, the Big Bang theory became a kind of scientific orthodoxy. But few people realized that it was really several theories, not one—the question of the curvature and ultimate fate of the universe was still wide open.

Over the 1960s and 1970s, a few physicists began to have misgivings about the Friedmann models. One centered on the assumption of homogeneity, which Friedmann (and Einstein) had made as a mathematical convenience. Though the microwave data had confirmed this nicely, there still was no good explanation for it. And, in fact, a careful look at the night sky seems to show the distribution of mass in the universe is not very uniform. Our immediate neighborhood is dominated by the Milky Way. The Milky Way is itself part of the Local Group of galaxies, which in turn lies on the

light-year the distance light travels within a vacuum in one year

fringes of the Virgo Cluster, whose center is 60 million **light-years** away. Astronomers have found evidence for even larger structures, such as a "Great Wall" of galaxies 300 million light-years across. How could a universe that started out uniform and featureless develop such big clumps?

A second problem was the "flatness problem." In cosmology, the branch of physics that deals with the beginnings (and endings) of the universe, the density of matter in the universe is denoted by Omega (abbreviated O or Ω). By convention, a flat (Euclidean) universe has a density of 1, and this is called the critical density. In more ordinary language, the critical density translates to about one hydrogen atom per cubic foot, or one poppy seed spread out over the volume of the Earth. It is an incredibly low density.

The trouble is that Omega does not stay constant unless it equals 1 or 0. If the universe started out with a density a little less than 1, then Omega would rapidly decrease until it was barely greater than 0. If it started out with a density of 1, then Omega would stay at 1. It is very difficult to create a universe where, after 10 billion years, Omega is still partway between 0 and 1. And yet that is what the observational data suggested: The most careful estimates to date put the density of ordinary matter, plus the density of dark matter that does not emit enough light to be seen through telescopes, at about 0.30. This seemed to require a very "fine-tuned" universe. Physicists view with suspicion any theory that suggests our universe looks the way it does because of a very specific, or lucky, choice of constants. They would rather deduce the story of our universe from general principles.

Revising the Big Bang Theory

In 1980, Alan Guth proposed a revised version of the Big Bang that has become the dominant theory, because it answers both the uniformity and flatness paradoxes. And unlike Friedmann's models, it makes a clear prediction about the geometry of the universe: The curvature is 0 (or very close to it), and the geometry is Euclidean.

Guth's theory, called inflation, actually came out of particle physics, not cosmology. At an unimaginably small time after the Big Bang—roughly a trillionth of a trillionth of a nanosecond—the fundamental forces that bind atoms together went through a "phase transition" that made them repel rather than attract. For just an instant, a billionth of a trillionth of a nanosecond, the universe flew apart at a phenomenal, faster-than-light rate. Then, just as suddenly, the inflationary epoch was over. The universe reached another phase transition, the direction of the nuclear forces reversed, and space resumed growing at the normal pace predicted by the ordinary Big Bang models.

Inflation solves the uniformity problem because everything we can see today came from an incredibly tiny, and therefore homogeneous, region of the pre-inflationary universe. And it solves the flatness problem because, like a balloon being stretched taut, the universe lost any curvature that it originally had. The prediction that space is flat has become one of the most important tests for inflation.

Until recently, the inflationary model has not been consistent with astronomical observations. Its supporters had to argue that the observed clumps of galaxies, up to hundreds of millions of light-years in size, can be

reconciled with a very homogeneous early universe. They also had to believe that something is out there to make up the difference between the observed Omega of 0.3 and the predicted Omega of 1.

A series of new measurements in 1999 and 2000, however, provided some real victories for inflation. First, they showed that the microwave background does have some slight variations, and the size of these variations is consistent with a flat universe. More surprisingly, the data suggest that most of the "missing matter" is not matter but *energy*, in an utterly mysterious form called dark energy or vacuum energy. In essence, this theory claims that there is something about the vacuum itself that pushes space apart. The extra energy causes space to become flat, but it also makes space expand faster and faster (rather than slower and slower, as in Friedmann's model).

Until these measurements, the theory of vacuum energy was not accepted. It had originally been proposed by Einstein, who later abandoned it and called it his greatest mistake. Yet years later, cosmologists said that the vacuum energy was the most accurately known constant in physics, to 120 decimal places, and composes 70 percent of the matter-energy in our universe.

There is no way to tell at this time whether vacuum energy is the final word or whether it will be discarded again. It is clear, though, that the geometry of space will continue to play a crucial role in understanding the most distant past and the most distant future of our universe. SEE ALSO COSMOS; EINSTEIN, ALBERT; EUCLID AND HIS CONTRIBUTIONS; GEOMETRY, SPHERICAL; SOLAR SYSTEM GEOMETRY, HISTORY OF; SOLAR SYSTEM GEOMETRY, MODERN UNDERSTANDINGS OF.

Dana Mackenzie

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Variation, Direct and Inverse

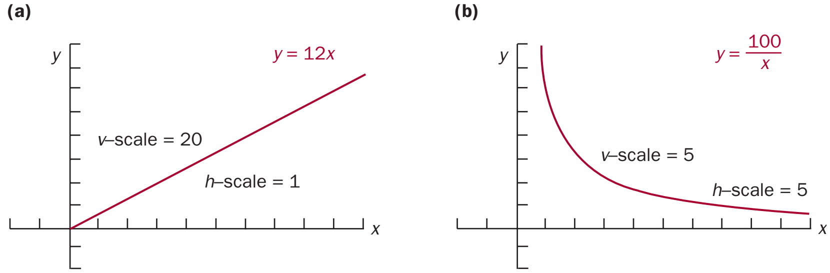
A variable is something that varies among components of a set or population, such as the height of high school students. Two types of relationships between variables are direct and inverse variation. In general, direct variation suggests that two variables change in the same direction. As one variable increases, the other also increases, and as one decreases, the other also decreases. In contrast, inverse variation suggests that variables change in opposite directions. As one increases, the other decreases and vice versa.

Direct Variation

Consider the case of someone who is paid an hourly wage. The amount of pay varies with the number of hours worked. If a person makes $12 per hour and works two hours, the pay is $24; for three hours worked, the pay is $36, and so on. If the number of hours worked doubles, say from 5 to 10, the pay doubles, in this case from $60 to $120. Also note that if the person works 0 hours, the pay is $0. This is an important component of direct variation: When one variable is 0, the other must be 0 as well.

So, if two variables vary directly and one variable is multiplied by a constant, then the other variable is also multiplied by the same constant. If one variable doubles, the other doubles; if one triples, the other triples; if one is cut in half, so is the other. Algebraically, the relationship between two variables that vary directly can be expressed as $y = kx$, where the variables are x and y, and k represents what is called the constant of proportionality. (Note that this relationship can also be expressed as $\frac{y}{x} = k$ or $\frac{x}{y} = \frac{1}{k}$, with $\frac{1}{k}$ also representing a constant.) In the preceding example, the equation is $y = 12x$, with x representing the number of hours worked, y representing the pay, and 12 representing the hourly rate, the constant of proportionality.

Graphically, the relationship between two variables that vary directly is represented by a ray that begins at the point (0, 0) and extends into the first quadrant. In other words, the relationship is linear, considering only positive values. See part (a) of the figure on the next page. The slope of the ray depends on the value of k, the constant of proportionality. The bigger k is, the steeper the graph, and vice versa.



Direct variation **Indirect variation**

Inverse Variation

When two variables vary inversely, one increases as the other decreases. As one variable is multiplied by a given factor, the other variable is divided by that factor, which is, of course, equivalent to being multiplied by the reciprocal (the multiplicative inverse) of the factor. For example, if one variable doubles, the other is divided by two (multiplied by one-half); if one triples, the other is divided by three (multiplied by one-third); if one is multiplied by two-thirds, the other is divided by two-thirds (multiplied by three-halves).

Consider a situation in which 100 miles are traveled. If traveling at an average rate of 5 miles per hour (mph), the trip takes 20 hours. If the average rate is doubled to 10 mph, then the trip time is halved to 10 hours. If the rate is doubled again, to 20 mph, the trip time is again halved, this time to 5 hours. If the average rate of speed is 60 mph, this is triple 20 mph. Therefore, if it takes 5 hours at 20 mph, 5 is divided by 3 to find the travel time at 60 mph. The travel time at 60 mph equals $\frac{5}{3}$, or $1\frac{2}{3}$ hours.

Algebraically, if x represents the rate (in miles per hour) and y represents the time it takes (in hours), this relationship can be expressed as $xy = 100$ or $y = \frac{100}{x}$ or $x = \frac{100}{y}$. In general, variables that vary inversely can be expressed in the following forms: $xy = k$, $y = \frac{k}{x}$, or $x = \frac{k}{y}$.

hyperbola a conic section; the locus of all points such that the absolute value of the difference in distance from two points called foci is a constant

asymptotic describes a line that a curve approaches but never reaches

The graph of the relationship between quantities that vary inversely is one branch of a **hyperbola**. See part (b) of the figure. The graph is **asymptotic** to both the positive x- and y-axes. In other words, if one of the quantities is 0, the other quantity must be infinite. For the example given here, if the average rate is 0 mph, it would take forever to go 100 miles; similarly, if the travel time is 0, the average rate must be infinite.

Many pairs of variables vary either directly or inversely. If variables vary directly, their quotient is constant; if variables vary inversely, their product is constant. In direct variation, as one variable increases, so too does the other; in inverse variation, as one variable increases, the other decreases. SEE ALSO Inverses; Ratio, Rate, and Proportion.

Bob Horton

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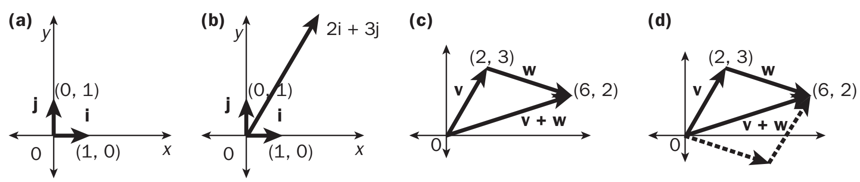
Vectors

A vector in the Cartesian plane is an ordered pair (a, b) of real numbers. This is the mathematician's concise definition of a two-dimensional vector. Physicists and engineers like to develop this concept a bit more for the purpose of applying vectors in their disciplines. Thus, they like to think of the mathematician's ordered pairs as representing displacements, velocities, accelerations, forces, and the like. Since such things have magnitude and direction, they like to imagine vectors as arrows in the plane whose magnitudes are their lengths and whose directions are the directions that the arrows are pointing. The two small dark arrows shown in part (a) of the drawing below form our point of departure in the study of vectors in the plane.

These are called the unit basis vectors, or unit vectors. From them all other vectors arise. To the mathematician, they are simply the ordered pairs (1, 0) and (0, 1). To the physicist, they really are the arrows extending from the origin to the points (1, 0) and (0, 1). The horizontal one is commonly named **i** and the vertical one is commonly named **j**. To say that all other vectors in the plane arise from these two means that all vectors are linear combinations of **i** and **j**. For instance, the mathematician's vector (2, 3) is $2(1, 0) + 3(0, 1)$, while the physicist's arrow with horizontal displacement of 2 units and vertical displacement 3 units is more compactly written as $2\mathbf{i} + 3\mathbf{j}$ and is represented as in part (b) below.



In the vocabulary of physics, the end of the arrow with the arrowhead is called the "head" of the vector, while the end without the arrowhead is called the "tail." The tail does not necessarily have to be at the origin as in the picture. When it is, the vector is said to be in standard position, but any arrow with horizontal displacement 2 and vertical displacement 3 is regarded by physicists as $2\mathbf{i} + 3\mathbf{j}$, as shown in part (b) of the drawing above. The 2 and 3 are called horizontal and vertical components (respectively) of the vector $2\mathbf{i} + 3\mathbf{j}$.

By definition, if

$\mathbf{v} = (a, b) = a\mathbf{i} + b\mathbf{j}$

and

$\mathbf{w} = (c, d) = c\mathbf{i} + d\mathbf{j}$,

then

$\mathbf{v} + \mathbf{w} = (a + c, b + d) = (a + c)\mathbf{i} + (b + d)\mathbf{j}$.

This definition of addition for vectors leads to a very convenient geometrical interpretation. In part (c) of the drawing above,

$\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$

and

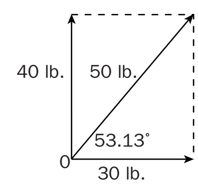
$w = 4\mathbf{i} - 1\mathbf{j}$.

Then $\mathbf{v} + \mathbf{w} = 6\mathbf{i} + 2\mathbf{j}$.

Physicists call their convention for adding geometric vectors "head-to-tail" addition. Notice in the part (d) of the drawing above that **v** is in standard position and extends to the point (2, 3). If the tail of **w** is placed at the head of **v**, then the head of **w** ends up at (6, 2). Now if we draw the arrow from the origin to (6, 2), we have an appropriate representation of **v** + **w** in standard position. The vector **v** + **w** is usually called the resultant vector of this addition. So, in general, the resultant of the addition of two vectors will be represented geometrically as an arrow extending from the tail of the first vector in the sum to the head of the second vector. This convention is often called the parallelogram law because the resultant vector always forms the diagonal of a parallelogram in which the two addend vectors lie along adjacent sides.

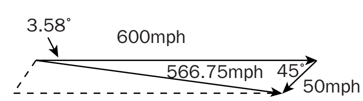
Applications of Vectors

As an example from physics, consider an object being acted upon by two forces: a 30 pound (lb) force acting horizontally and a 40 lb force acting vertically. The physicist wants to know the magnitude and direction of the resultant force. The schematic diagram below represents this situation.



By the parallelogram law, the resultant of the two forces is the diagonal of the rectangle. The Pythagorean Theorem may be used to calculate that this diagonal has a length of 50, representing a 50-lb. resultant force. The inverse cosine of 30/50 is 53.13°, giving the angle to the horizontal at which the resultant force acts on the object.

Consider another example in which vectors represent velocities. An airplane is attempting to fly due east at 600 mph (miles per hour), but a 50-mph wind is blowing from the northeast at a 45° angle to the intended due east flight path. If the pilot does not take corrective action, in what direction and with what velocity will the plane fly? The drawing below is a vector representation of the situation.



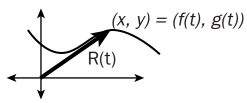
Law of Cosines for a triangle with angles *A*, *B*, *C* and sides *a*, *b*, *c*, $a^2 = b^2 + c^2 - 2bc \cos A$

Law of Sines if a triangle has sides *a*, *b*, and *c* and opposite angles *A*, *B*, and *C*, then $\sin A/a = \sin B/b = \sin C/c$

The vector representing the plane's intended velocity points due east and is labeled 600 mph, while the vector representing the wind velocity points from the northeast at a 45° angle to the line heading due east. Note the head-to-tail positioning of these two vectors. Now the parallelogram law gives the actual velocity vector for the plane, the resultant vector, as the diagonal of the parallelogram with two sides formed by these two vectors. In this case, the Pythagorean Theorem may not be used, because the triangle formed by the three vectors is not a right triangle. Fortunately, some advanced trigonometry using the so-called **Law of Cosines** and **Law of Sines**

can be used to determine that the plane's actual velocity relative to the ground is 566.75 mph at an angle of 3.58° south of the line representing due east. In navigation, angles are typically measured clockwise from due north, so a navigator might report that this plane was traveling at 566.75 mph on a heading, or bearing, of 93.58°.

As another example of how mathematicians and physicists use vectors, consider a point moving in the *xy*-coordinate plane so that it traces out some curve as the path of its motion. As the point moves along this curve, the x and y coordinates are changing as functions of time. Suppose that $x = f(t)$ and $y = g(t)$. Now the mathematician will say that the position at any time t is $(f(t), g(t))$ and that the position vector for the point is $\mathbf{R}(t) = (f(t), g(t)) = f(t)\mathbf{i} + g(t)\mathbf{j}$. The physicist will say that the position vector $\mathbf{R}(t)$ is an arrow starting at the origin and ending with the head of the arrow at the point $(f(t), g(t))$. (See the figure below.) It is now possible to define the velocity and acceleration vectors for this motion in terms of ideas from calculus, which are beyond the scope of this article.



The mathematician's definition of a vector may be extended to three or more dimensions as needed for applications in higher dimensional space. For example, in three dimensions, a vector is defined as an ordered triple of real numbers. So the vector $\mathbf{R}(t) = (f(t), g(t), h(t)) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ could be a position vector that traces out a curve in three-dimensional space. SEE ALSO Flight, Measurements of; Numbers, Complex.

Stephen Robinson

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Virtual Reality

"Virtual Reality," or VR (also known as "artificial reality" (AR) or "cyberspace"), is the creation of an interactive computer-generated spatial environment. This simulated environment can represent what one might encounter in everyday experience, represent pure fantasy, or be a combination of both.

Early "first-generation" computer interfaces handled only simple one-dimensional (1D) streams of text. The second generation, developed in the late 1970s and early 1980s for two-dimensional (2D) environments, started to use a computer screen's windows, icons, menus, and pointers (WIMP), including sliders, clicking to select, dragging to move, and so on. The third generation of interfaces is characterized by three-dimensional (3D) models and expressiveness.

GENERATIONS AND DIMENSIONS OF COMPUTER–HUMAN USER INTERFACES

Generation/ Dimensions	Mode	Input	Output
First/1D	textual	keyboard line editor	teletype monaural sound
Second/2D	planar	screen editor mouse joystick trackball touchpad light pen	image-based visuals stereo panning
Third/3D	aural	speech understanding head-tracking	speech synthesis MIDI spatial sound
	haptic: tactile and kinesthetic	3D joystick, spaceball DataGlove mouse, bird, bat, wand gesture recognition handwriting recognition	tactile displays Braille devices force-feedback displays motion platforms
	olfactory	gas detectors	smell emitters
	gustatory	??	?
	visual	head- and eye-tracking	3D graphic-based visuals stereoscopic systems; head-worn displays holograms vibrating mirrors

realtime occuring immediately, allowing interaction without significant delay

position-tracking sensing the location and/or orientation of an object

multimodal input/output (I/O) multimedia control and display that uses various senses and interaction styles

stereographics presenting slightly different views to left and right eyes, so that graphic scenes acquire depth

spatial sound audio channels endowed with directional and positional attributes (like azimuth, elevation, and range) and room effects (like echoes and reverberation)

bits binary digits: the smallest units of information that a digital computer can represent (high/low, on/off, true/false)

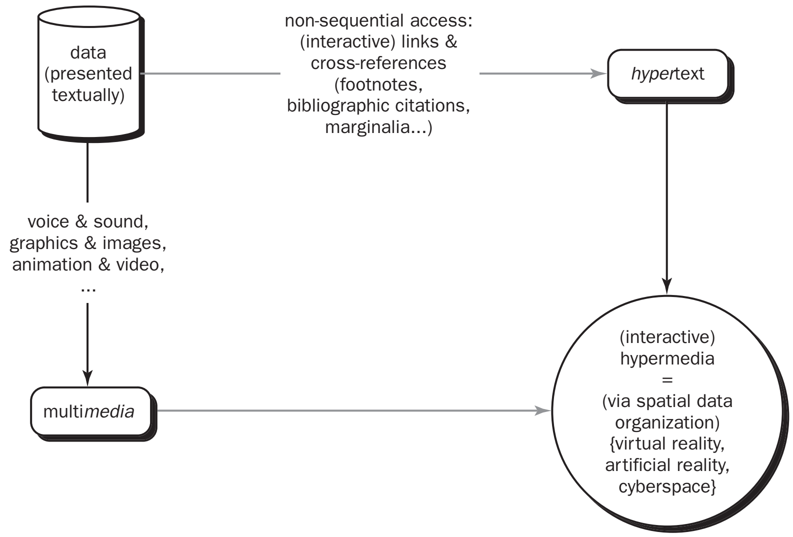
VR uses various computer techniques, including **realtime** 3D computer graphics and animation, **position-tracking**, and **multimodal input/output (I/O)**, especially **stereographics** and **spatial sound**. Fully developed VR will use all the senses: vision (seeing), audition (hearing), haptics (feeling, including pressure, position, temperature, texture, vibration), and eventually olfaction (smell) and gustation (taste).

Humans have the capacity to absorb a great deal of data, but traditional two-dimensional computer interfaces rarely generate more than a few hundred **bits** per second worth of data. Traditional computer interfaces force a user to interact graphically, in the plane of the screen. Virtual reality, however, opens up this interaction between user and computer by creating an 3D-environment in which the user can manipulate volumetric objects and navigate through spaces.

Virtual Reality and Immersive Hypermedia

If simple, linear (1D) text, such as a story, is augmented with extra media—such as sound, graphics, images, animation, video, and so on—it becomes multimedia. If this same story is extended with nonlinear attributes—such as annotations, cross-references, footnotes, marginalia, bibliographic citations, and hyperlinks—it becomes hypertext. The combination of multi*media* and *hyper*text is called "hypermedia."

VR is interactive hypermedia because it gives users a sense of immersion or presence in a virtual space, including a flexible perspective or point of view, with the ability to move about.



Multi*media* x *Hyper*text = *Hypermedia*

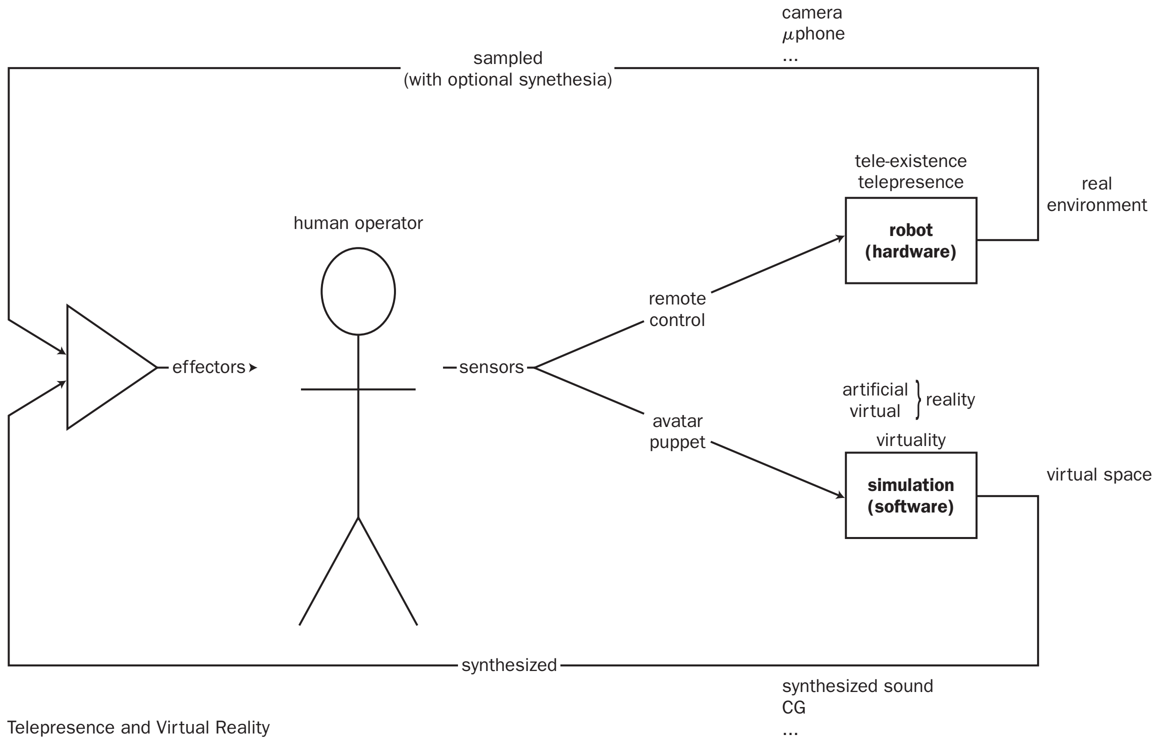
"Classic" VR uses a head-worn display (HWD), also known as a head-mounted display (HMD), that presents a synthetic environment via stereophonic audio and miniature displays in front of the eyes. Such a helmet, or "brain bucket," often uses a position tracker to determine the orientation of the wearer and adjust the displays accordingly. Users may also wear "DataGloves," sets of finger and hand sensors that can be used to virtually grasp, manipulate, and move virtual objects. Full-body VR-suits are manufactured and sold, but they are less common because they are still expensive and cumbersome.

Mixed Reality

Reality and virtuality are not opposites, but rather two ends of a spectrum. What is usually thought of as "reality" is actually filled with sources of virtual information (such as telephones, televisions, and computers), and virtuality has many artifacts of the real world, such as gravity.

Along the reality-virtuality spectrum are techniques called "mixed reality," also variously known as or associated with augmented, enhanced, hybrid, mediated, or virtualized reality/virtuality. In a mixed reality system, sampled data (from the "real world") and synthesized data (generated by computer) are combined. An example of a mixed reality is a computer graphic scene that includes images captured by a camera.

Head-mounted displays (HMDs) are important for mixed reality systems. A typical mixed reality system adds simulation to reality by either overlaying computer graphics on a transparent display ("optical see-through," which uses half-silvered mirrors) or mixing computer graphics with a video signal captured by a camera mounted in front of the face, presenting the result via an opaque display ("video see-through"). Similarly, computer-generated spatial sound can be added to the user's environment to provide navigation cues. Mixed reality systems encourage "synesthesia," or cross-sensory experiences. For example, infrared heat sensor data could be shifted into the visible spectrum.



Telepresence and Virtual Reality

Liquid Presence

A rich multimedia environment, like that enabled by (but unfortunately not always provided by) VR, carries the danger of overwhelming a user with too much information. Control mechanisms are needed to limit the media streams and help the user focus attention. For example, "radio buttons," like those used in a car to select a station, automatically cancel any previous selection (since only one station is listened to at once). On an audio mixing console, controls are associated with every channel, so they may be selectively disabled with "mute" and exclusively heard with "solo" (in the spirit of "anything not mandatory is forbidden"). Predicate calculus provides a mathematical notation for describing logical relations. For these mixing console functions, the relation can be written:

$$\text{active}(\text{source}_x) = \neg\ \text{mute}(\text{source}_x) \wedge (\exists y\ \text{solo}(\text{source}_y) \Rightarrow \text{solo}(\text{source}_x)\ ),$$

where $\neg$ means "not," $\wedge$ means "and," $\exists$ means "there exists," and $\Rightarrow$ means "implies."

This expression means that a particular source channel x is active unless it has been explicitly excluded (turned off) or another channel y has been included (focused upon) when the original source x is ignored. Symmetrically, the opposite of an information or media source is a sink. In an articulated sound system, equivalents of mute and solo are deafen and attend:

$$\text{active}(\text{sink}_x) = \neg\ \text{deafen}(\text{sink}_x) \wedge (\exists y\ \text{attend}(\text{sink}_y) \Rightarrow \text{attend}(\text{sink}_x)\ ).$$

In general, a user might want to listen to multiple audio channels simultaneously. For example, one might want to have a private conversation with a small number of people while simultaneously monitoring an ongoing conference as well as a nursery intercom, installing pairs of virtual ears in each interesting space. Such an omnipresence capability is equally useful

for other sensory modalities. For instance, a guard might generally have access to many video channels from security cameras but may sometimes want to focus on some subset of them or disable some of them (for privacy).

Virtual environments allow users to control the quality and quantity of their presence. For example, some interfaces allow a user to cut/paste a representative icon between windows symbolizing rooms, using the pasteboard as a science-fiction teleporter. Coupling such capability with a metaphorical replicator, a user can copy/paste their icon, allowing them to clone their **avatar** and distribute their presence. A general predicate calculus expression modeling multimedia attributes and such liquid presence is:

$$\text{active}(x) = \neg\ \text{exclude}(x) \wedge (\exists y\ \text{include}(y) \Rightarrow \text{include}(x)).$$

avatar representation of user in virtual space (after the Hindu idea of an incarnation of a deity in human form)

Related Technology and Approaches

Important core technologies for VR include computer graphics and animation, multimedia, hypermedia, spatial sound, and force-feedback displays. Trackers (including magnetic, inertial, ultrasonic, and optical), along with global positioning systems (GPSs), are needed to sense users' positions and, in the case of mixed reality systems, align the displays with the real world. VR systems are often integrated with other advanced interface technology, like speech understanding and synthesis. Virtual Reality is also encouraged by Internet technology, including Java (and Java3D), XML (eXtensible Markup Language), MPEG-4 (for low bit rate videoconferencing and videophony), and QTVR (QuickTime for Virtual Reality, enabling panoramic image-based rendering), as well as the VRML (Virtual Reality Modeling Language) and its successor X3D.

Expressive figures are increasingly important in virtual environments, and virtual humans, also known as "vactors" (virtual actors), integrate natural body motion (sometimes using "mocap," or motion capture, along with kinematics and physics), facial expressions, skin, muscles, hair, and clothes. As such technology matures, its realtime performance will approach in realism the prerendered special effects of Hollywood movies. A related idea is "A-life" (for "artificial life"), the programming of organic or natural-seeming processes, including genetic algorithms, cellular automata, and artificial intelligence.

Applications

Applications of virtual reality and mixed reality are unlimited, but are currently focused on allowing users to explore and design spaces before they are built; "infoviz," or information visualization (of financial or scientific data, for example); simulations (of vehicles, factories, etc.); entertainment and games (like *m*assively *m*ultiplayer *o*nline *r*ole *p*laying *g*ames [MMORPG], modern equivalents of the classic *Dungeons and Dragons* games); conferencing (chatspaces and collaborative systems); and, especially for mixed reality, medicine.

In the Future

Future goals of virtual reality include a whole-body user interface paradigm and ultimately a system that allows people to enjoy completely virtual worlds without any restrictions, like the Holodeck on the television show *Star Trek:*

The helmet, visor, and glove in this virtual reality (VR) simulation give the user the sensation of driving a car.

Next Generation. Current VR systems are more like 3D extensions to 2D interfaces, in which the world has become a mouse pad and the user has become a mouse. For example, the Vivid Mandala system (www.vivid.com) uses "mirror VR," in which users watch a **chromakey** reflection of themselves placed in computer graphics and photographic context, gesturing through fantasy scenarios.

chromakey photographing an object shot against a known color, which can be replaced with an arbitrary background (like the weather maps on television newscasts)

On the horizon are full-immersion photorealistic and sonorealistic interfaces in shared virtual environments via high-bandwidth wireless Internet connections, perhaps using visual display technology that writes images directly to the retina from one's eyewear, or head-mounted projective systems that reflect images back to each wearer via retroreflective surfaces in a room. Virtual reality will be extended by mixed reality and complemented by pervasive or ubiquitous computing (also known as "ubicomp")—that exploits an environment saturated with networked computers and transparent interfaces, including information furniture and appliances sensitive to human intentions—and wearable computers. SEE ALSO COMPUTER ANIMATION; COMPUTERS, FUTURE.

Michael Cohen

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Java3D: 3D framework for Java. <http://java.sun.com/products/java-media/3D/>.

MPEG Home Page. <http://www.cselt.it/mpeg>.

Presence: Teleoperators and Virtual Environments. <http://mitpress.mit.edu/journal-home.tcl?issn=10547460>.

QuickTime for Virtual Reality. <http://www.apple.com/quicktime/qtvr/>.

SIGGRAPH: Special Interest Group on Computer Graphics. <http://www.siggraph.org>.

Web 3D Consortium. <http://www.web3d.org>.

XML: eXtensible Markup Language. <http://www.w3.org/XML/>.

Vision, Measurement of

Many people wear glasses or contact lenses. In most cases, this is to correct some inherent defect in vision. The three most common vision defects are farsightedness, nearsightedness, and astigmatism.

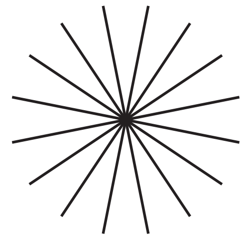
Detecting Vision Defects

The most common test for detecting vision defects is the visual acuity test. This is the well-known eye chart with a big letter E at the top. The test is conducted by having the patient stand or sit a certain distance from the chart. The person being tested covers one eye at a time and reads the smallest line of text she or he can see clearly. Each line of text is identified by a pair of numbers such as 20/15, 20/40, and so on. The first number is the distance (in feet) from the eye chart. The second number is the distance at which a person with "normal" vision can read that line. So a visual acuity of 20/40 means that a person with normal vision could read that line from 40 feet away.

Normal vision refers to average vision for the general population and is considered reasonably good vision. Many people have vision that is significantly better than normal. A visual acuity of 20/15 (common in young people) means that the person can read a line at 20 feet that a person with normal vision could only read at 15 feet.

Metric units are used in countries that have adopted the metric system. In those countries, normal visual acuity would be reported as 6/6 (that is, in units of meters where 1 meter equals about 3.28 feet). Minimum levels of visual acuity (for example, required for driving without corrective lenses) would be 6/12.

Astigmatism can be easily detected by having the patient examine a chart such as the one shown on the next page. Astigmatism will cause some of the lines to appear darker or lighter. Some lines may appear to be blurry or double.



Presbyopia is a common condition among older adults. This is the progressive inability to focus on nearby objects. This condition is evident when a person tries to read something by holding it at arm's length. The condition occurs when the lens in the eye becomes less flexible with age.

Corrective Lenses

If a person has visual acuity that is less than adequate, corrective lenses may be appropriate. Eyeglasses and contact lenses are available to correct nearsightedness, farsightedness, and astigmatism.

The human eye is much like a camera. It has a lens and a diaphragm (called the iris) for controlling the amount of light that enters the eye. Instead of film or a Charge Coupled Device (CCD) chip, the eye has a layer of light-sensitive nerve tissue at the back called the retina. The eye does not have a shutter. However, the human nervous system acts much like a shutter by processing individual images received on the retina at the rate of about 30 images per second.

The retina is highly curved (instead of flat, like the film in a camera) but our nervous system is trained from infancy to correctly interpret images from this curved surface. So straight lines are seen as straight lines, vertical lines appear vertical, and the horizon looks like a horizontal line.

The bending of light rays necessary to get the image correctly focused on the curved retina is done mostly by the cornea at the front of the eye. Modern surgical techniques have been developed for altering the shape of the cornea to correct many vision defects without the need for corrective lenses. The lens of the eye provides the small adjustments necessary for focusing at different distances. To focus on distant objects, the ciliary muscles around the lens relax and the lens gets thinner. This lengthens the **focal length** of the lens so that distant objects are correctly focused. To focus on objects that are near, the ciliary muscles contract, forcing the lens to get thicker and shortening its focal length.

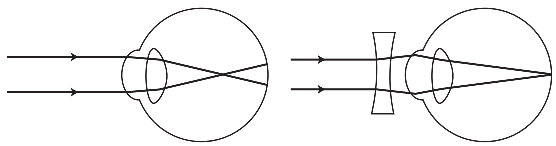
focal length the distance from the focal point (the principle point of focus) to the surface of a lens or concave mirror

The closest distance at which objects can be sharply focused is called the near point. The far point is the maximum distance at which objects can be seen clearly. An eye is considered normal with a near point of 25 cm (centimeters) and a far point at infinity (able to focus parallel rays). However, these distances change with age. Children can focus on objects as close as 10 cm, while older adults may be able to focus on objects no nearer than 50 cm. Calling any eye "normal" is somewhat misleading, because 67 percent of adults in the United States use glasses or other corrective lenses.

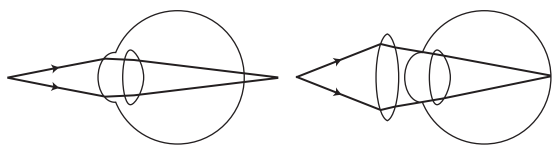
Farsightedness, or hyperopia, refers to a vision defect that causes the eye to be unable to focus on nearby objects because the eye is either too short or the cornea is the wrong shape. For persons with hyperopia, the near point is greater than 25 cm. A similar vision defect is known as pres-

byopia. Hyperopia is due to a misshapen eyeball or cornea, whereas presbyopia is due to the gradual stiffening of the crystalline lens. Presbyopia also prevents people from being able to focus on nearby objects, but it affects only the near point. Converging lenses (magnifying lenses) will correct for both hyperopia and presbyopia by making light rays appear to come from objects that are farther away.

Nearsightedness, or myopia, refers to the vision defect that causes an eye to be able to only focus on nearby objects. The far point will be less than infinity. It is caused by an eye that is too long or by a cornea that is incorrectly shaped. In both cases, the images of distant objects fall in front of the retina and appear blurry. Diverging lenses will correct this problem, because they cause parallel rays to diverge as if they came from an object that is closer.



Nearsighted eye/corrected with diverging lens



Farsighted eye/corrected with converging lens

Astigmatism is caused by a lens or cornea that is somewhat distorted so that point objects appear as short lines. The image may be spread out horizontally, vertically, or obliquely. Astigmatism is corrected by using a lens that is also "out of round." The lens is somewhat cylindrical instead of spherical. Frequently, astigmatism occurs along with hyperopia or myopia, so the lens must be ground to compensate for both problems.

Contact lenses work in a way very similar to eyeglasses. However, they have one advantage over eyeglasses. Since the contact lens is much closer to the eye, less correction is needed. This gives the user of contact lenses a wider field of view.

Calculating the power of corrective lenses can be done in some cases using the familiar lens formula. If the focal length of a lens is f, the distance from the lens to the object is d_o, and the distance from the lens to the image is d_i, then:

$$\frac{1}{f} = \frac{1}{d_i} + \frac{1}{d_o}$$

As an example, suppose a farsighted person has a near point of 100 cm. What power of glasses will this person need to be able to read a newspaper at 25 cm? To solve this we must recognize that the image will be virtual (as in a magnifying glass) so d_i, will have a negative value. The glasses must provide an image that appears to be 100 cm away when the object is 25 cm away.

$$\frac{1}{f} = \frac{1}{d_i} + \frac{1}{d_o} = \frac{1}{-100} + \frac{1}{25} = \frac{1}{33}$$

diopter a measure of the power of a lens or a prism, equal to the reciprocal of its focal length in meters

The power, P, of a lens in **diopters** (D) is $P = \frac{1}{f}$ when f is in meters, so $P = \frac{1}{f} = \frac{1}{0.3333} = 3.0\text{D}$. SEE ALSO Light.

Elliot Richmond

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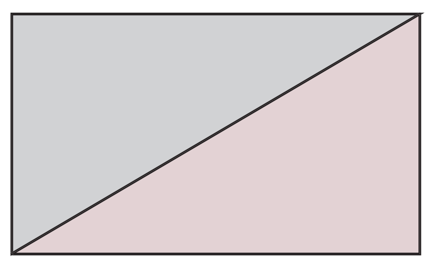
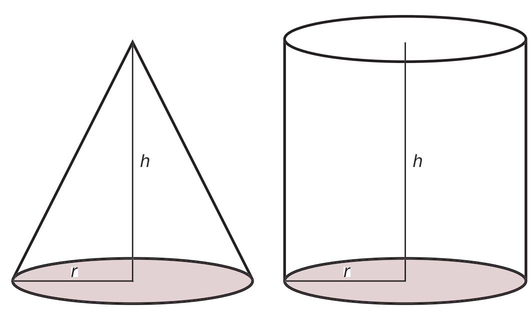
Volume of Cone and Cylinder

Picture a rectangle divided into two right triangles by a diagonal. How is the area of the right triangle formed by the diagonal related to the area of the rectangle? The area of any rectangle is the product of its width and length. For example, if a rectangle is 3 inches wide and 5 inches long, its area is 15 square inches (length times width). The figure below shows a rectangle "split" along a diagonal, demonstrating that the rectangle can be thought of as two equal right triangles joined together. The areas of rectangles and right triangles are proportional to one another: a rectangle has twice the area of the right triangle formed by its diagonal.



In a similar way, the volumes of a cone and a cylinder that have identical bases and heights are proportional. If a cone and a cylinder have bases (shown in color) with equal areas, and both have identical heights, then the volume of the cone is one-third the volume of the cylinder.

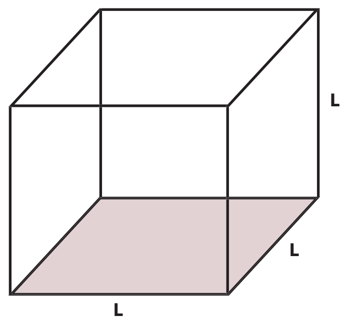


Imagine turning the cone in the figure upside down, with its point downward. If the cone were hollow with its top open, it could be filled with a liquid just like an ice cream cone. One would have to fill and pour the contents of the cone into the cylinder three times in order to fill up the cylinder.

The figure above also illustrates the terms height and radius for a cone and a cylinder. The base of the cone is a circle of radius r. The height of

the cone is the length h of the straight line from the cone's tip to the center of its circular base. Both ends of a cylinder are circles, each of radius r. The height of the cylinder is the length h between the centers of the two ends.

The volume relationship between these cones and cylinders with equal bases and heights can be expressed mathematically. The volume of an object is the amount of space enclosed within it. For example, the volume of a cube is the area of one side times its height. The figure below shows a cube. The area of its base is indicated in color. Multiplying this (colored) area by the height L of the cube gives its volume. And since each dimension (length, width and height) of a cube is identical, its volume is $L \times L \times L$, or L^3, where L is the length of each side.



The same procedure can be applied to finding the volume of a cylinder. That is, the area of the base of the cylinder times the height of the cylinder gives its volume. The bases of the cylinder and cone shown previously are circles. The area of a circle is πr^2, where r is the radius of the circle. Therefore, the volume V_{cyl} is given by the equation: $V_{cyl} = \pi r^2 h$ (area of its circular base times its height) where r is the radius of the cylinder and h is its height. The volume of the cone (V_{cone}) is one-third that of a cylinder that has the same base and height: $V_{cone} = \frac{1}{3}\pi r^2 h$.

The cones and cylinders shown previously are right circular cones and right circular cylinders, which means that the central axis of each is perpendicular to the base. There are other types of cylinders and cones, and the proportions and equations that have been developed above also apply to these other types of cylinders and cones.

Philip Edward Koth (with William Arthur Atkins)

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Weather Forecasting Models

The weather has an astonishing impact on our lives, ranging from the frustration of being caught in a sudden downpour to the trillions of dollars spent in weather-sensitive businesses. Consequently, a great deal of time, effort, money, and technology is used to predict the weather. In the attempt to improve weather prediction, **meteorologists** rely on increasingly sophisticated computers and computer models.

meteorologists people who study the atmosphere in order to understand weather and climate

When researchers created the first computer models of Earth's atmosphere in the mid-twentieth century, they worked with computers that were extremely limited compared to the supercomputers that exist today. As a result, the first weather models were oversimplified, although they still provided valuable insights. As computer technology advanced, more of the factors that influenced the atmosphere, such as physical variables, could be taken into account, and the complexity and accuracy of weather forecast models increased.

Today's computer-enhanced weather forecasts can be successful only if the data observations are accurate and complete. Ground-based weather stations are equipped with many instruments, including barometers (which measure atmospheric pressure, or the weight of the air); wind vanes (which measure wind direction), anemometers (which measure wind velocity, or speed); hygrometers (which measure the moisture, or humidity, in the air); thermometers; and rain gauges. Data obtained through these instruments are combined with data from aircraft, ships, satellites, and radar networks. These data are then put into complex mathematical equations that model the physical laws governing atmospheric conditions. Solving these equations with the aid of computers yields a description—a forecast—of the atmosphere derived from its current state (that is, the initial values). This can then be interpreted in terms of weather—rain, temperature, sunshine, and wind.

Early computer models relied heavily on weather station data and measurements obtained from weather balloons. A gondola carries instruments that measure wind speed, temperature, humidity, and other atmospheric conditions.

Early Attempts to Forecast Weather

Several decades prior to the advent of the modern computer, Vilhelm Bjerknes, a Norwegian physicist-turned-meteorologist, advocated a mathematical approach to weather forecasting. This approach was based on his belief that it was possible to bring together the full range of observation and theory to predict weather changes. British scientist Lewis Fry Richardson was the first person to attempt to work out Bjerknes's program. During and

algorithm a rule or procedure used to solve a mathematical problem

shortly after World War I, he went on to devise an **algorithmic** scheme of weather prediction. Unfortunately, Richardson's method required 6 weeks to calculate a 6-hour advance in the weather. Moreover, the results from his method were inaccurate. Consequently, Richardson's work, which was widely noticed, convinced contemporary meteorologists that a computational approach to weather prediction was completely impractical. Yet it laid the foundation for numerical weather prediction.

Following World War II, a Hungarian-American mathematician named John von Neumann began making plans to build a powerful and versatile electromechanical computer devoted to the advancement of the mathematical sciences. Von Neumann's objective was to demonstrate, through a particular scientific problem, the revolutionary potential of the computer. He chose weather prediction for that problem, and in 1946 established the Meteorology Project at the Institute for Advanced Study in Princeton, New Jersey, to work on its resolution.

The Chaos Connection. By the 1960s the work of American atmospheric sciences researcher Edward Lorenz showed that no matter how good the data or forecasting methods, accurate day-by-day forecasts of the weather may not be possible for future periods beyond about two weeks. Because Earth's atmosphere follows the complex, nonlinear rules of **fluid dynamics**, the smallest error in the determination of initial conditions (such as wind, temperature, and pressure) will amplify because of the nonlinear dynamics of atmospheric processes, so that a forecast at some point becomes nearly useless. This dependence of prediction on the precision of input values is a basic tenet of mathematical **chaos theory**.

fluid dynamics the science of fluids in motion

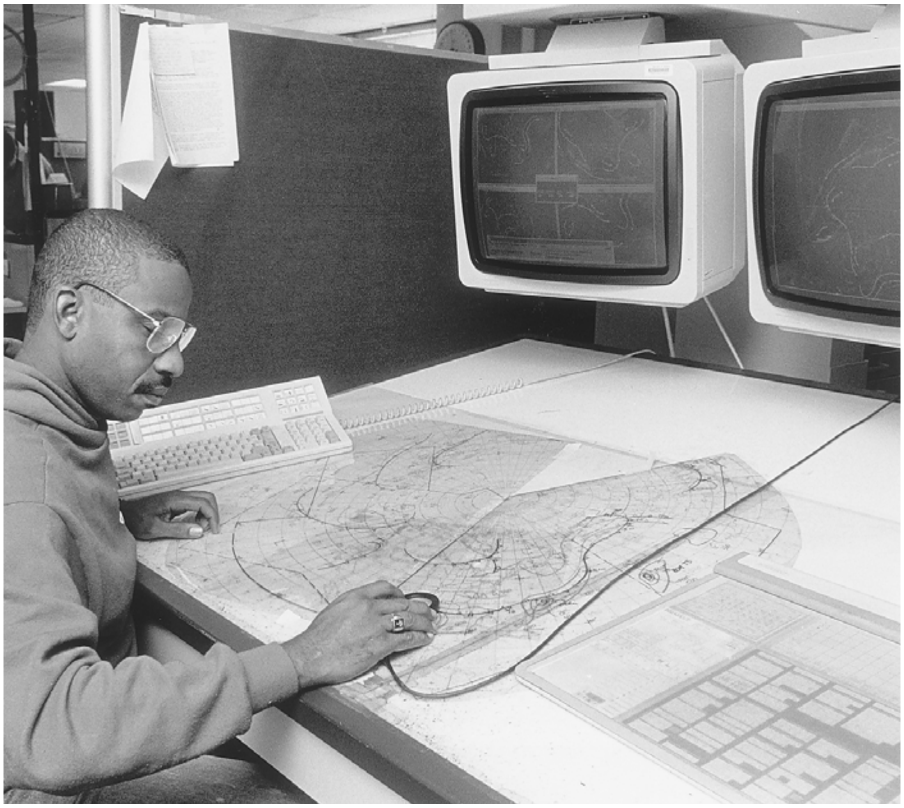
chaos theory the qualitative study of unstable aperiodic behavior in deterministic nonlinear dynamical systems

The fundamental ideas of chaos theory are captured in what is known as the "butterfly effect," which, roughly stated, holds that a butterfly flapping its wings can set in motion a storm on the other side of the world. Yet with the next flap, nothing of meteorological significance may happen. Hence, chaos places a limit on the accuracy of computer-enhanced weather forecasts as the forecast period and geographic scale increases.

Relationship between Forecasting and Computers

The desire to make better weather predictions coincided with the evolution of today's modern electronic computers. In fact, computers and meteorology have helped shape one another. Moreover, weather forecasting has helped drive the development of better multimedia software, both for forecasting purposes and for communicating forecasts to specific audiences.

Forecasting the weather by computer is called numerical weather prediction. To make a skillful forecast, weather forecasters must understand the strengths and limitations of computer models and recognize that models can still be mistaken. Forecasters study the output from models over time and compare the forecast output to the weather that actually occurs. This is how they determine the biases, strengths, and weaknesses of the models. They also may modify what they learn from computer models based on their own experience forecasting the weather through physical and dynamical processes. Hence, good forecasters must not only have knowledge of meteorological variables, but they must also be able to use their intuition.



Increasingly powerful computers have enabled more sophisticated models of numerical weather prediction. Here a geophysicist studies weather patterns using maps and visualization software.

Improvements in Forecasts. Continued improvements in computer technology allows more of the dynamical and physical factors influencing the atmosphere to be added to models, thereby improving weather forecasts, including those for specific conditions. For example, some forecasting models help predict tropical weather features such as hurricanes, typhoons, and monsoons. Other models help forecast smaller-scale features such as thunderstorms and lightning. As these forecasting models improve, forecasters should be able to issue more accurate and timely storm warnings and advisories.

Improvements in Communication. Even if weather data were extraordinarily precise, weather forecasts would have little meaning if consumers of the data did not receive them in an effective and timely manner. Local and national weather reports demonstrate the strong relationship between weather forecasting and computer technology. Today's state-of-the-art visualization techniques for weather forecasts, including three-dimensional data, time-series, and virtual scenery, have made the weather segment a highly watchable part of television newscasts. A three-dimensional representation of weather patterns is typically superimposed on local and national geographic maps. With a click of a computer mouse, cloud formations and fronts appear, and predictions about temperature, precipitation, and other weather-related data march across the screen.

The twentieth century saw the inventions of radio broadcasting, television, and the Internet. The twenty-first century undoubtedly will bring other innovations in technology and media that will play an equally significant role

in the dissemination of weather information. SEE ALSO Chaos; Computer Simulations; Predictions; Temperature, Measurement of; Weather, Violent.

Marilyn K. Simon

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Weather, Violent

Violent weather can occur anywhere. Depending on where an outbreak occurs, there will be different types of storms. However, not all storms are violent. There are, for example, thunderstorms that are not severe, and tornadoes that do not cause extensive damage. Common measurement scales do not provide an understanding of the strength of a weather disturbance. Consequently, **meteorologists** must use different scales to measure the strength of weather systems, and to make predictions about when violent weather may strike.

meteorologist a person who studies the atmosphere in order to understand weather and climate

Thunderstorms

There are three ingredients necessary for a thunderstorm to form:

1. moisture must be present in the lower levels of the atmosphere;
2. cold air must be present in the upper atmosphere; and
3. there must be a catalyst to push the warm air into the cold air.

This catalyst is usually in the form of a front, which is the interface between air masses at different temperatures. As the warm air rises it cools and some of the water vapor turns into clouds. Eventually the air mass will reach an area where it is the same temperature as the area surrounding it, and thunderstorms can occur.

Thunderstorms can be measured as strong or severe. Some thunderstorms, though, may be neither. Severe thunderstorms are classified as having winds greater than or equal to 58 mph (miles per hour), and/or hail greater than or equal to three-quarters of an inch. If a severe thunderstorm is strong enough, it can become a supercell thunderstorm. A supercell thunderstorm is the type that is most likely to spawn a tornado. In a supercell, as the air moves upward it begins to rotate, which does not happen in other types of thunderstorms. When the whole cell rotates, it is called a mesocyclone.

Tracking Thunderstorms. Many meteorologists use Doppler radar to track thunderstorms. Christian Doppler suggested the idea behind Doppler radar over 150 years ago. It differs from traditional weather radar because it can detect not only where a storm is located, but also how fast the air inside the storm is moving.

Doppler radar transmits pulses of waves that are reflected by water droplets in the air. Some of the energy reflected by the droplets then returns to the radar unit and shows how the storm is moving and what it looks like. Doppler radar also shows where moisture is located in a storm because where there is a higher moisture content, there is greater reflectivity since there is more for the waves to reflect off of.

The ability to determine the structure of storms is particularly useful in the detection of severe weather. For example, on radar, areas of a storm containing hail will display denser than areas with just rain. Doppler radar is therefore an invaluable tool in storm detection and prediction. Meteorologists can monitor a storm across large areas and make sure the public stays informed about its progress.

Tornadoes

Tornadoes are formed in thunderstorms when winds meet at varying angles. If low-level winds flow in a southerly direction and upper-level wind flows towards the west, the air masses can interact violently. The two paths will begin to swirl against each other, forming a vortex. As this vortex increases it will appear as a funnel to someone on the ground. Once this funnel touches the ground, it becomes a tornado.

One of the recent scales created to measure the severity of a tornado is the Fujita scale. Developed in 1971, the Fujita scale is used to measure the intensity of a tornado by examining the damage caused after it has passed over man-made structures. Tetsuya Theodore Fujita (1920–1998) took into account the wind speed of a tornado and the damage it caused, and from that he was able to assign a number to represent the intensity of the tornado.

On the Fujita scale, an F2 tornado is classified as a significant tornado with winds between 113 and 157 mph. An F2 tornado can cause considerable damage. Roofs may be torn off frame houses, mobile homes may be demolished, and large trees can be uprooted. In an F5 tornado winds can reach between 261 and 318 mph. A tornado of this strength can lift frame houses from their foundation and carry them considerable distances, send automobile-sized objects through the air, and cause structural damage to steel-reinforced structures.

Not long after the inception of the Fujita scale, it was combined with the work of Allen Pearson to also consider the components of path length and width. Adding these two measurements together provides more accurate tornado readings.

The Fujita scale is not perfect. Wind speed does not always accurately correlate with the amount of damage caused by a tornado. For example, a tornado with a great intensity that touches down in a rural area may rank lower on the Fujita scale than a less intense tornado that touches down in an urban area. This is because a tornado is a rural area could cause less damage

HOW REALISTIC IS HOLLYWOOD?

It is possible to measure a tornado from the inside, just as the characters were trying to do in the 1996 movie *Twister*. The idea for the device that they used, Dorothy, actually came from a similar instrument, TOTO (TOtable Tornado Observatory), invented by Howie Bluestein. TOTO was retired after 1987 and is now on display in Silver Spring, Maryland, at a facility of the National Oceanic and Atmospheric Administration.

Scientists use current data measurements and historical information to predict the likelihood of tornadoes.

since there is less to be damaged. An accurate F-scale rating often depends on the experience of the person who assigns the rating.

Though the Fujita scale does have flaws, it is an improvement over the rating system that had previously been in place. The scale is simple enough to be put into daily practice without substantial added expense, and a rating can be assigned in a reasonable amount of time.

Predicting Tornadoes. While tornadoes can occur during any time of year in the United States, they most frequently occur during the months of March through May. Organizations such as the National Weather Service keep detailed records of when and where tornadoes occur. From these records, they can make predictions about when the most favorable times are for tornadoes to occur in different areas of the country. Different areas have different times when tornado development is most likely. For example, by keeping records, it has been shown that the mid-section of the country is much more likely to have tornadoes than the west. The amount of tornadoes seen in the mid-section has earned it the nickname "tornado alley."

Hurricanes

Hurricanes form when there is a pre-existing weather disturbance, warm tropical oceans, and light winds. If these conditions persist, they can combine and produce violent wind, waves, and torrential rain.

A hurricane is a type of tropical cyclone. These cyclones are classified by their top wind speeds as a tropical depression, tropical storm, or a hurricane. In a hurricane, walls of wind and rain form a circle of spinning wind and water. In the very center of the storm there is a calm called the "eye." At the edges of the eye, the winds are the strongest. Hurricanes cover huge areas of water and are the source of considerable damage when they come onshore.

Like tornadoes, hurricanes are classified according to their wind speed. They are measured on the Saffir-Simpson hurricane scale. The number rating assigned from the scale is based on a hurricane's intensity. A category one hurricane has winds between 74 and 95 mph. This type of hurricane will not have a very high storm surge, and will cause minimal damage to building structures. In a category five hurricane, top sustained winds will be greater than 155 mph. The **storm surge** in a hurricane like this can be greater than 18 feet above normal, flooding will occur, and evacuation will be necessary in low-lying areas.

storm surge the front of a hurricane, which bulges because of strong winds; can be the most damaging part of a hurricane

The number assigned to a hurricane is not always a reliable indicator of how much damage a hurricane can cause. Lower-category storms can cause more damage than high-category storms depending on where they make landfall. In most hurricanes, loss of life occurs due to flooding, and lesser hurricanes, and even tropical storms, can often bring more extensive rainfall than a higher-category hurricane.

Predicting Hurricanes. For years meteorologists have been tracking and keeping records on where and when hurricanes occur. The official start of hurricane season in the Atlantic Basin is June 1. The peak time for hurricane formation is from mid-August to late October. Hurricanes can form outside of the season, but it is uncommon. Scientists have been able to designate a season for hurricanes from careful observation of storms over many years.

Not only can scientists predict the favorable time of year for hurricanes to form, but they can also make predictions on the path a hurricane may take depending on where it forms at certain times in the season. To make these predictions, scientists must use statistical analysis to find out probabilities on where and when a hurricane can form.

Floods

Floods happen when the ground becomes so saturated with moisture that it cannot hold any more. Often, floods occur when too much rain falls in a short period of time, or when snow melts quickly. Initially the ground can absorb the moisture, but eventually there is a point where the soil cannot absorb anymore. At this point, water begins to collect above ground, and flooding occurs.

A flood is measured by how much the water is above the flood stage. Flood stages are different everywhere. Measurements of a flood can be done in inches, feet, or larger units depending on the severity. We have also begun to quantify a flood by the economic damage it causes.

Predicting Floods. The prediction of floods can be statistically described by frequency curves. Flood frequency is expressed as the probability that a flood of a given magnitude will be equaled or exceeded in any one year.

The 100-year flood, for example, is the peak discharge that is expected to be equaled or exceeded on the average of once in 100 years. In other words, there is a 1 percent chance that a flood of peak magnitude will occur in a given year. Similarly, a 50-year flood has a 2 percent chance of occurring in any given year, and a 25-year flood has a 4 percent chance. This recurrence interval represents a long-term average, and these types of floods

could occur with a greater or lesser frequency. Many floods that people consider to be record-breaking are actually much lower in magnitude.

Violent weather is unavoidable. With reliable measurements and predictions, though, humans can increase their preparedness to minimize the damage and destruction violent weather can cause. SEE ALSO Predictions; Weather Forcasting Models.

Brook Ellen Hall

Internet Resources

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Web Designer

World Wide Web the part of the Internet allowing users to examine graphic "web" pages

A web designer puts information on the **World Wide Web** in the form of web pages for use by individuals and businesses. Web designers use the principles that a graphic artist would use to create their artwork. The difference, however, is that a web designer's tool is a computer, and their "paints" are computer software.

geometry the branch of mathematics that deals with the properties and relationships of points, lines, angles, surfaces, planes, and solids

Web designers use mathematics in a variety of ways. The drawing applications used by web designers require a thorough knowledge of **geometry**. Additional applications require knowledge of spatial measurements and the ability to fit text and pictures in a page layout.

Hypertext Markup Language the computer markup language used to create documents for the World Wide Web

Web designers who use source codes such as **Hypertext Markup Language** (HTML) need to know basic math for calculating the widths and heights of objects and pages. Accounting and bookkeeping are important areas of math that web designers should study if they plan to establish their own businesses.

algebra the branch of mathematics that deals with variables or unknowns representing the arithmetic numbers

Because web designers must be able to plan, design, program, and maintain web sites, a knowledge of computers and how they work is very important. Most computer science courses require **algebra** and geometry as prerequisites. SEE ALSO Computer Graphic Artist; Internet.

Marilyn Schwader

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Zero

The idea of nothingness and emptiness has inspired and puzzled mathematicians, physicists, and even philosophers. What does empty space mean? If the space is empty, does it have any physical meaning or purpose?

From the mathematical point of view, the concept of zero has eluded humans for a very long time. In his book, *The Nothing That Is*, author Robert Kaplan writes, "Zero's path through time and thought has been as full of intrigue, disguise and mistaken identity as were the careers of the travellers who first brought it to the West." But our own familiarity with zero makes it difficult to imagine a time when the concept of zero did not exist. When the last pancake is devoured and the plate is empty, there are zero pancakes left. This simple example illustrates the connection between counting and zero.

Counting is a universal human activity. Many ancient cultures, such as the Sumerians, Indians, Chinese, Egyptians, Romans, and Greeks, developed different symbols and rules for counting. But the concept of zero did not appear in number systems for a long time; and even then, the Roman number system had no symbol for zero. Sometime between the sixth and third centuries B.C.E., zero made its appearance in the Sumerian number system as a slanted double wedge.

To appreciate the significance of zero in counting, compare the decimal and Roman number system. In the decimal system, all numbers are composed of ten digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. After counting to nine, the digits are repeated in different sequences so that any number can be written with just ten digits. Also, the position of the number indicates the value of the number. For example, in 407, 4 stands for four hundreds, 0 stands for no tens, and 7 stands for seven.

The Roman number system consists of the following few basic symbols: I for 1, V for 5, and X for 10. Here are some examples of numbers written with Roman numerals.

IV = 4	XV = 15
VIII = 8	XX = 20
XIII = 13	XXX = 30

Without a symbol for zero, it becomes very awkward to write large numbers. For 50, instead of writing five Xs, the Roman system has a new symbol, L.

Performing a simple addition, such as 33 + 22, in both number systems further shows the efficiency of the decimal system. In the decimal number system, the two numbers are aligned right on top of each other and the corresponding digits are added.

$$\begin{array}{r} 33 \\ +\ 22 \\ \hline 55 \end{array}$$

In the Roman number system, the same problem is expressed as XXXIII + XXII, and the answer is expressed as LV. Placing the two Roman numbers on top of each other does not give the digits LV, and therefore when adding, it is easier to find the sum with the decimal system.

Properties of Zero

All real numbers, except 0, are either positive ($x > 0$) or negative ($x < 0$). But 0 is neither positive nor negative. Zero has many unique and curious properties, listed below.

Additive Identity: Adding 0 to any number x equals x. That is, $x + 0 = x$. Zero is called the additive identity.

Multiplication property: Multiplying any number b by 0 gives 0. That is, $b \times 0 = 0$. Therefore, the square of 0 is equal to zero ($0^2 = 0$).

Exponent property: Any number other than zero raised to the power 0 equals 1. That is, $b^0 = 1$.

Division property: A number cannot be divided by 0. Consider the problem $12/0 = x$. This means that $0 \times x$ must be equal to 12. No value of x will make $0 \times x = 12$. Therefore, division by 0 is undefined.

Undefined Division

Because division by 0 is undefined, many functions in which the denominator becomes 0 are not defined at certain points in their domain sets. For instance, $f(x) = \frac{1}{x}$ is not defined at $x = 0$; $g(x) = \frac{1}{(x - 1)}$ is not defined at $x = 1$; and $h(x) = \frac{1}{(x^2 - 1)}$ is not defined at either $x = 1$ or $x = -1$.

Even though the function $f(x) = \frac{1}{x}$ is not defined at 0, it is possible to see the behavior of the function around 0. Points can be chosen close to 0; for instance, x equal to 0.001, 0.0001, and 0.00001. The function values at these points are $f(0.001) = 1/0.001 = 1{,}000$; $f(0.0001) = 10{,}000$; and $f(0.00001) = 100{,}000$.

As x becomes smaller and approaches 0, the function values become larger. In fact, the function $f(x) = \frac{1}{x}$ grows without bound; that is, the function values has no upper ceiling, or limit, at $x = 0$. In mathematics, this behavior is described by saying that as x approaches 0, the function $f(x) = \frac{1}{x}$ approaches infinity.

Approaching Zero

Consider a sequence of numbers $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots, \frac{1}{n}$, which in decimal notation is expressed as 1, 0.5, 0.33, 0.25, 0.2, 0.16, 0.14, and so on.

Each number in the sequence $\{\frac{1}{n}\}$ is called a term. As n becomes larger, $\frac{1}{n}$ becomes increasingly smaller. When $n = 10{,}000$, $\frac{1}{n}$ is 0.0001.

The sequence $\frac{1}{n}$ approaches 0, but its terms never equals 0. However, the terms of the sequence can be as close to 0 as wanted. For instance, it is possible for the terms of the sequence to get close enough to 0 so that the difference between the two is less than a billionth, $\frac{1}{1{,}000{,}000}$ or 10^{-6}. If one takes $n > \frac{1}{10^{-6}} = 1{,}000{,}000$, then the sequence terms will be smaller than 10^{-6}. SEE ALSO Division by Zero; Limit.

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Glossary

abscissa: the x-coordinate of a point in a Cartesian coordinate plane

absolute: standing alone, without reference to arbitrary standards of measurement

absolute dating: determining the date of an artifact by measuring some physical parameter independent of context

absolute value: the non-negative value of a number regardless of sign

absolute zero: the coldest possible temperature on any temperature scale; −273° Celsius

abstract: having only intrinsic form

abstract algebra: the branch of algebra dealing with groups, rings, fields, Galois sets, and number theory

acceleration: the rate of change of an object's velocity

accelerometer: a device that measures acceleration

acute: sharp, pointed; in geometry, an angle whose measure is less than 90 degrees

additive inverse: any two numbers that add to equal 1

advection: a local change in a property of a system

aerial photography: photographs of the ground taken from an airplane or balloon; used in mapping and surveying

aerodynamics: the study of what makes things fly; the engineering discipline specializing in aircraft design

aesthetic: having to do with beauty or artistry

aesthetic value: the value associated with beauty or attractiveness; distinct from monetary value

algebra: the branch of mathematics that deals with variables or unknowns representing the arithmetic numbers

algorithm: a rule or procedure used to solve a mathematical problem

algorithmic: pertaining to an algorithm

ambiguity: the quality of doubtfulness or uncertainty

analog encoding: encoding information using continuous values of some physical quantity

analogy: comparing two things similar in some respects and inferring they are also similar in other respects

analytical geometry: describes the study of geometric properties by using algebraic operations

anergy: spent energy transferred to the environment

angle of elevation: the angle formed by a line of sight above the horizontal

angle of rotation: the angle measured from an initial position a rotating object has moved through

anti-aliasing: introducing shades of gray or other intermediate shades around an image to make the edge appear to be smoother

applications: collections of general-purpose software such as word processors and database programs used on modern personal computers

arc: a continuous portion of a circle; the portion of a circle between two line segments originating at the center of the circle

areagraph: a fine-scale rectangular grid used for determining the area of irregular plots

artifact: something made by a human and left in an archaeological context

artificial intelligence: the field of research attempting the duplication of the human thought process with digital computers or similar devices; also includes expert systems research

ASCII: an acronym that stands for American Standard Code for Information Interchange; assigns a unique 8-bit binary number to every letter of the alphabet, the digits, and most keyboard symbols

assets: real, tangible property held by a business corporation including collectible debts to the corporation

asteroid: a small object or "minor planet" orbiting the Sun, usually in the space between Mars and Jupiter

astigmatism: a defect of a lens, such as within an eye, that prevents focusing on sharply defined objects

astrolabe: a device used to measure the angle between an astronomical object and the horizon

astronomical unit (AU): the average distance of Earth from the Sun; the semi-major axis of Earth's orbit

asymptote: the line that a curve approaches but never reaches

asymptotic: pertaining to an asymptote

atmosphere (unit): a unit of pressure equal to 14.7 lbs/in^2, which is the air pressure at mean sea level

atomic weight: the relative mass of an atom based on a scale in which a specific carbon atom (carbon-12) is assigned a mass value of 12

autogiro: a rotating wing aircraft with a powered propellor to provide thrust and an unpowered rotor for lift; also spelled "autogyro"

avatar: representation of user in virtual space (after the Hindu idea of an incarnation of a deity in human form)

average rate of change: how one variable changes as the other variable increases by a single unit

axiom: a statement regarded as self-evident; accepted without proof

axiomatic system: a system of logic based on certain axioms and definitions that are accepted as true without proof

axis: an imaginary line about which an object rotates

axon: fiber of a nerve cell that carries action potentials (electrochemical impulses)

azimuth: the angle, measured along the horizon, between north and the position of an object or direction of movement

azimuthal projections: a projection of a curved surface onto a flat plane

bandwidth: a range within a band of wavelengths or frequencies

base-10: a number system in which each place represents a power of 10 larger than the place to its right

base-2: a binary number system in which each place represents a power of 2 larger than the place to its right

base-20: a number system in which each place represents a power of 20 larger than the place to the right

base-60: a number system used by ancient Mesopotamian cultures for some calculations in which each place represents a power of 60 larger than the place to its right

baseline: the distance between two points used in parallax measurements or other triangulation techniques

Bernoulli's Equation: a first order, nonlinear differential equation with many applications in fluid dynamics

biased sampling: obtaining a nonrandom sample; choosing a sample to represent a particular viewpoint instead of the whole population

bidirectional frame: in compressed video, a frame between two other frames; the information is based on what changed from the previous frame as well as what will change in the next frame

bifurcation value: the numerical value near which small changes in the initial value of a variable can cause a function to take on widely different values or even completely different behaviors after several iterations

Big Bang: the singular event thought by most cosmologists to represent the beginning of our universe; at the moment of the big bang, all matter, energy, space, and time were concentrated into a single point

binary: existing in only two states, such as "off" or "on," "one" or "zero"

binary arithmetic: the arithmetic of binary numbers; base two arithmetic; internal arithmetic of electronic digital logic

binary number: a base-2 number; a number that uses only the binary digits 1 and 0

binary signal: a form of signal with only two states, such as two different values of voltage, or "on" and "off" states

binary system: a system of two stars that orbit their common center of mass; any system of two things

binomial: an expression with two terms

binomial coefficients: coefficients in the expansion of $(x + y^n$, where n is a positive integer

binomial distribution: the distribution of a binomial random variable

binomial theorem: a theorem giving the procedure by which a binomial expression may be raised to any power without using successive multiplications

bioengineering: the study of biological systems such as the human body using principles of engineering

biomechanics: the study of biological systems using engineering principles

bioturbation: disturbance of the strata in an archaeological site by biological factors such as rodent burrows, root action, or human activity

bit: a single binary digit, 1 or 0

bitmap: representing a graphic image in the memory of a computer by storing information about the color and shade of each individual picture element (or pixel)

Boolean algebra: a logic system developed by George Boole that deals with the theorems of undefined symbols and axioms concerning those symbols

Boolean operators: the set of operators used to perform operations on sets; includes the logical operators AND, OR, NOT

byte: a group of eight binary digits; represents a single character of text

cadaver: a corpse intended for medical research or training

caisson: a large cylinder or box that allows workers to perform construction tasks below the water surface, may be open at the top or sealed and pressurized

calculus: a method of dealing mathematically with variables that may be changing continuously with respect to each other

calibrate: act of systematically adjusting, checking, or standardizing the graduation of a measuring instrument

carrying capacity: in an ecosystem, the number of individuals of a species that can remain in a stable, sustainable relationship with the available resources

Cartesian coordinate system: a way of measuring the positions of points in a plane using two perpendicular lines as axes

Cartesian plane: a mathematical plane defined by the x and y axes or the ordinate and abscissa in a Cartesian coordinate system

cartographers: persons who make maps

catenary curve: the curve approximated by a free-hanging chain supported at each end; the curve generated by a point on a parabola rolling along a line

causal relations: responses to input that do not depend on values of the input at later times

celestial: relating to the stars, planets, and other heavenly bodies

celestial body: any natural object in space, defined as above Earth's atmosphere; the Moon, the Sun, the planets, asteroids, stars, galaxies, nebulae

central processor: the part of a computer that performs computations and controls and coordinates other parts of the computer

centrifugal: the outwardly directed force a spinning object exerts on its restraint; also the perceived force felt by persons in a rotating frame of reference

cesium: a chemical element, symbol Cs, atomic number 55

Chandrasekhar limit: the 1.4 solar mass limit imposed on a white dwarf by quantum mechanics; a white dwarf with greater than 1.4 solar masses will collapse to a neutron star

chaos theory: the qualitative study of unstable aperiodic behavior in deterministic nonlinear dynamical systems

chaotic attractor: a set of points such that all nearby trajectories converge to it

chert: material consisting of amorphous or cryptocrystalline silicon dioxide; fine-grained chert is indistinguishable from flint

chi-square test: a generalization of a test for significant differences between a binomial population and a multinomial population

chlorofluorocarbons: compounds similar to hydrocarbons in which one or more of the hydrogen atoms has been replaced by a chlorine or fluorine atom

chord: a straight line connecting the end points of an arc of a circle

chromakey: photographing an object shot against a known color, which can be replaced with an arbitrary background (like the weather maps on television newscasts)

chromosphere: the transparent layer of gas that resides above the photosphere in the atmosphere of the Sun

chronometer: an extremely precise timepiece

ciphered: coded; encrypyted

circumference: the distance around a circle

circumnavigation: the act of sailing completely around the globe

circumscribed: bounded, as by a circle

circumspheres: spheres that touch all the "outside" faces of a regular polyhedron

client: an individual, business, or agency for whom services are provided by another individual, business, or industry; a patron or customer

clones: computers assembled of generic components designed to use a standard operation system

codomain: for a given function *f*, the set of all possible values of the function; the range is a subset of the codomain

cold dark matter: hypothetical form of matter proposed to explain the 90 percent of mass in most galaxies that cannot be detected because it does not emit or reflect radiation

coma: the cloud of gas that first surrounds the nucleus of a comet as it begins to warm up

combinations: a group of elements from a set in which order is not important

combustion: chemical reaction combining fuel with oxygen accompanied by the release of light and heat

comet: a lump of frozen gas and dust that approaches the Sun in a highly elliptical orbit forming a coma and one or two tails

command: a particular instruction given to a computer, usually as part of a list of instructions comprising a program

commodities: anything having economic value, such as agricultural products or valuable metals

compendium: a summary of a larger work or collection of works

compiler: a computer program that translates symbolic instructions into machine code

complex plane: the mathematical abstraction on which complex numbers can be graphed; the *x*-axis is the real component and the *y*-axis is the imaginary component

composite number: an integer that is not prime

compression: reducing the size of a computer file by replacing long strings of identical bits with short instructions about the number of bits; the information is restored before the file is used

compression algorithm: the procedure used, such as comparing one frame in a movie to the next, to compress and reduce the size of electronic files

concave: hollowed out or curved inward

concentric: sets of circles or other geometric objects sharing the same center

conductive: having the ability to conduct or transmit

confidence interval: a range of values having a predetermined probability that the value of some measurement of a population lies within it

congruent: exactly the same everywhere; having exactly the same size and shape

conic: of or relating to a cone, that surface generated by a straight line, passing through a fixed point, and moving along the intersection with a fixed curve

conic sections: the curves generated by an imaginary plane slicing through an imaginary cone

continuous quantities: amounts composed of continuous and undistinguishable parts

converge: come together; to approach the same numerical value

convex: curved outward, bulging

coordinate geometry: the concept and use of a coordinate system with respect to the study of geometry

coordinate plane: an imaginary two-dimensional plane defined as the plane containing the x- and y-axes; all points on the plane have coordinates that can be expressed as x, y

coordinates: the set of n numbers that uniquely identifies the location of a point in n-dimensional space

corona: the upper, very rarefied atmosphere of the Sun that becomes visible around the darkened Sun during a total solar eclipse

corpus: Latin for "body"; used to describe a collection of artifacts

correlate: to establish a mutual or reciprocal relation between two things or sets of things

correlation: the process of establishing a mutual or reciprocal relation between two things or sets of things

cosine: if a unit circle is drawn with its center at the origin and a line segment is drawn from the origin at angle theta so that the line segment intersects the circle at (x, y), then x is the cosine of theta

cosmological distance: the distance a galaxy would have to have in order for its red shift to be due to Hubble expansion of the universe

cosmology: the study of the origin and evolution of the universe

cosmonaut: the term used by the Soviet Union and now used by the Russian Federation to refer to persons trained to go into space; synonomous with astronaut

cotton gin: a machine that separates the seeds, hulls, and other undesired material from cotton

cowcatcher: a plow-shaped device attached to the front of a train to quickly remove obstacles on railroad tracks

cryptography: the science of encrypting information for secure transmission

cubit: an ancient unit of length equal to the distance from the elbow to the tip of the middle finger; usually about 18 inches

culling: removing inferior plants or animals while keeping the best; also known as "thinning"

curved space: the notion suggested by Albert Einstein to explain the properties of space near a massive object, space acts as if it were curved in four dimensions

deduction: a conclusion arrived at through reasoning, especially a conclusion about some particular instance derived from general principles

deductive reasoning: a type of reasoning in which a conclusion necessarily follows from a set of axioms; reasoning from the general to the particular

degree: 1/360 of a circle or complete rotation

degree of significance: a determination, usually in advance, of the importance of measured differences in statistical variables

demographics: statistical data about people—including age, income, and gender—that are often used in marketing

dendrite: branched and short fiber of a neuron that carries information to the neuron

dependent variable: in the equation $y = f(x)$, if the function f assigns a single value of y to each value of x, then y is the output variable (or the dependent variable)

depreciate: to lessen in value

deregulation: the process of removing legal restrictions on the behavior of individuals or corporations

derivative: the derivative of a function is the limit of the ratio of the change in the function; the change is produced by a small variation in the variable as the change in the variable is allowed to approach zero; an inverse operation to calculating an integral

determinant: a square matrix with a single numerical value determined by a unique set of mathematical operations performed on the entries

determinate algebra: the study and analysis of equations that have one or a few well-defined solutions

deterministic: mathematical or other problems that have a single, well-defined solution

diameter: the chord formed by an arc of one-half of a circle

differential: a mathematical quantity representing a small change in one variable as used in a differential equation

differential calculus: the branch of mathematics primarily dealing with the solution of differential equations to find lengths, areas, and volumes of functions

differential equation: an equation that expresses the relationship between two variables that change in respect to each other, expressed in terms of the rate of change

digit: one of the symbols used in a number system to represent the multiplier of each place

digital: describes information technology that uses discrete values of a physical quantity to transmit information

digital encoding: encoding information by using discrete values of some physical quantity

digital logic: rules of logic as applied to systems that can exist in only discrete states (usually two)

dihedral: a geometric figure formed by two half-planes that are bounded by the same straight line

Diophantine equation: polynomial equations of several variables, with integer coefficients, whose solutions are to be integers

diopter: a measure of the power of a lens or a prism, equal to the reciprocal of its focal length in meters

directed distance: the distance from the pole to a point in the polar coordinate plane

discrete: composed of distinct elements

discrete quantities: amounts composed of separate and distinct parts

distributive property: property such that the result of an operation on the various parts collected into a whole is the same as the operation performed separately on the parts before collection into the whole

diverge: to go in different directions from the same starting point

dividend: the number to be divided; the numerator in a fraction

divisor: the number by which a dividend is divided; the denominator of a fraction

DNA fingerprinting: the process of isolating and amplifying segments of DNA in order to uniquely identify the source of the DNA

domain: the set of all values of a variable used in a function

double star: a binary star; two stars orbiting a common center of gravity

duodecimal: a numbering system based on 12

dynamometer: a device that measures mechanical or electrical power

eccentric: having a center of motion different from the geometric center of a circle

eclipse: occurrence when an object passes in front of another and blocks the view of the second object; most often used to refer to the phenomenon

that occurs when the Moon passes in front of the Sun or when the Moon passes through Earth's shadow

ecliptic: the plane of the Earth's orbit around the Sun

eigenvalue: if there exists a vector space such that a linear transformation onto itself produces a new vector equal to a scalar times the original vector, then that scalar is called an eigenfunction

eigenvector: if there exists a vector space such that a linear transformation onto itself produces a new vector equal to a scalar times the original vector, then that vector is called an eigenvector

Einstein's General Theory of Relativity: Albert Einstein's generalization of relativity to include systems accelerated with respect to one another; a theory of gravity

electromagnetic radiation: the form of energy, including light, that transfers information through space

elements: the members of a set

ellipse: one of the conic sections, it is defined as the locus of all points such that the sum of the distances from two points called the foci is constant

elliptical: a closed geometric curve where the sum of the distances of a point on the curve to two fixed points (foci) is constant

elliptical orbit: a planet, comet, or satellite follows a curved path known as an ellipse when it is in the gravitational field of the Sun or another object; the Sun or other object is at one focus of the ellipse

empirical law: a mathematical summary of experimental results

empiricism: the view that the experience of the senses is the single source of knowledge

encoding tree: a collection of dots with edges connecting them that have no looping paths

endangered species: a species with a population too small to be viable

epicenter: the point on Earth's surface directly above the site of an earthquake

epicycle: the curved path followed by planets in Ptolemey's model of the solar system; planets moved along a circle called the epicycle, whose center moved along a circular orbit around the sun

epicylic: having the property of moving along an epicycle

equatorial bulge: the increase in diameter or circumference of an object when measured around its equator usually due to rotation, all planets and the sun have equatorial bulges

equidistant: at the same distance

equilateral: having the property that all sides are equal; a square is an equilateral rectangle

equilateral triangle: a triangle whose sides and angles are equal

equilibrium: a state of balance between opposing forces

equinox points: two points on the celestial sphere at which the ecliptic intersects the celestial equator

escape speed: the minimum speed an object must attain so that it will not fall back to the surface of a planet

Euclidean geometry: the geometry of points, lines, angles, polygons, and curves confined to a plane

exergy: the measure of the ability of a system to produce work; maximum potential work output of a system

exosphere: the outermost layer of the atmosphere extending from the ionosphere upward

exponent: the symbol written above and to the right of an expression indicating the power to which the expression is to be raised

exponential: an expression in which the variable appears as an exponent

exponential power series: the series by which *e* to the *x* power may be approximated; $e^x = 1 + x + x^{2/2!} + x^{3/3!} + \ldots$

exponents: symbols written above and to the right of expressions indicating the power to which an expression is to be raised or the number of times the expression is to be multiplied by itself

externality: a factor that is not part of a system but still affects it

extrapolate: to extend beyond the observations; to infer values of a variable outside the range of the observations

farsightedness: describes the inability to see close objects clearly

fiber-optic: a long, thin strand of glass fiber; internal reflections in the fiber assure that light entering one end is transmitted to the other end with only small losses in intensity; used widely in transmitting digital information

fibrillation: a potentially fatal malfunction of heart muscle where the muscle rapidly and ineffectually twitches instead of pulsing regularly

fidelity: in information theory a measure of how close the information received is to the information sent

finite: having definite and definable limits; countable

fire: the reaction of a neuron when excited by the reception of a neurotransmitter

fission: the splitting of the nucleus of a heavy atom, which releases kinetic energy that is carried away by the fission fragments and two or three neutrons

fixed term: for a definite length of time determined in advance

fixed-wing aircraft: an aircraft that obtains lift from the flow of air over a nonmovable wing

floating-point operations: arithmetic operations on a number with a decimal point

fluctuate: to vary irregularly

flue: a pipe designed to remove exhaust gases from a fireplace, stove, or burner

fluid dynamics: the science of fluids in motion

focal length: the distance from the focal point (the principle point of focus) to the surface of a lens or concave mirror

focus: one of the two points that define an ellipse; in a planetary orbit, the Sun is at one focus and nothing is at the other focus

formula analysis: a method of analysis of the Boolean formulas used in computer programming

Fourier series: an infinite series consisting of cosine and sine functions of integral multiples of the variable each multiplied by a constant; if the series is finite, the expression is known as a Fourier polynomial

fractal: a type of geometric figure possessing the properties of self-similarity (any part resembles a larger or smaller part at any scale) and a measure that increases without bound as the unit of measure approaches zero

fractal forgery: creating a natural landscape by using fractals to simulate trees, mountains, clouds, or other features

fractal geometry: the study of the geometric figures produced by infinite iterations

futures exchange: a type of exchange where contracts are negotiated to deliver commodites at some fixed price at some time in the future

g: a common measure of acceleration; for example 1 *g* is the acceleration due to gravity at the Earth's surface, roughly 32 feet per second per second

game theory: a discipline that combines elements of mathematics, logic, social and behavioral sciences, and philosophy

gametes: mature male or female sexual reproductive cells

gaming: playing games or relating to the theory of game playing

gamma ray: a high-energy photon

general relativity: generalization of Albert Einstein's theory of relativity to include accelerated frames of reference; presents gravity as a curvature of four-dimensional space-time

generalized inverse: an extension of the concept of the inverse of a matrix to include matrices that are not square

generalizing: making a broad statement that includes many different special cases

genus: the taxonomic classification one step more general than species; the first name in the binomial nomenclature of all species

geoboard: a square board with pegs and holes for pegs used to create geometric figures

geocentric: Earth-centered

geodetic: of or relating to geodesy, which is the branch of applied mathematics dealing with the size and shape of the earth, including the precise location of points on its surface

geometer: a person who uses the principles of geometry to aid in making measurements

geometric: relating to the principles of geometry, a branch of mathematics related to the properties and relationships of points, lines, angles, surfaces, planes, and solids

geometric sequence: a sequence of numbers in which each number in the sequence is larger than the previous by some constant ratio

geometric series: a series in which each number is larger than the previous by some constant ratio; the sum of a geometric sequence

geometric solid: one of the solids whose faces are regular polygons

geometry: the branch of mathematics that deals with the properties and relationships of points, lines, angles, surfaces, planes, and solids

geostationary orbit: an Earth orbit made by an artificial satellite that has a period equal to the Earth's period of rotation on its axis (about 24 hours)

geysers: springs that occasionally spew streams of steam and hot water

glide reflection: a rigid motion of the plane that consists of a reflection followed by a translation parallel to the mirror axis

grade: the amount of increase in elevation per horizontal distance, usually expressed as a percent; the slope of a road

gradient: a unit used for measuring angles, in which the circle is divided into 400 equal units, called gradients

graphical user interface: a device designed to display information graphically on a screen; a modern computer interface system

Greenwich Mean Time: the time at Greenwich, England; used as the basis for universal time throughout the world

Gross Domestic Product: a measure in the change in the market value of goods, services, and structures produced in the economy

group theory: study of the properties of groups, the mathematical systems consisting of elements of a set and operations that can be performed on that set such that the results of the operations are always members of the same set

gyroscope: a device typically consisting of a spinning wheel or disk, whose spin-axis turns between two low-friction supports; it maintains its angular orientation with respect to inertial conditions when not subjected to external forces

Hagia Sophia: Instanbul's most famous landmark, built by the emperor Justinian I in 537 C.E. and converted to a mosque in 1453 C.E.

Hamming codes: a method of error correction in digital information

headwind: a wind blowing in the opposite direction as that of the course of a vehicle

Heisenberg Uncertainty Principle: the principle in physics that asserts it is impossible to know simultaneously and with complete accuracy the values of certain pairs of physical quantities such as position and momentum

heliocentric: Sun-centered

hemoglobin: the oxygen-bearing, iron-containing conjugated protein in vertebrate red blood cells

heuristics: a procedure that serves to guide investigation but that has not been proven

hominid: a member of family Hominidae; *Homo sapiens* are the only surviving species

Huffman encoding: a method of efficiently encoding digital information

hydrocarbon: a compound of carbon and hydrogen

hydrodynamics: the study of the behavior of moving fluids

hydrograph: a tabular or graphical display of stream flow or water runoff

hydroscope: a device designed to allow a person to see below the surface of water

hydrostatics: the study of the properties of fluids not in motion

hyperbola: a conic section; the locus of all points such that the absolute value of the difference in distance from two points called foci is a constant

hyperbolic: an open geometric curve where the difference of the distances of a point on the curve to two fixed points (foci) is constant

Hypertext Markup Language: the computer markup language used to create documents on the World Wide Web

hypertext: the text that contains hyperlinks, that is, links to other places in the same document or other documents or multimedia files

hypotenuse: the long side of a right triangle; the side opposite the right angle

hypothesis: a proposition that is assumed to be true for the purpose of proving other propositions

ice age: one of the broad spans of time when great sheets of ice covered the Northern parts of North America and Europe; the most recent ice age was about 16,000 years ago

identity: a mathematical statement much stronger than equality, which asserts that two expressions are the same for all values of the variables

implode: violently collapse; fall in

inclination: a slant or angle formed by a line or plane with the horizontal axis or plane

inclined: sloping, slanting, or leaning

incomplete interpretation: a statistical flaw

independent variable: in the equation $y = f(x)$, the input variable is x (or the independent variable)

indeterminate algebra: study and analysis of solution strategies for equations that do not have fixed or unique solutions

indeterminate equation: an equation in which more than one variable is unknown

index (number): a number that allows tracking of a quantity in economics by comparing it to a standard, the consumer price index is the best known example

inductive reasoning: drawing general conclusions based on specific instances or observations; for example, a theory might be based on the outcomes of several experiments

Industrial Revolution: beginning in Great Britain around 1730, a period in the eighteenth and nineteenth centuries when nations in Europe, Asia, and the Americas moved from agrarian-based to industry-based economies

inertia: tendency of a body that is at rest to remain at rest, or the tendency of a body that is in motion to remain in motion

inferences: the act or process of deriving a conclusion from given facts or premises

inferential statistics: analysis and interpretation of data in order to make predictions

infinite: having no limit; boundless, unlimited, endless; uncountable

infinitesimals: functions with values arbitrarily close to zero

infinity: the quality of unboundedness; a quantity beyond measure; an unbounded quantity

information database: an array of information related to a specific subject or group of subjects and arranged so that any individual bit of information can be easily found and recovered

information theory: the science that deals with how to separate information from noise in a signal or how to trace the flow of information through a complex system

infrastructure: the foundation or permanent installations necessary for a structure or system to operate

initial conditions: the values of variables at the beginning of an experiment or of a set at the beginning of a simulation; chaos theory reveals that small changes in initial conditions can produce widely divergent results

input: information provided to a computer or other computation system

inspheres: spheres that touch all the "inside" faces of a regular polyhedron; also called "enspheres"

integer: a positive whole number, its negative counterpart, or zero

integral: a mathematical operation similar to summation; the area between the curve of a function, the x-axis, and two bounds such as $x = a$ and $x = b$; an inverse operation to finding the derivative

integral calculus: the branch of mathematics dealing with the rate of change of functions with respect to their variables

integral number: integer; that is, a positive whole number, its negative counterpart, or zero

integral solutions: solutions to an equation or set of equations that are all integers

integrated circuit: a circuit with the transistors, resistors, and other circuit elements etched into the surface of a single chip of silicon

integration: solving a differential equation; determining the area under a curve between two boundaries

intensity: the brightness of radiation or energy contained in a wave

intergalactic: between galaxies; the space between the galaxies

interplanetary: between planets; the space between the planets

interpolation: filling in; estimating unknown values of a function between known values

intersection: a set containing all of the elements that are members of two other sets

interstellar: between stars; the space between stars

intraframe: the compression applied to still images, interframe compression compares one image to the next and only stores the elements that have changed

intrinsic: of itself; the essential nature of a thing; originating within the thing

inverse: opposite; the mathematical function that expresses the independent variable of another function in terms of the dependent variable

inverse operations: operations that undo each other, such as addition and subtraction

inverse square law: a given physical quality varies with the distance from the source inversely as the square of the distance

inverse tangent: the value of the argument of the tangent function that produces a given value of the function; the angle that produces a particular value of the tangent

invert: to turn upside down or to turn inside out; in mathematics, to rewrite as the inverse function

inverted: upside down; turned over

ionized: an atom that has lost one or more of its electrons and has become a charged particle

ionosphere: a layer in Earth's atmosphere above 80 kilometers characterized by the existence of ions and free electrons

irrational number: a real number that cannot be written as a fraction of the form *a/b*, where *a* and *b* are both integers and *b* is not zero; when expressed in decimal form, an irrational number is infinite and nonrepeating

isometry: equality of measure

isosceles triangle: a triangle with two sides and two angles equal

isotope: one of several species of an atom that has the same number of protons and the same chemical properties, but different numbers of neutrons

iteration: repetition; a repeated mathematical operation in which the output of one cycle becomes the input for the next cycle

iterative: relating to a computational procedure to produce a desired result by replication of a series of operations

iterator: the mathematical operation producing the result used in iteration

kinetic energy: the energy an object has as a consequence of its motion

kinetic theory of gases: the idea that all gases are composed of widely separated particles (atoms and molecules) that exert only small forces on each other and that are in constant motion

knot: nautical mile per hour

Lagrange points: two positions in which the motion of a body of negligible mass is stable under the gravitational influence of two much larger bodies (where one larger body is moving)

latitude: the number of degrees on Earth's surface north or south of the equator; the equator is latitude zero

law: a principle of science that is highly reliable, has great predictive power, and represents the mathematical summary of experimental results

law of cosines: for a triangle with angles A, B, C and sides a, b, c, $a^2 = b^2 + c^2 - 2bc \cos A$

law of sines: if a triangle has sides a, b, and c and opposite angles A, B, and C, then $\sin A/a = \sin B/b = \sin C/c$

laws of probability: set of principles that govern the use of probability in determining the truth or falsehood of a hypothesis

light-year: the distance light travels within a vaccuum in one year

limit: a mathematical concept in which numerical values get closer and closer to a given value

linear algebra: the study of vector spaces and linear transformations

linear equation: an equation in which all variables are raised to the first power

linear function: a function whose graph on the *x-y* plane is a straight line or line segment

litmus test: a test that uses a single indicator to prompt a decision

locus (pl: loci): in geometry, the set of all points, lines, or surfaces that satisfies a particular requirement

logarithm: the power to which a certain number called the base is to be raised to produce a particular number

logarithmic coordinates: the x and y coordinates of a point on a cartesian plane using logarithmic scales on the x- and y-axes.

logarithmic scale: a scale in which the distances that numbers are positioned, from a reference point, are proportional to their logarithms

logic circuits: circuits used to perform logical operations and containing one or more logic elements: devices that maintain a state based on previous input to determine current and future output

logistic difference equation: the equation $x_{(n+1)} = r \times x_{n(1-xn)}$ is used to study variability in animal populations

longitude: one of the imaginary great circles beginning at the poles and extending around Earth; the geographic position east or west of the prime meridian

machine code: the set of instructions used to direct the internal operation of a computer or other information-processing system

machine language: electronic code the computer can utilize

magnetic trap: a magnetic field configured in such a way that an ion or other charged particle can be held in place for an extended period of time

magnetosphere: an asymmetric region surrounding the Earth in which charged particles are trapped, their behavior being dominated by Earth's magnetic field

magnitude: size; the measure or extent of a mathematical or physical quantity

mainframes: large computers used by businesses and government agencies to process massive amounts of data; generally faster and more powerful than desktops but usually requiring specialized software

malfunctioning: not functioning correctly; performing badly

malleability: the ability or capability of being shaped or formed

margin of error: the difference between the estimated maximum and minimum values a given measurement could have

mathematical probability: the mathematical computation of probabilities of outcomes based on rules of logic

matrix: a rectangular array of data in rows and columns

mean: the arithmetic average of a set of data

median: the middle of a set of data when values are sorted from smallest to largest (or largest to smallest)

megabyte: term used to refer to one million bytes of memory storage, where each byte consists of eight bits; the actual value is 1,048,576 (2^{20})

memory: a device in a computer designed to temporarily or permanently store information in the form of binomial states of certain circuit elements

meridian: a great circle passing through Earth's poles and a particular location

metallurgy: the study of the properties of metals; the chemistry of metals and alloys

meteorologist: a person who studies the atmosphere in order to understand weather and climate

methanol: an alcohol consisting of a single carbon bonded to three hydrogen atoms and an O–H group

microcomputers: an older term used to designate small computers designed to sit on a desktop and to be used by one person; replaced by the term personal computer

microgravity: the apparent weightless condition of objects in free fall

microkelvin: one-millionth of a kelvin

minicomputers: a computer midway in size between a desktop computer and a main frame computer; most modern desktops are much more powerful than the older minicomputers and they have been phased out

minimum viable population: the smallest number of individuals of a species in a particular area that can survive and maintain genetic diversity

mission specialist: an individual trained by NASA to perform a specific task or set of tasks onboard a spacecraft, whose duties do not include piloting the spacecraft

mnemonic: a device or process that aids one's memory

mode: a kind of average or measure of central tendency equal to the number that occurs most often in a set of data

monomial: an expression with one term

Morse code: a binary code designed to allow text information to be transmitted by telegraph consisting of "dots" and "dashes"

mouse: a handheld pointing device used to manipulate an indicator on a screen

moving average: a method of averaging recent trends in relation to long term averages, it uses recent data (for example, the last 10 days) to calculate an average that changes but still smooths out daily variations

multimodal input/output (I/O): multimedia control and display that uses various senses and interaction styles

multiprocessing: a computer that has two or more central processers which have common access to main storage

nanometers: billionths of a meter

nearsightedness: describes the inability to see distant objects clearly

negative exponential: an exponential function of the form $y = e^{-x}$

net force: the final, or resultant, influence on a body that causes it to accelerate

neuron: a nerve cell

neurotransmitters: the substance released by a neuron that diffuses across the synapse

neutron: an elementary particle with approximately the same mass as a proton and neutral charge

Newtonian: a person who, like Isaac Newton, thinks the universe can be understood in terms of numbers and mathematical operations

nominal scales: a method for sorting objects into categories according to some distinguishing characteristic, then attaching a label to each category

non-Euclidean geometry: a branch of geometry defined by posing an alternate to Euclid's fifth postulate

nonlinear transformation: a transformation of a function that changes the shape of a curve or geometric figure

nonlinear transformations: transformations of functions that change the shape of a curve or geometric figure

nuclear fission: a reaction in which an atomic nucleus splits into fragments

nuclear fusion: mechanism of energy formation in a star; lighter nuclei are combined into heavier nuclei, releasing energy in the process

nucleotides: the basic chemical unit in a molecule of nucleic acid

nucleus: the dense, positive core of an atom that contains protons and neutrons

null hypothesis: the theory that there is no validity to the specific claim that two variations of the same thing can be distinguished by a specific procedure

number theory: the study of the properties of the natural numbers, including prime numbers, the number theorem, and Fermat's Last Theorem

numerical differentiation: approximating the mathematical process of differentiation using a digital computer

nutrient: a food substance or mineral required for the completion of the life cycle of an organism

oblate spheroid: a spheroid that bulges at the equator; the surface created by rotating an ellipse 360 degrees around its minor axis

omnidirectional: a device that transmits or receives energy in all directions

Öort cloud: a cloud of millions of comets and other material forming a spherical shell around the solar system far beyond the orbit of Neptune

orbital period: the period required for a planet or any other orbiting object to complete one complete orbit

orbital velocity: the speed and direction necessary for a body to circle a celestial body, such as Earth, in a stable manner

ordinate: the *y*-coordinate of a point on a Cartesian plane

organic: having to do with life, growing naturally, or dealing with the chemical compounds found in or produced by living organisms

oscillating: moving back and forth

outliers: extreme values in a data set

output: information received from a computer or other computation system based on the information it has received

overdubs: adding voice tracks to an existing film or tape

oxidant: a chemical reagent that combines with oxygen

oxidizer: the chemical that combines with oxygen or is made into an oxide

pace: an ancient measure of length equal to normal stride length

parabola: a conic section; the locus of all points such that the distance from a fixed point called the focus is equal to the perpendicular distance from a line

parabolic: an open geometric curve where the distance of a point on the curve to a fixed point (focus) and a fixed line (directrix) is the same

paradigm: an example, pattern, or way of thinking

parallax: the apparent motion of a nearby object when viewed against the background of more distant objects due to a change in the observer's position

parallel operations: separating the parts of a problem and working on different parts at the same time

parallel processing: using at least two different computers or working at least two different central processing units in the same computer at the same time or "in parallel" to solve problems or to perform calculation

parallelogram: a quadrilateral with opposite sides equal and opposite angles equal

parameter: an independent variable, such as time, that can be used to rewrite an expression as two separate functions

parity bits: extra bits inserted into digital signals that can be used to determine if the signal was accurately received

partial sum: with respect to infinite series, the sum of its first *n* terms for some *n*

pattern recognition: a process used by some artificial-intelligence systems to identify a variety of patterns, including visual patterns, information patterns buried in a noisy signal, and word patterns imbedded in text

payload specialist: an individual selected by NASA, another government agency, another government, or a private business, and trained by NASA to operate a specific piece of equipment onboard a spacecraft

payloads: the passengers, crew, instruments, or equipment carried by an aircraft, spacecraft, or rocket

perceptual noise shaping: a process of improving signal-to-noise ratio by looking for the patterns made by the signal, such as speech

perimeter: the distance around an area; in fractal geometry, some figures have a finite area but infinite perimeter

peripheral vision: outer area of the visual field

permutation: any arrangement, or ordering, of items in a set

perpendicular: forming a right angle with a line or plane

perspective: the point of view; a drawing constructed in such a way that an appearance of three dimensionality is achieved

perturbations: small displacements in an orbit

phonograph: a device used to recover the information recorded in analog form as waves or wiggles in a spiral grove on a flat disc of vinyl, rubber, or some other substance

photosphere: the very bright portion of the Sun visible to the unaided eye; the portion around the Sun that marks the boundary between the dense interior gases and the more diffuse

photosynthesis: the chemical process used by plants and some other organisms to harvest light energy by converting carbon dioxide and water to carbohydrates and oxygen

pixel: a single picture element on a video screen; one of the individual dots making up a picture on a video screen or digital image

place value: in a number system, the power of the base assigned to each place; in base-10, the ones place, the tens place, the hundreds place, and so on

plane: generally considered an undefinable term, a plane is a flat surface extending in all directions without end, and that has no thickness

plane geometry: the study of geometric figures, points, lines, and angles and their relationships when confined to a single plane

planetary: having to do with one of the planets

planisphere: a projection of the celestial sphere onto a plane with adjustable circles to demonstrate celestial phenomena

plates: the crustal segments on Earth's surface, which are constantly moving and rotating with respect to each other

plumb-bob: a heavy, conical-shaped weight, supported point-down on its axis by a strong cord, used to determine verticality in construction or surveying

pneumatic drill: a drill operated by compressed air

pneumatic tire: air-filled tire, usually rubber or synthetic

polar axis: the axis from which angles are measured in a polar coordinate system

pole: the origin of a polar coordinate system

poll: a survey designed to gather information about a subject

pollen analysis: microscopic examination of pollen grains to determine the genus and species of the plant producing the pollen; also known as palynology

polyconic projections: a type of map projection of a globe onto a plane that produces a distorted image but preserves correct distances along each meridian

polygon: a geometric figure bounded by line segments

polyhedron: a solid formed with all plane faces

polynomial: an expression with more than one term

polynomial function: a functional expression written in terms of a polynomial

position tracking: sensing the location and/or orientation of an object

power: the number of times a number is to be multiplied by itself in an expression

precalculus: the set of subjects and mathematical skills generally necessary to understand calculus

predicted frame: in compressed video, the next frame in a sequence of images; the information is based on what changed from the previous frame

prime: relating to, or being, a prime number (that is, a number that has no factors other than itself and 1)

Prime Meridian: the meridian that passes through Greenwich, England

prime number: a number that has no factors other than itself and 1

privatization: the process of converting a service traditionally offered by a government or public agency into a service provided by a private corporation or other private entity

proactive: taking action based on prediction of future situations

probability: the likelihood an event will occur when compared to other possible outcomes

probability density function: a function used to estimate the likelihood of spotting an organism while walking a transect

probability theory: the branch of mathematics that deals with quantities having random distributions

processor: an electronic device used to process a signal or to process a flow of information

profit margin: the difference between the total cost of a good or service and the actual selling cost of that good or service, usually expressed as a percentage

program: a set of instructions given to a computer that allows it to perform tasks; software

programming language processor: a program designed to recognize and process other programs

proliferation: growing rapidly

proportion: the mathematical relation between one part and another part, or between a part and the whole; the equality of two ratios

proportionately: divided or distributed according to a proportion; proportional

protractor: a device used for measuring angles, usually consisting of a half circle marked in degrees

pseudorandom numbers: numbers generated by a process that does not guarantee randomness; numbers produced by a computer using some highly complex function that simulates true randomness

Ptolemaic theory: the theory that asserted Earth was a spherical object at the center of the universe surrounded by other spheres carrying the various celestial objects

Pythagorean Theorem: a mathematical statement relating the sides of right triangles; the square of the hypotenuse is equal to the sums of the squares of the other two sides

Pythagorean triples: any set of three numbers obeying the Pythogorean relation such that the square of one is equal to the sum of the squares of the other two

quadrant: one-fourth of a circle; also a device used to measure angles above the horizon

quadratic: involving at least one term raised to the second power

quadratic equation: an equation in which the variable is raised to the second power in at least one term when the equation is written in its simplest form

quadratic form: the form of a function written so that the independent variable is raised to the second power

quantitative: of, relating to, or expressible in terms of quantity

quantum: a small packet of energy (matter and energy are equivalent)

quantum mechanics: the study of the interactions of matter with radiation on an atomic or smaller scale, whereby the granularity of energy and radiation becomes apparent

quantum theory: the study of the interactions of matter with radiation on an atomic or smaller scale, whereby the granularity of energy and radiation becomes apparent

quaternion: a form of complex number consisting of a real scalar and an imaginary vector component with three dimensions

quipus: knotted cords used by the Incas and other Andean cultures to encode numeric and other information

radian: an angle measure approximately equal to 57.3 degrees, it is the angle that subtends an arc of a circle equal to one radius

radicand: the quantity under the radical sign; the argument of the square root function

radius: the line segment originating at the center of a circle or sphere and terminating on the circle or sphere; also the measure of that line segment

radius vector: a line segment with both magnitude and direction that begins at the center of a circle or sphere and runs to a point on the circle or sphere

random: without order

random walks: a mathematical process in a plane of moving a random distance in a random direction then turning through a random angle and repeating the process indefinitely

range: the set of all values of a variable in a function mapped to the values in the domain of the independent variable; also called range set

rate (interest): the portion of the principal, usually expressed as a percentage, paid on a loan or investment during each time interval

ratio of similitude: the ratio of the corresponding sides of similar figures

rational number: a number that can be written in the form a/b, where a and b are intergers and b is not equal to zero

rations: the portion of feed that is given to a particular animal

ray: half line; line segment that originates at a point and extends without bound

real number: a number that has no imaginary part; a set composed of all the rational and irrational numbers

real number set: the combined set of all rational and irrational numbers, the set of numbers representing all points on the number line

realtime: occuring immediately, allowing interaction without significant delay

reapportionment: the process of redistributing the seats of the U. S. House of Representatives, based on each state's proportion of the national population

recalibration: process of resetting a measuring instrument so as to provide more accurate measurements

reciprocal: one of a pair of numbers that multiply to equal 1; a number's reciprocal is 1 divided by the number

red shift: motion-induced change in the frequency of light emitted by a source moving away from the observer

reflected: light or soundwaves returned from a surface

reflection: a rigid motion of the plane that fixes one line (the mirror axis) and moves every other point to its mirror image on the opposite side of the line

reflexive: directed back or turning back on itself

refraction: the change in direction of a wave as it passes from one medium to another

refrigerants: fluid circulating in a refrigerator that is successively compressed, cooled, allowed to expand, and warmed in the refrigeration cycle

regular hexagon: a hexagon whose sides are all equal and whose angles are all equal

relative: defined in terms of or in relation to other quantities

relative dating: determining the date of an archaeological artifact based on its position in the archaeological context relative to other artifacts

relativity: the assertion that measurements of certain physical quantities such as mass, length, and time depend on the relative motion of the object and observer

remediate: to provide a remedy; to heal or to correct a wrong or a deficiency

retrograde: apparent motion of a planet from east to west, the reverse of normal motion; for the outer planets, due to the more rapid motion of Earth as it overtakes an outer planet

revenue: the income produced by a source such as an investment or some other activity; the income produced by taxes and other sources and collected by a governmental unit

rhomboid: a parallelogram whose sides are equal

right angle: the angle formed by perpendicular lines; it measures 90 degrees

RNA: ribonucleic acid

robot arm: a sophisticated device that is standard equipment on space shuttles and on the International Space Station; used to deploy and retrieve satellites or perform other functions

Roche limit: an imaginary surface around a star in a binary system; outside the Roche limit, the gravitational attraction of the companion will pull matter away from a star

root: a number that when multiplied by itself a certain number of times forms a product equal to a specified number

rotary-wing design: an aircraft design that uses a rotating wing to produce lift; helicopter or autogiro (also spelled autogyro)

rotation: a rigid motion of the plane that fixes one point (the center of rotation) and moves every other point around a circle centered at that point

rotational: having to do with rotation

round: also to round off, the systematic process of reducing the number of decimal places for a given number

rounding: process of giving an approximate number

sample: a randomly selected subset of a larger population used to represent the larger population in statistical analysis

sampling: selecting a subset of a group or population in such a way that valid conclusions can be made about the whole set or population

scale (map): the numerical ratio between the dimensions of an object and the dimensions of the two or three dimensional representation of that object

scale drawing: a drawing in which all of the dimensions are reduced by some constant factor so that the proportions are preserved

scaling: the process of reducing or increasing a drawing or some physical process so that proper proportions are retained between the parts

schematic diagram: a diagram that uses symbols for elements and arranges these elements in a logical pattern rather than a practical physical arrangement

schematic diagrams: wiring diagrams that use symbols for circuit elements and arranges these elements in a logical pattern rather than a practical physical arrangement

search engine: software designed to search the Internet for occurences of a word, phrase, or picture, usually provided at no cost to the user as an advertising vehicle

secant: the ratio of the side adjacent to an acute angle in a right triangle to the side opposite; given a unit circle, the ratio of the *x* coordinate to the *y* coordinate of any point on the circle

seismic: subjected to, or caused by an earthquake or earth tremor

self-similarity: the term used to describe fractals where a part of the geometric figure resembles a larger or smaller part at any scale chosen

semantic: the study of how words acquire meaning and how those meanings change over time

semi-major axis: one-half of the long axis of an ellipse; also equal to the average distance of a planet or any satellite from the object it is orbiting

semiconductor: one of the elements with characteristics intermediate between the metals and nonmetals

set: a collection of objects defined by a rule such that it is possible to determine exactly which objects are members of the set

set dancing: a form of dance in which dancers are guided through a series of moves by a caller

set theory: the branch of mathematics that deals with the well-defined collections of objects known as sets

sextant: a device for measuring altitudes of celestial objects

signal processor: a device designed to convert information from one form to another so that it can be sent or received

significant difference: to distinguish greatly between two parameters

significant digits: the digits reported in a measure that accurately reflect the precision of the measurement

silicon: element number 14, it belongs in the category of elements known as metalloids or semiconductors

similar: in mathematics, having sides or parts in constant proportion; two items that resemble each other but are not identical

sine: if a unit circle is drawn with its center at the origin and a line segment is drawn from the origin at angle theta so that the line segment intersects the circle at (x, y), then y is the sine of theta

skepticism: a tendency towards doubt

skew: to cause lack of symmetry in the shape of a frequency distribution

slope: the ratio of the vertical change to the corresponding horizontal change

software: the set of instructions given to a computer that allows it to perform tasks

solar masses: dimensionless units in which mass, radius, luminosity, and other physical properties of stars can be expressed in terms of the Sun's characteristics

solar wind: a stream of particles and radiation constantly pouring out of the Sun at high velocities; partially responsible for the formation of the tails of comets

solid geometry: the geometry of solid figures, spheres, and polyhedrons; the geometry of points, lines, surfaces, and solids in three-dimensional space

spatial sound: audio channels endowed with directional and positional attributes (like azimuth, elevation, and range) and room effects (like echoes and reverberation)

spectra: the ranges of frequencies of light emitted or absorbed by objects

spectrum: the range of frequencies of light emitted or absorbed by an object

sphere: the locus of points in three-dimensional space that are all equidistant from a single point called the center

spin: to rotate on an axis or turn around

square: a quadrilateral with four equal sides and four right angles

square root: with respect to real or complex numbers s, the number t for which $t^2 = s$

stade: an ancient Greek measurement of length, one stade is approximately 559 feet (about 170 meters)

standard deviation: a measure of the average amount by which individual items of data might be expected to vary from the arithmetic mean of all data

static: without movement; stationary

statistical analysis: a set of methods for analyzing numerical data

statistics: the branch of mathematics that analyzes and interprets sets of numerical data

stellar: having to do with stars

sterographics: presenting slightly different views to left and right eyes, so that graphic scenes acquire depth

stochastic: random, or relating to a variable at each moment

Stonehenge: a large circle of standing stones on the Salisbury plain in England, thought by some to be an astronomical or calendrical marker

storm surge: the front of a hurricane, which bulges because of strong winds; can be the most damaging part of a hurricane

stratopause: the boundary in the atmosphere between the stratosphere and the mesosphere usually around 55 kilometers in altitude

stratosphere: the layer of Earth's atmosphere from 15 kilometers to about 50 kilometers, usually unaffected by weather and lacking clouds or moisture

sublimate: change of phase from a solid to a gas

sublunary: "below the moon"; term used by Aristotle and others to describe things that were nearer to Earth than the Moon and so not necessarily heavenly in origin or composition

subtend: to extend past and mark off a chord or arc

sunspot activity: one of the powerful magnetic storms on the surface of the Sun, which causes it to appear to have dark spots; sunspot activity varies on an 11-year cycle

superconduction: the flow of electric current without resistance in certain metals and alloys while at temperatures near absolute zero

superposition: the placing of one thing on top of another

suspension bridge: a bridge held up by a system of cables or cables and rods in tension; usually having two or more tall towers with heavy cables anchored at the ends and strung between the towers and lighter vertical cables extending downward to support the roadway

symmetric: to have balanced proportions; in bilateral symmetry, opposite sides are mirror images of each other

symmetry: a correspondence or equivalence between or among constituents of a system

synapse: the narrow gap between the terminal of one neuron and the dendrites of the next

tactile: relating to the sense of touch

tailwind: a wind blowing in the same direction of that of the course of a vehicle

tangent: a line that intersects a curve at one and only one point in a local region

tectonic plates: large segments of Earth's crust that move in relation to one another

telecommuting: working from home or another offsite location

tenable: defensible, reasonable

terrestrial refraction: the apparent raising or lowering of a distant object on Earth's surface due to variations in atmospheric temperature

tessellation: a mosaic of tiles or other objects composed of identical repeated elements with no gaps

tesseract: a four-dimensional cube, formed by connecting all of the vertices of two three-dimensional cubes separated by the length of one side in four-dimensional space

theodolite: a surveying instrument designed to measure both horizontal and vertical angles

theorem: a statement in mathematics that can be demonstrated to be true given that certain assumptions and definitions (called axioms) are accepted as true

threatened species: a species whose population is viable but diminishing or has limited habitat

time dilation: the principle of general relativity which predicts that to an outside observer, clocks would appear to run more slowly in a powerful gravitational field

topology: the study of those properties of geometric figures that do not change under such nonlinear transformations as stretching or bending

topspin: spin placed on a baseball, tennis ball, bowling ball, or other object so that the axis of rotation is horizontal and perpendicular to the line of flight and the top of the object is rotating in the same direction as the motion of the object

trajectory: the path followed by a projectile; in chaotic systems, the trajectory is ordered and unpredictable

transcendental: a real number that cannot be the root of a polynomial with rational coefficients

transect: to divide by cutting transversly

transfinite: surpassing the finite

transformation: changing one mathematical expression into another by translation, mapping, or rotation according to some mathematical rule

transistor: an electronic device consisting of two different kinds of semiconductor material, which can be used as a switch or amplifier

transit: a surveyor's instrument with a rotating telescope that is used to measure angles and elevations

transitive: having the mathematical property that if the first expression in a series is equal to the second and the second is equal to the third, then the first is equal to the third

translate: to move from one place to another without rotation

translation: a rigid motion of the plane that moves each point in the same direction and by the same distance

tree: a collection of dots with edges connecting them that have no looping paths

triangulation: the process of determining the distance to an object by measuring the length of the base and two angles of a triangle

trigonometric ratio: a ratio formed from the lengths of the sides of right triangles

trigonometry: the branch of mathematics that studies triangles and trigonometric functions

tropopause: the boundry in Earth's atmosphere between the troposphere and the stratosphere at an altitude of 14 to 15 kilometers

troposphere: the lowest layer of Earth's atmosphere extending from the surface up to about 15 kilometers; the layer where most weather phenomena occur

ultra-violet radiation: electromagnetic radiation with wavelength shorter than visible light, in the range of 1 nanometer to about 400 nanometer

unbiased sample: a random sample selected from a larger population in such a way that each member of the larger population has an equal chance of being in the sample

underspin: spin placed on a baseball, tennis ball, bowling ball, or other object so that the axis of rotation is horizontal and perpendicular to the line of flight and the top of the object is rotating in the opposite direction from the motion of the object

Unicode: a newer system than ASCII for assigning binary numbers to keyboard symbols that includes most other alphabets; uses 16-bit symbol sets

union: a set containing all of the members of two other sets

upper bound: the maximum value of a function

vaccuum: theoretically, a space in which there is no matter

variable: a symbol, such as letters, that may assume any one of a set of values known as the domain

variable star: a star whose brightness noticeably varies over time

vector: a quantity which has both magnitude and direction

velocity: distance traveled per unit of time in a specific direction

verify: confirm; establish the truth of a statement or proposition

vernal equinox: the moment when the Sun crosses the celestial equator marking the first day of spring; occurs around March 22 for the northern hemisphere and September 21 for the southern hemisphere

vertex: a point of a graph; a node; the point on a triangle or polygon where two sides come together; the point at which a conic section intersects its axis of symmetry

viable: capable of living, growing, and developing

wavelengths: the distance in a periodic wave between two points of corresponding phase in consecutive cycles

whole numbers: the positive integers and zero

World Wide Web: the part of the Internet allowing users to examine graphic "web" pages

yield (interest): the actual amount of interest earned, which may be different than the rate

zenith: the point on the celestial sphere vertically above a given position

zenith angle: from an observer's viewpoint, the angle between the line of sight to a celestial body (such as the Sun) and the line from the observer to the zenith point

zero pair: one positive integer and one negative integer

ziggurat: a tower built in ancient Babylonia with a pyramidal shape and stepped sides

Topic Outline

APPLICATIONS

Agriculture
Architecture
Athletics, Technology in
City Planning
Computer-Aided Design
Computer Animation
Cryptology
Cycling, Measurements of
Economic Indicators
Flight, Measurements of
Gaming
Grades, Highway
Heating and Air Conditioning
Interest
Maps and Mapmaking
Mass Media, Mathematics and the
Morgan, Julia
Navigation
Population Mathematics
Roebling, Emily Warren
Solid Waste, Measuring
Space, Comercialization of
Space, Growing Old in
Stock Market
Tessellations, Making

CAREERS

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Agriculture
Archaeologist
Architect
Artist
Astronaut
Astronomer
Carpenter
Cartographer
Ceramicist
City Planner
Computer Analyst
Computer Graphic Artist
Computer Programmer
Conservationist
Data Analyst
Electronics Repair Technician
Financial Planner
Insurance Agent
Interior Decorator
Landscape Architect
Marketer
Mathematics Teacher
Music Recording Technician
Nutritionist
Pharmacist
Photographer
Radio Disc Jockey
Restaurant Manager
Roller Coaster Designer
Stone Mason
Web Designer

DATA ANALYSIS

Census
Central Tendency, Measures of
Consumer Data
Cryptology
Data Collection and Interpretation
Economic Indicators
Endangered Species, Measuring
Gaming
Internet Data, Reliability of
Lotteries, State
Numbers, Tyranny of
Polls and Polling
Population Mathematics
Population of Pets
Predictions
Sports Data
Standardized Tests
Statistical Analysis
Stock Market
Television Ratings
Weather Forecasting Models

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Absolute Value
Algorithms for Arithmetic
Division by Zero

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Cumulative Index

Page numbers in boldface indicate article titles. Those in italics indicate illustrations and photographs; those with an italic "f" after the page numbers indicate figures, and those with an italic "t" after the page numbers indicate tables. The boldface number preceding the colon indicates the volume number.

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